Scattering from an Impurity: Lax-Phillips approach *

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Abstract

A difference equation describing scattering from an impurity in a crystal is investigated. The time evolution consistent with the stationary operator leads to a difference wave equation, or linear version of the Hirota equation. Lax-Phillips scattering theory is developed for this evolution equation using a dynamical approach. It is shown that the Lax-Phillips scattering matrix for this special choice of the evolution equation coincides with the standard stationary scattering matrix.

1 Introduction.

Scattering from an impurity in a crystal can be described by the Schrödinger equation with a periodic potential $V(x)$ perturbed by a decreasing at infinity potential $W(x)$

$$(-\Delta + V(x) + W(x))\, \psi(x,t) = \frac{1}{i}\frac{\partial \psi}{\partial t}. \tag{1}$$

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The scattering problem amounts to comparing the evolution given by (1) with the unperturbed evolution given by the same equation, but with $W(x) = 0$. The existence and completeness of the wave operators can be shown for suitable W [1, 5, 7, 20]. The scattering matrix is an unitary function on the absolutely continuous spectrum of the periodic problem. An important problem is to understand the analytical properties of the scattering matrix on the Riemann surface of the quasimomentum. This Riemann surface has complicated structure because the corresponding Schrödinger operator has continuous band spectrum. Harmonic analysis for such Riemann surfaces has been developed by B.Pavlov and S.Fedorov [4, 15, 17, 18]. Analytical properties of the scattering matrix for the problems without periodic background can be studied with the help of the Lax-Phillips scattering theory [11]. This theory can not be applied to the problem (1) directly. The main reason is the absence of incoming and outgoing subspaces for this evolution. Such subspaces can be observed if we substitute the first derivative operator in the *rhs* of the equation (1) by another differential operator, which has the same essential spectrum as the stationary operator. This idea was proposed first by V.Evstratov and the author in 1989, but has never been developed. We present here this approach applied to partial difference operators. Lax-Phillips scattering theory is modified for the case of the stationary operator with the band continuous spectrum.

The importance of the discrete equations is explained by their numerous applications in theoretical physics (see [12] for references). Close relations between the continuous limits of the nonlinear partial difference equations and nonlinear partial differential equations, such as KdV and sinh-Gordon equations, have been discovered [6]. It appears that solutions to the discrete partial differential equations can be constructed with the help of the inverse scattering method. These equations became a useful tool during investigation of the chaotic phenomena. Exact solutions to these equations can be used to study transitions to chaos. We believe that construction of the Lax-Phillips theory for the scattering problems on the lattice can be used during the investigation of the geometrical and analytical properties of the simplectic maps, defined by nonlinear partial difference equations [3, 14].

The discrete stationary operator defined by a periodic selfadjoint Jacobi matrix is considered in this paper. Such operator with the band continuous spectrum is a discrete analog of the unperturbed Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$. The impurity is described by an additive perturbation. Spectral and resonance properties of these operators have been studied by B.Pavlov and S.Fedorov[15, 16, 18, 19]. Their approach is based on the harmonic analysis on the Riemann surface of finite genius. We develop here the dynamical approach, based on a special form of the time evolution consistent with the stationary operator. The unperturbed evolution is defined on the discrete space-time lattice by the difference analog of the wave equation. This equation can be considered as a linear analogue of the Hirota equation [3, 6]. It is shown, that the difference wave equation has d'Alembert solutions. Incoming - outgoing translational representation of the evolution operator is constructed explicitly.

The evolution, described by the perturbed discrete wave equation is defined in the space of initial data with indefinite energy form. The Lax-Phillips scattering theory for this evolution group is developed in the spirit of [10]. The case of indefinite energy form for the operators with the band spectrum has not been considered. It is shown, that Lax-Phillips scattering theory can be constructed for the evolution restricted on the subspace, where the energy form is positive definite. Incoming and outgoing translational representations of the evolution are constructed explicitly using the dynamical approach. The scattering matrix is calculated in the spectral representation.

We show in Section 5, that the Lax-Phillips and the standard stationary scattering matrices differ by an inessential factor.

2 Time evolution for periodic Jacobi matrices, difference wave equation.

We consider in this paper the stationary operator defined by a periodic selfadjoint Jacobi matrix $\mathcal{A}$ in ℓ_2. Matrix coefficients satisfy the following conditions

$\mathcal{A}_{m,n} = 0$ if $|m-n| > M$ for certain M;

$\mathcal{A}_{m+q,n+q} = \mathcal{A}_{m,n}$ for some fixed q;

$\bar{\mathcal{A}}_{m,n} = \mathcal{A}_{n,m}$.

Every such infinite matrix $\mathcal{A}$ can be presented in the tridiagonal block form

$$\mathcal{A} = \begin{pmatrix} ... & ... & ... & ... & ... & ... & ... \\ ... & A & \Gamma & 0 & 0 & 0 & ... \\ ... & \Gamma^* & A & \Gamma & 0 & 0 & ... \\ ... & 0 & \Gamma^* & A & \Gamma & 0 & ... \\ ... & 0 & 0 & \Gamma^* & A & \Gamma & ... \\ ... & 0 & 0 & 0 & \Gamma^* & A & ... \\ ... & ... & ... & ... & ... & ... & ... \end{pmatrix}$$

with A, Γ $N \times N$ matrices. The dimension N of the matrices can be chosen equal to $N = Mq$. It is natural to consider this operator in the space $\ell_2(\mathbf{C}^N) = \ell_2 \otimes \mathbf{C}^N$. This operator can be written in the following form using the shift operator $(T_\pm \psi)(n) = \psi(n \mp 1)$

$$\mathcal{A} = A + T_- \Gamma + T_+ \Gamma^*. \tag{2}$$

We restrict our consideration to the case of selfadjoint matrices $\Gamma = \Gamma^*$.

Additive perturbations of the operator $\mathcal{A}$ by finite dimensional matrices $\mathcal{B}$ will be investigated. Only one special one dimensional family of such matrices will be considered $\mathcal{B}_{n,m} = \delta_{n,0}\delta_{m,0}\alpha\Gamma,\ \alpha \in \mathbf{R}$. Thus the following stationary discrete operator will be studied

$$\mathcal{A}_\alpha = A + \alpha\Gamma\delta + (T_+ + T_-)\Gamma, \tag{3}$$

where δ denotes the indicator of the origin: $(\delta\psi)(n) = \delta_{n,0}\psi(0)$.

Consistent with the stationary operator (2) a time evolution can be defined on the discrete space-time lattice $\Lambda = \{(n,t), n \in \mathbf{Z}, t \in \mathbf{Z}\}$ by the following partial difference equation

$$(A + (T_- + T_+)\Gamma)\, f = (A + (\mathcal{T}_- + \mathcal{T}_+)\Gamma)\, f, \tag{4}$$

where $\mathcal{T}_\pm$ defines the shift operators in the time direction $(\mathcal{T}_\pm f)(n,t) = f(n, t \mp 1)$. The latter equation can be simplified to get the difference wave equation

$$(T_- + T_+) f = (\mathcal{T}_- + \mathcal{T}_+) f. \tag{5}$$

Different components of the N-dimensional vectors $f(n,t)$ on the space-time lattice are independent. That's why we are going to study this equation for one dimensional vectors $f(n,t)$ instead of working all the time in the tensor representation. This equation will be called *difference wave equation*. It is a linear version of the Hirota equation [3, 6]. Lax-Phillips scattering theory for this equation will be constructed in Section 3.

The time evolution for the perturbed problem will be introduced in a similar way using the same unperturbed operator to connect the wave function at different time moments

$$(A + \alpha\Gamma\delta + \Gamma(T_+ + T_-))\, f = (A + \Gamma(\mathcal{T}_+ + \mathcal{T}_-))\, f. \tag{6}$$

This equation can be simplified as follows

$$(\alpha\delta + (T_+ + T_-)) f = (\mathcal{T}_+ + \mathcal{T}_-) f. \tag{7}$$

The latter equation will be called *the perturbed difference wave equation*. It is possible to simplify eq. (6) due to the special form of the chosen perturbation. Different components of the N-dimensional vectors $f(n,t)$ are independent again and we are going to study this equation in the case $N = 1$. The case of arbitrary N can be considered in a similar way. The Lax-Phillips approach to this equation will be developed in Section 4.

3 Lax-Phillips theory for the difference wave equation.

We consider the Lax-Phillips scattering theory for the evolution governed by the discrete wave equation (5). This equation defines the unperturbed evolution group $\{U_0^T\}$ for the discrete values of the time parameter T. Initial data for the evolution at the moment t are two infinite vectors $F(t) = \begin{pmatrix} \{f(n,t)\} \\ \{f(n,t+1)\} \end{pmatrix}$. The evolution equation (5) connects values of the solution at the points (n,t) with the same evenness of $n+t$. The evolution can be considered independently on the even and odd sublattices $\Lambda = \Lambda^{even} \bigcup \Lambda^{odd}, \Lambda^{even} = \{(n,t) : (-1)^{n+t} = 1\}, \Lambda^{odd} = \{(n,t) : (-1)^{n+t} = -1\}$. Let us introduce the following indicator of the even lattice $\chi(n,m) = \frac{1}{2}((-1)^{n+m}+1)$. The solution of the evolution equation can be written as

$$f(n,t+T) = \sum_{\alpha=0,1}(-1)^{\alpha+T+1}\sum_{|m-n|<T}\chi(n-m,T-\alpha)f(m,t+\alpha), \quad T\geq 2. \tag{8}$$

Only initial data at the points $(m,t),(m,t-1)$ with the same evenness as $(n,t+T)$ from the region $|m-n|<T$ define the value of the solution $f(n,t+T)$. This property of the solution can be called "hyperbolicity" [3]. The same equation describes the evolution in the space direction. The problem under consideration is similar to one studied in [2].

The energy norm for this evolution equation is given by the following expression

$$\| F \|^2_{\mathcal{E}_0}(t) = \sum_{n\in\mathbf{Z}}\left(|f(n,t+1)-f(n+1,t)|^2+|f(n,t+1)-f(n-1,t)|^2\right). \tag{9}$$

This quadratic form is nonnegative and vanishes on the functions which are equal to constants on the sublattices

$$K = \{F = \begin{pmatrix} f(n,0), \\ f(n,1) \end{pmatrix} : f(n,\alpha) = a^{even}\chi(n,\alpha)+a^{odd}\chi(n+1,\alpha), \alpha=0,1\}.$$

We are going to consider the space $\mathcal{H}_0$ of all initial data with the finite energy factorized by the kernel K. The energy form defines positive functional which can be used as the norm in this space. The generator of the unperturbed evolution group

$$\mathbf{U}_0 = \begin{pmatrix} 0 & 1 \\ -1 & T_+ + T_- \end{pmatrix}. \tag{10}$$

is an unitary operator in the space $\mathcal{H}_0$ with the energy norm.

Lemma 1 *Every solution of the evolution equation (5) can be represented by the linear combination of d'Alembert waves*

$$f(n,t) = v^{\rightarrow}(t-n)+v^{\leftarrow}(t+n). \tag{11}$$

Proof. It is clear that every function $f(n,t)$ possessing the representation (11) satisfies equation (5). Substitution of the ansatz (11) into the initial conditions

$$\begin{cases} f(n,0) &= f^0(n) \\ f(n,1) &= f^1(n) \end{cases}$$

gives the following equation for the functions $v^{\rightarrow}$ and $v^{\leftarrow}$

$$v^{\rightarrow}(-n)+v^{\leftarrow}(n) = f^0(n);$$

$$v^{\rightarrow}(1-n)+v^{\leftarrow}(1+n) = f^1(n). \tag{12}$$

The latter equation does not define the functions $v^{\rightarrow}$ and $v^{\leftarrow}$ uniquely and two constants can be chosen arbitrarily, since the kernel K is two dimensional. One can put for example

$$v^{\leftarrow}(0) = v^{\rightarrow}(0) = \frac{f^0(0)}{2}, \quad v^{\leftarrow}(1) = v^{\rightarrow}(1) = \frac{f^1(0)}{2} \tag{13}$$

Difference derivatives of the functional invariant solutions on the even and odd sublattices can be calculated using equation (12)

$$\begin{array}{l} Dv^{\rightarrow}(n) = v^{\rightarrow}(n+2) - v^{\rightarrow}(n) = f^1(-n-1) - f^0(-n); \\ Dv^{\leftarrow}(n) = v^{\leftarrow}(n+2) - v^{\leftarrow}(n) = f^1(n+1) - f^0(n). \end{array} \tag{14}$$

Functions $v^{\rightarrow}$ and $v^{\leftarrow}$ can be reconstructed from their derivatives and values at the points $s = 0, 1$:

$$\left\{ \begin{array}{l} v(2n) = v(0) + \sum_{l=1}^{n} (Dv)(2l-2); \\ v(-2n) = v(0) - \sum_{l=1}^{n} (Dv)(-2l); \\ v(2n+1) = v(1) + \sum_{l=1}^{n} (Dv)(2l-1); \\ v(-2n+1) = v(1) - \sum_{l=1}^{n} (Dv)(-2l+1) \end{array} \right. \qquad n > 0. \tag{15}$$

Lemma 1 is proven. □

The norm of the derivatives of the functional invariant solutions coincides with the energy norm of initial data:

$$\begin{array}{rcl} \| Dv^{\rightarrow} \|_{\ell_2}^2 + \| Dv^{\leftarrow} \|_{\ell_2}^2 & = & \sum_{n \in \mathbf{Z}} \left(|f^1(-n-1) - f^0(-n)|^2 + |f^1(n+1) - f^0(n)|^2 \right) \\ & = & \sum_{n \in \mathbf{Z}} \left(|f^1(n) - f^0(n+1)|^2 + |f^1(n) - f^0(n-1)|^2 \right) = \\ & = & \| (f^0, f^1) \|_{\mathcal{E}}^2 \, . \end{array} \tag{16}$$

Theorem 1 *The mapping*

$$\Phi_0 : (f^0, f^1) \Rightarrow h(s) = \left(\begin{array}{c} (Dv^{\rightarrow})(-s) \\ (Dv^{\leftarrow})(-s) \end{array} \right)$$

defines a unitary translational representation for the group $\{\mathbf{U}_0^t\}$, *which is both incoming and outgoing.*

Proof. Here $h(s)$ is an element of $\ell_2(\mathbf{Z}; \mathbf{C}^2)$. It is necessary to show that

$$\Phi_0 \ \mathbf{U}_0^T = \mathbf{T}_+^T \ \Phi_0$$

to prove that this representation is a translational one. We introduce the following notation

$$u_T(n,t) = u(n, t+T).$$

Then

$$u_T(n,t) = v_T^{\rightarrow}(t-n) + v_T^{\leftarrow}(t+n),$$

where functions $v_T^{\rightarrow}(s)$ and $v_T^{\leftarrow}(s)$ can be defined using the following formulae for even s

$$v_T^{\leftarrow}(s) = v^{\leftarrow}(s+T) + \frac{1}{2}\left(v^{\rightarrow}(T) - v^{\leftarrow}(T)\right);$$

$$v_T^{\rightarrow}(s) = v^{\rightarrow}(s+T) - \frac{1}{2}\left(v^{\rightarrow}(T) - v^{\leftarrow}(T)\right)$$

and for odd s

$$v_T^{\leftarrow}(s) = v^{\leftarrow}(s+T) + \frac{1}{2}\left(v^{\rightarrow}(1+T) - v^{\leftarrow}(1+T)\right);$$

$$v_T^{\rightarrow}(s) = v^{\rightarrow}(s+T) - \frac{1}{2}\left(v^{\rightarrow}(1+T) - v^{\leftarrow}(1+T)\right).$$

Functions $v_T^{\rightarrow}(s)$ and $v_T^{\leftarrow}(s)$ can be obtained from the functions $v^{\rightarrow}(s)$ and $v^{\leftarrow}(s)$ by shifting the argument and extracting some constant, which does not give any contribution into the derivative. It follows that $h_T(s) = h(s-T)$.

We shall prove now that subspace $\ell_2(1,\infty;\mathbf{C}^2)$ is outgoing. Consider first the even sublattice and suppose, that $h(0) = h(-2) = h(-4) = ... = 0$. Both d'Alembert waves are equal to a certain constant on the negative part of the even sublattice

$$v^{\rightarrow}(0) = v^{\leftarrow}(0) = v^{\rightarrow}(2) = v^{\leftarrow}(2) = v^{\rightarrow}(4) = ...$$

and this common value can be denoted by a^{even}. It implies that $u(n,t) = 2a^{even}$ for $|n| \le t$ and this solution is equivalent to the solution vanishing on the even sublattice in the region $|n| \le t$. The odd sublattice can be considered in a similar way. The unitary character of the considered representation follows from the formula (16). $\square$

Such defined incoming and outgoing subspaces lead to the trivial Lax-Phillips scattering matrix. It shows that the chosen time evolution is consistent with the stationary operator. We note that consistent time evolution is not defined in a unique way.

4 Lax-Phillips theory for the perturbed difference wave equation.

We consider in this section the Lax-Phillips scattering theory for the evolution, described by the perturbed difference wave equation (7). The perturbed evolution group defined by this equation will be denoted by $\{\mathbf{U}^T\}$. Evolutions on the even and odd sublattices of the space-time lattice are not independent anymore.

We introduce the bilinear energy form

$$\ll F, G \gg_{\mathcal{E}} (t) = \sum_{n\in\mathbf{Z}} (< f(n,t+1) - f(n+1,t), g(n,t+1) - g(n+1,t) > +$$

$$+ < f(n,t+1) - f(n-1,t), g(n,t+1) - g(n-1,t) >) - \tag{17}$$
$$-\alpha\,(< f(0,t+1), g(0,t) > + < f(0,t), g(0,t+1) > .$$

The corresponding quadratic form is preserved during the evolution, but it is not positive definite. We are going to consider the case of positive α only. Negative α can be studied in a similar way.

The generator of the perturbed evolution is equal to

$$\mathbf{U} = \begin{pmatrix} 0 & 1 \\ -1 & T_+ + T_- + \alpha\{\delta_{n,0}\} \end{pmatrix}. \tag{18}$$

This operator has two eigenvalues with nonunit norm $z_\pm = e^{\mp b}$, where $e^b - e^{-b} = \alpha$. The corresponding eigenvectors are equal to

$$\Phi^\pm = \begin{pmatrix} \{e^{-b|n|}\} \\ e^{\mp b}\{e^{-b|n|}\} \end{pmatrix}.$$

These eigenvectors are related to the eigenvectors of the stationary operator (see section 5).

Lemma 2 *Eigenvectors $\Phi^\pm$ satisfy the following conditions with respect to the energy form $\ll *, * \gg_{\mathcal{E}}$:*

$$\ll \Phi^\pm, \Phi^\pm \gg_{\mathcal{E}} = 0, \tag{19}$$

$$\ll \Phi^\pm, \Phi^\mp \gg_{\mathcal{E}} = -e^{2b} + e^{-2b} \neq 0. \tag{20}$$

Proof. The first two equalities (19) follow from the fact, that $\Phi^\pm$ are eigenvectors of the evolution operator corresponding to the nonunit eigenvalues.

Equality (20) follows from the following calculations

$$\ll \Phi^+, \Phi^- \gg_{\mathcal{E}} = \alpha(-2 + (e^{-b} - 1)(e^b - 1)) = -(e^{2b} - e^{-2b}) \neq 0.$$

The lemma is proven. □

We'll use the following definition

Definition 1 *Subspace B of initial data is a finite dimensional subspace of all linear combinations of the vectors $\Phi^\pm$.*

There is no element in this subspace which is orthogonal to the whole B, i.e. the energy form is nondegenerate on this subspace. Subspace B is invariant subspace for the evolution operator $\mathbf{U}^T$. The evolution operator restricted on this subspace is a multiplication by diagonal matrix. Let $F \in B$, then

$$F(t) = f^+(t)\Phi^+ + f^-(t)\Phi^-.$$

The operator in this basis is equal to

$$\begin{pmatrix} f^+(t+T) \\ f^-(t+T) \end{pmatrix} = \begin{pmatrix} e^{-bT} & 0 \\ 0 & e^{bT} \end{pmatrix} \begin{pmatrix} f^+(t) \\ f^-(t) \end{pmatrix}.$$

Lemma 3 *The subspace B consists of all elements from $\ell_2 \oplus \ell_2$ of the following form*

$$B = \{e^{-b|n|} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} : g_0, g_1 \in \mathbf{C}\}.$$

Proof. Every element from B possesses the desired representation. Consider an arbitrary element $F = e^{-b|n|} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$. If it belongs to the subspace B , then the coordinates of this vector f^+, f^- should satisfy the following linear equation

$$\begin{cases} f^+ + f^- = f_0 \\ e^{-b} f^+ + e^b f^- = f_1 \end{cases} .$$

The determinant of the linear system is equal to $e^b - b^{-b} = |\alpha|$. Thus the coordinates $f^\pm$ can be calculated and F belongs to B. □

We are going to consider the evolution group restricted on the following subspace.

Definition 2 *Subspace H' is the subspace of all initial data which are $\mathcal{E}-$ orthogonal to B.*

This subspace is invariant with respect to the evolution operator because the subspace B is invariant with respect to the reverse evolution. Every element from H has unique $\mathcal{E}$-projection on H'.Orthogonal projector on the space H' will be denoted by $\mathbf{P}$.

Lemma 4 *The invariant subspace H' consists of the elements from the space H satisfying the following conditions*

$$\sum_n f(n,0)e^{-b|n|} = 0,$$
$$\sum_n f(n,1)e^{-b|n|} = 0. \qquad (21)$$

Proof. Let $F \in H'$, then it is orthogonal to all elements G from B. This condition is equivalent to the orthogonality condition with the elements $G_0 = e^{-b|n|}(1,0)$ and $G_1 = e^{-b|n|}(0,1)$

$$2\sum_n f(n,0)e^{-b|n|} - (e^{-b} + e^b)\sum_n f(n,1)e^{-b|n|} = 0, \qquad (22)$$

$$-(e^{-b} + e^b)\sum_n f(n,0)e^{-b|n|} + 2\sum_n f(n,1)e^{-b|n|} = 0. \qquad (23)$$

We get a system of linear equations for the elements $\sum_n f(n,0)e^{-b|n|}$ and $\sum_n f(n,1)e^{-b|n|}$. The determinant of the linear system (22,23) is equal to $-\alpha^2 \neq 0$. It follows that the unique solution of this linear system is trivial, which leads to the relations (21). □

Theorem 2 *The quadratic energy form is nonnegative on the subspace H' and it is equivalent to the unperturbed energy norm.*

Proof. Lemma 4 shows, that the values of the elements $F = (\{f(n,0), f(n,1)\})$ at the origin can be calculated from their difference derivatives. Conditions (21) can be written in the following form using notation $z = e^{-b}$

$$
\begin{aligned}
0 &= f(0,0) + f(1,0)z + f(2,0)z^2 + ... + f(-1,0)z + f(-2,0)z^2 + ... \\
&= f(0,0)(1 + \frac{2z^2}{1-z^2}) + f(0,1)\frac{2z}{1-z^2} \\
&\quad + \sum_{n\geq 1}(f(n,0) - f(n-1,1) + f(-n,0) - f(-n+1,1))\frac{z^n}{1-z^2} \\
&\quad + \sum_{n\geq 1}(f(n,1) - f(n-1,0) + f(-n,1) - f(-n+1,0))\frac{z^{n+1}}{1-z^2},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
0 &= f(0,1)(1 + \frac{2z^2}{1-z^2}) + f(0,0)\frac{2z}{1-z^2} \\
&\quad + \sum_{n\geq 1}(f(n,1) - f(n-1,0) + f(-n,1) - f(-n+1,0))\frac{z^n}{1-z^2} \\
&\quad + \sum_{n\geq 1}(f(n,0) - f(n-1,1) + f(-n,0) - f(-n+1,1))\frac{z^{n+1}}{1-z^2}.
\end{aligned} \tag{25}
$$

These conditions are equivalent to the linear system

$$
\begin{cases}
(1+z^2)f(0,0) + 2zf(0,1) = -zA - z^2B \\
2zf(0,0) + (1+z^2)f(0,1) = -z^2A - zB
\end{cases},
$$

where the following notations have been used:

$$A = \sum_{n\geq 1}(f(n,1) - f(n-1,0) + f(-n,1) - f(-n+1,0))z^{n-1};$$

$$B = \sum_{n\geq 1}(f(n,0) - f(n-1,1) + f(-n,0) - f(-n+1,1))z^{n-1}.$$

This linear system for $f(0,0), f(1,0)$ can be easily solved

$$
\begin{cases}
f(0,0) = \dfrac{-zA + z^2B}{1-z^2} \\
f(0,1) = \dfrac{z^2A - zB}{1-z^2}
\end{cases}.
$$

Then the following is valid for the perturbed energy form

$$\begin{aligned}
\ll F,F \gg_{\mathcal{E}} &= \ll F,F \gg_{\mathcal{E}_0} - \frac{1}{z(1-z^2)}\left(-2z^3(|A|^2+|B|^2)+z^2(z^2+1)(A\bar{B}+B\bar{A})\right) \\
&\geq \ll F,F \gg_{\mathcal{E}_0} - \frac{z(1-z)}{1+z}(|A|^2+|B|^2) \\
&\geq \left(1-\frac{2z(1-z)}{1+z}\right) \ll F,F \gg_{\mathcal{E}_0} \\
&= \frac{(4z-1)^2+7}{8(1+z)} \ll F,F \gg_{\mathcal{E}_0} \\
&\geq 0.
\end{aligned}$$

We used here the following estimate $|A|^2+|B|^2 \leq 2 \ll F,F \gg_{\mathcal{E}_0}$. The upper estimate for the quadratic form can be proven as follows

$$\begin{aligned}
\ll F,F \gg_{\mathcal{E}} &\leq \ll F,F \gg_{\mathcal{E}_0} + \frac{z}{1-z^2}\left|-2z(|A|^2+|B|^2)+(z^2+1)(A\bar{B}+B\bar{A})\right| \\
&\leq \ll F,F \gg_{\mathcal{E}_0} + z\frac{(1+z)^2}{(1-z^2)}(|A|^2+|B|^2) \\
&\leq \frac{(4z+1)^2+7}{8(1-z)} \ll F,F \gg_{\mathcal{E}_0} .
\end{aligned}$$

This finishes proof of the theorem. □

Consider the subspaces D_{inc} and $D_{out} \in H$ formed by the initial data which define solutions of the evolution equation, vanishing in the corresponding regions $|n| \leq -t,\ t<0$ and $|n| \leq t,\ t>0$. These subspaces do not belong to H'. Projections of these subspaces on the space H' will define the incoming and outgoing subspaces for the restricted evolution.

Lemma 5 *The subspace D_{inc} consists of all initial data satisfying the following conditions*

$$\begin{array}{ll}
f(n,0)=f(n+1,1), n\leq -2; & f(n,0)=f(n-1,1), n\geq 2; \\
f(0,1)=f(-1,0)+f(1,0); & f(0,0)=0.
\end{array}$$

The subspace D_{out} consists of all initial data satisfying the conditions

$$\begin{array}{ll}
f(n,0)=f(n-1,1), n\leq -1; & f(n,0)=f(n+1,1), n\geq 1; \\
f(0,0)=f(-1,1)+f(1,1); & f(0,1)=0.
\end{array}$$

Proof can be carried out by induction starting from the origin.

These two subspaces are invariant with respect to the evolution and reverse evolution operators correspondingly

$$\mathbf{U}D_{out} \subset D_{out}, \quad \mathbf{U}^{-1}D_{inc} \subset D_{inc}.$$

The energy norm is positive definite on each of these subspaces. It is possible to define special bases in the subspaces D_{inc} and D_{out}, such that the evolution operator will act as translation in the corresponding representations. Consider the orthogonal complement of the subspace $\mathbf{U}D_{out}$ in the subspace D_{out} : $N_{out} = D_{out} \ominus \mathbf{U}D_{out}$. Subspace N_{out} is a finite dimensional subspace. Vectors

$$E_{-1}^{out,r} = \begin{pmatrix} \Theta(m)\chi(0,m) \\ \Theta(m-1)\chi(1,m) \end{pmatrix}, \quad E_{-1}^{out,l} = \begin{pmatrix} \Theta(-m)\chi(0,m) \\ \Theta(-m-1)\chi(1,m) \end{pmatrix}$$

form a basis in the subspace N_{out}. Then the basis associated with the outgoing subspace can be chosen equal to the following one for all $n \geq -1$

$$E_n^{out,l} = \begin{pmatrix} e_n^{out,l}(m,0) \\ e_n^{out,l}(m,1) \end{pmatrix} = \begin{pmatrix} \Theta(-n-m-1)\chi(n,m+1) \\ \Theta(-n-m-2)\chi(n,m) \end{pmatrix}, n = -1, 0, 1, 2, 3, ...;$$

$$E_n^{out,r} = \begin{pmatrix} e_n^{out,r}(m,0) \\ e_n^{out,r}(m,1) \end{pmatrix} = \begin{pmatrix} \Theta(-n+m-1)\chi(n,m+1) \\ \Theta(-n+m-2)\chi(n,m) \end{pmatrix}, n = -1, 0, 1, 2, 3, ...$$

This outgoing basis can be extended for all negative n using the translational property, which is valid for all $n \geq 0$: $\quad E_{n-1}^{out} = \mathbf{U}^{-1}E_n^{out}$.

Similar translational basis in the subspace D_{inc} is formed by the elements

$$E_n^{inc,l} = \begin{pmatrix} e_n^{inc,l}(m,0) \\ e_n^{inc,l}(m,1) \end{pmatrix} = \begin{pmatrix} \Theta(n-m-1)\chi(n,m+1) \\ \Theta(n-m)\chi(n,m) \end{pmatrix}, n = 0, -1, -2, -3, ...;$$

$$E_n^{inc,r} = \begin{pmatrix} e_n^{inc,r}(m,0) \\ e_n^{inc,r}(m,1) \end{pmatrix} = \begin{pmatrix} \Theta(n+m-1)\chi(n,m+1) \\ \Theta(n+m)\chi(n,m) \end{pmatrix}, n = 0, -1, -2, -3, ...$$

One can use the translational property to define the incoming basis vectors for all $n \in \mathbf{Z}$.

Consider the projections of these subspaces and corresponding bases on H'. The following notations will be used in the future

$$D'_{out} = \mathbf{P}D_{out}; \; D'_{inc} = \mathbf{P}D_{inc};$$

$$\mathcal{E}_n^{inc,l} = \mathbf{P}E_n^{inc,l}; \; \mathcal{E}_n^{inc,r} = \mathbf{P}E_n^{inc,r}; \; \mathcal{E}_n^{out,l} = \mathbf{P}E_n^{out,l}; \; \mathcal{E}_n^{out,r} = \mathbf{P}E_n^{out,r}.$$

Lemma 6 *The subspaces D'_{inc} and D'_{out} span the space H'.*

Proof. To prove this Lemma, it is enough to show, that subspaces D_{inc}, D_{out} span the space H. Consider arbitrary element F from H and solution of the homogeneous equation with these initial data. Every such solution can be presented by the combination of the d'Alembert waves (Lemma 1) and certain vectors from the kernel $\mathbf{K}$

$$F = \begin{pmatrix} v^{\rightarrow}(-n) + v^{\leftarrow}(n) + a^{even}\chi(n,0) + a^{odd}\chi(n,1) \\ v^{\rightarrow}(1-n) + v^{\leftarrow}(1+n) + a^{even}\chi(n,1) + a^{odd}\chi(n,0) \end{pmatrix}.$$

Every element from the kernel $\mathbf{K}$ and d'Alembert waves are equal to the linear combination of the basis elements $E^{out}_{n-1}, E^{inc}_{-n}, n = 0, 1, 2, ...$:

$$\begin{aligned} F \;=\; & v^{\rightarrow}(0)E^{out,r}_{-1} + v^{\rightarrow}(-1)E^{out,r}_{0} + \sum_{n\geq 1}(v^{\rightarrow}(-n-1) - v^{\rightarrow}(-n+1))E^{out,r}_{n} \\ & + v^{\rightarrow}(1)E^{inc,l}_{0} + v^{\rightarrow}(2)E^{inc,l}_{-1} + \sum_{n\geq 2}(v^{\rightarrow}(n+1) - v^{\rightarrow}(n-1))E^{inc,l}_{-n} \\ & + v^{\leftarrow}(0)E^{out,l}_{-1} + v^{\leftarrow}(-1)E^{out,l}_{0} + \sum_{n\geq 1}(v^{\leftarrow}(-n-1) - v^{\leftarrow}(-n+1))E^{out,l}_{n} \\ & + v^{\leftarrow}(1)E^{inc,r}_{0} + v^{\leftarrow}(2)E^{inc,r}_{-1} + \sum_{n\geq 2}(v^{\leftarrow}(n+1) - v^{\leftarrow}(n-1))E^{inc,r}_{-n} \\ & + a^{odd}(E^{inc,l}_{0} + E^{out,r}_{0}) + a^{even}(E^{out,r}_{-1} + E^{inc,l}_{-1}). \end{aligned} \tag{26}$$

The last sum converges for every element with the finite energy norm. It follows, that the spaces D'_{out}, D'_{inc} span together the space H'. $\square$

Theorem 3 *The subspaces D'_{inc} and D'_{out} are correspondingly incoming and outgoing subspaces for the perturbed evolution in H'.*

Proof. We are going to prove, that subspace D'_{out} is outgoing in the Lax-Phillips sense. We have to show, that

1. $\mathbf{U}D'_{out} \subset D'_{out}$,
2. $\cap \mathbf{U}^T D'_{out} = \{0\}$,
3. $\cup \mathbf{U}^T D'_{out}$ is dense in H'.

The first two properties can be proven using similar properties of the subspace D_{out}. The perturbed evolution restricted on the subspace D_{out} coincides with the unperturbed one. Hence the first two properties follow from the similar results for the unperturbed evolution.

Consider the third property now. The subspaces D'_{out} and D'_{inc} span the space H'. Hence it is enough to show that every vector from D'_{inc} is an element of $\cup \mathbf{U}^T D'_{out}$. We are going to show, that every basis element $\mathcal{E}^{inc,l}_{n}$ belongs to the closure of $\cup \mathbf{U}^T D'_{out}$ in the energy norm, i.e. that there exist constants a_n^{ll}, a_n^{lr}, such that the following representation converges in the energy norm

$$\mathcal{E}_n^{inc,l} = \sum_k \left(a_{n-k}^{ll} \mathcal{E}_k^{out,l} + a_{n-k}^{lr} \mathcal{E}_k^{out,r} \right). \tag{27}$$

Coefficients in the last representation depend on the difference $n-k$ because the evolution operator acts as translation in both incoming and outgoing bases. The same result for the vectors $\mathcal{E}_n^{inc,r}$ will follow from the symmetry of the problem with respect to the point zero. Suppose, that representation (27) is proven, then the following formula is valid also

$$\mathcal{E}_n^{inc,r} = \sum_k \left(a_{n-k}^{rr} \mathcal{E}_k^{out,r} + a_{n-k}^{rl} \mathcal{E}_k^{out,l} \right), \quad a_k^{ll} = a_k^{rr}, \ a_k^{rl} = a_k^{lr}. \tag{28}$$

We prove the existence of such representations for $n = 0$ first. The same property for arbitrary n will follow from the fact, that these bases define translational representations of the evolution operator.

Consider first the following property of the basis elements

$$E_0^{inc,l} - E_0^{inc,r} = E_0^{out,l} - E_0^{out,r}.$$

The same property is valid for their projections and we get the following condition on the constants a^{ll}, a^{rl}

$$a_k^{ll} - a_k^{rl} = \delta_{k,0}. \tag{29}$$

Consider now the elements from the subspace B. The following representations are valid for $\Phi^\pm$

$$\begin{aligned} \Phi^+ &= \Delta_0 - e^{-b}\Delta_1 + e^{-b}(E_0^{inc,r} + E_0^{inc,l}) \\ &\quad + e^{-2b}(E_{-1}^{inc,r} + E_{-1}^{inc,l}) + e^{-b} \sum_{m \geq 2} (E_{-m}^{inc,r} + E_{-m}^{inc,l}) e^{-mb}(1 - e^{2b}); \end{aligned} \tag{30}$$

$$\begin{aligned} \Phi^- &= -\Delta_0 + e^{b}\Delta_1 + (E_{-1}^{out,r} + E_{-1}^{out,l}) \\ &\quad + e^{-b}(E_0^{out,r} + E_0^{out,l}) + e^{-b} \sum_{m \geq 1} (E_m^{out,r} + E_m^{out,l}) e^{-mb}(1 - e^{2b}), \end{aligned} \tag{31}$$

where $\Delta_0 = \begin{pmatrix} \{\delta_{n,0}\} \\ 0 \end{pmatrix}; \Delta_1 = \begin{pmatrix} 0 \\ \{\delta_{n,0}\} \end{pmatrix}$.

Projections of $\Phi^\pm$ on H' are equal to zero and projections of the vectors Δ_0, Δ_1 can be calculated

$$\begin{aligned} \mathbf{P}\Delta_1 &= \frac{-1}{\alpha} \Big\{ e^{-b}(\mathcal{E}_0^{inc,r} + \mathcal{E}_0^{inc,l}) + e^{-2b}(\mathcal{E}_{-1}^{inc,r} + \mathcal{E}_{-1}^{inc,l}) \\ &\quad + e^{-b} \sum_{m \geq 2} (\mathcal{E}_{-m}^{inc,r} + \mathcal{E}_{-m}^{inc,l}) e^{-mb}(1 - e^{2b}) + (\mathcal{E}_{-1}^{out,r} + \mathcal{E}_{-1}^{out,l}) \\ &\quad + e^{-b}(\mathcal{E}_0^{out,r} + \mathcal{E}_0^{out,l}) + e^{-b} \sum_{m \geq 1} (\mathcal{E}_m^{out,r} + \mathcal{E}_m^{out,l}) e^{-mb}(1 - e^{2b}) \Big\}; \end{aligned} \tag{32}$$

$$
\begin{aligned}
\mathbf{P}\Delta_0 \quad = \quad & \frac{-1}{\alpha}\Big\{(\mathcal{E}_0^{inc,r}+\mathcal{E}_0^{inc,l})+e^{-b}(\mathcal{E}_{-1}^{inc,r}+\mathcal{E}_{-1}^{inc,l}) \\
& +\sum_{m\geq 2}(\mathcal{E}_{-m}^{inc,r}+\mathcal{E}_{-m}^{inc,l})e^{-mb}(1-e^{2b})+e^{-b}(\mathcal{E}_{-1}^{out,r}+\mathcal{E}_{-1}^{out,l}) \\
& +e^{-2b}(\mathcal{E}_0^{out,r}+\mathcal{E}_0^{out,l})+e^{-2b}\sum_{m\geq 1}(\mathcal{E}_m^{out,r}+\mathcal{E}_m^{out,l})e^{-mb}(1-e^{2b})\Big\}.
\end{aligned} \tag{33}
$$

We use the following identity for the basis elements

$$
E_0^{inc,l}+E_0^{inc,r}+E_{-1}^{inc,l}+E_{-1}^{inc,r}=E_{-1}^{out,l}+E_{-1}^{out,r}+E_0^{out,l}+E_0^{out,r}+2(\Delta_1-\Delta_0). \tag{34}
$$

The latter identity is valid also for the projections of basis elements. Substitution of the representation (27) into the last equality together with the property (29) leads to the following equation on the coefficients a_n^{rl}

$$
\begin{aligned}
& \sum_k(\delta_{k,0}+2a_{-k}^{rl})(\mathcal{E}_k^{out,l}+\mathcal{E}_k^{out,r})+\sum_k(\delta_{k+1,0}+2a_{-1-k}^{rl})(\mathcal{E}_k^{out,l}+\mathcal{E}_k^{out,r})- \\
& -(\mathcal{E}_{-1}^{out,l}+\mathcal{E}_{-1}^{out,r})-(\mathcal{E}_0^{out,l}+\mathcal{E}_0^{out,r})= \\
& =\frac{2(1-e^b)}{\alpha}\Big\{\sum_k(\delta_{k,0}+2a_{-k}^{rl})(\mathcal{E}_k^{out,l}+\mathcal{E}_k^{out,r})+e^{-b}\sum_k(\delta_{k+1,0}+2a_{-1-k}^{rl})(\mathcal{E}_k^{out,l}+\mathcal{E}_k^{out,r})+ \\
& +\sum_{m\geq 2}(1-e^{2b})e^{-mb}\sum_k(\delta_{k+m,0}+2a_{-m-k}^{rl})(\mathcal{E}_k^{out,l}+\mathcal{E}_k^{out,r})- \\
& -(\mathcal{E}_{-1}^{out,r}+\mathcal{E}_{-1}^{out,l})-e^{-b}(\mathcal{E}_0^{out,r}+\mathcal{E}_0^{out,l})-e^{-b}\sum_{m\geq 1}(\mathcal{E}_m^{out,r}+\mathcal{E}_m^{out,l})(1-e^{2b})e^{-mb}\Big\}
\end{aligned} \tag{35}
$$

If the coefficients in front of all terms $(\mathcal{E}_n^{out,r}+\mathcal{E}_n^{out,l})$ at the *lhs* and at the *rhs* of the last equation are equal, then this equality holds. We get infinite linear system on $\{a_n^{rl}\}$

$$
\sum_m Q_{nm}(\delta_{m,0}+2a_m^{rl})=q_n, \tag{36}
$$

where

$$
\begin{array}{lll}
Q_{nm}=Q_{n-m}; & Q_m=0, m\geq 1; & Q_0=1-e^b; \\
Q_{-1}=2e^{-b}-1-e^b; & Q_m=2(1-e^{2b})e^{mb}, m\leq -2 & q_m=Q_{-m-1}.
\end{array}
$$

This system can easily be solved using the Fourier representation $\hat{q}(\varphi)=\sum_n q_n e^{in\varphi}$. The infinite matrix Q in the Fourier representation is the multiplication operator and equation (36) is transformed to the following one

$$
\frac{(1-e^b)(1+e^{-i\varphi})(1+e^{-b}e^{-i\varphi})}{(1-e^{-b}e^{-i\varphi})}(1+2\hat{a}^{rl}(\varphi))=\frac{e^{-i\varphi}(1-e^b)(1+e^{i\varphi})(1+e^{-b}e^{i\varphi})}{(1-e^{-b}e^{i\varphi})}. \tag{37}
$$

The coefficients $a_n^{ll}, a_n^{rl}, a_n^{lr}, a_n^{rr}$ can easily be calculated from their Fourier representations

$$\hat{a}^{rl}(\varphi) = \hat{a}^{lr}(\varphi) = \frac{2i\sin\varphi}{\alpha - 2i\sin\varphi}$$

$$\hat{a}^{rr}(\varphi) = \hat{a}^{ll}(\varphi) = \frac{\alpha}{\alpha - 2i\sin\varphi}. \tag{38}$$

It is proven, that representation (27) holds and corresponding sums converge in the energy norm. It accomplishes the proof of the Theorem. □

Corrolary. The sets of elements $\{\mathcal{E}_n^{inc,l}, \mathcal{E}_n^{inc,r}\}_{n=-\infty}^{+\infty}$ and $\{\mathcal{E}_n^{out,l}, \mathcal{E}_n^{out,r}\}_{n=-\infty}^{+\infty}$ are orthogonal bases in the space H'.

Incomimg and outgoing translation representations of the evolution operator in the space H' can be defined.

Theorem 4 *The Lax-Phillips scattering matrix is meromorphic on the Riemann surface of the quasimomentum and it is equal to the following multiplication operator in the spectral representation*

$$\hat{\mathbf{S}} = \begin{pmatrix} \dfrac{-2i\sin\varphi}{\alpha + 2i\sin\varphi} & \dfrac{\alpha}{\alpha + 2i\sin\varphi} \\ \dfrac{\alpha}{\alpha + 2i\sin\varphi} & \dfrac{-2i\sin\varphi}{\alpha + 2i\sin\varphi} \end{pmatrix}. \tag{39}$$

Proof. The Lax-Phillips scattering matrix connects the incoming and outgoing translational representations for the evolution operator. Suppose $F \in H'$, then it can be presented by one of the following sums

$$F = \sum_n \left(f_n^{inc,l}\mathcal{E}_n^{inc,l} + f_n^{inc,r}\mathcal{E}_n^{inc,r} \right) = \sum_n \left(f_n^{out,r}\mathcal{E}_n^{out,r} + f_n^{out,l}\mathcal{E}_n^{out,l} \right).$$

The scattering matrix is the operator, which connects these two representations

$$\mathbf{S} : \begin{pmatrix} f^{inc,l} \\ f^{inc,r} \end{pmatrix} \Rightarrow \begin{pmatrix} f^{out,r} \\ f^{out,l} \end{pmatrix}.$$

This operator can be calculated using the decomposition of the basis vectors $\mathcal{E}_n^{inc,l}$ and $\mathcal{E}_n^{inc,r}$ in the outgoing basis $\mathcal{E}_n^{out,l}, \mathcal{E}_n^{out,l}$ (27,28). Proof of Theorem 3 shows, that it is more convenient to work in the spectral representation $\hat{f}(\varphi) = \sum_n f_n e^{in\varphi}$. The following formula connects the incoming and outgoing spectral representations

$$\begin{pmatrix} \hat{f}^{out,l}(\varphi) \\ \hat{f}^{out,r}(\varphi) \end{pmatrix} = \begin{pmatrix} \hat{a}^{lr}(-\varphi) & \hat{a}^{rr}(-\varphi) \\ \hat{a}^{ll}(-\varphi) & \hat{a}^{rl}(-\varphi) \end{pmatrix} \begin{pmatrix} \hat{f}^{inc,l}(\varphi) \\ \hat{f}^{inc,r}(\varphi) \end{pmatrix}.$$

It follows, that the scattering matrix in the spectral representation is given by the formula (39). Analytical properties of the scattering matrix follow immediately from this formula. □

5 Stationary scattering matrix.

We consider in this section stationary approach to the scattering problem, described by the stationary operators $\mathcal{A}$ and $\mathcal{A}_\alpha$ (2,3). We show, that Lax-Phillips scattering matrix coincides with the stationary one in the simplest nontrivial case of $N = 2$ and following matrices A, Γ

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \lambda \in \mathbf{R}_+; \quad \Gamma = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}, \gamma \in \mathbf{R}.$$

Eigenfunctions of the operator $\mathcal{A}$ are of Bloch type

$$\psi(n+1) = e^{ip}\psi(n). \tag{40}$$

Dispersion equation, which connects the quasimomentum $p \in [0, 2\pi)$ and the energy E of the eigenfunction

$$E = \pm E(p), \; E(p) = \sqrt{\lambda^2 + 4\gamma^2 \cos^2 p}$$

can be obtained by substituting ansatz (40) into the eigenfunction equation

$$A\psi(n) + \Gamma(\psi(n+1) + \psi(n-1)) = E\psi(n).$$

The dispersion equation defines two bands of continuous spectrum situated symmetrically with respect to the origin $[-\sqrt{\lambda^2 + 4\gamma^2}, -\lambda] \bigcup [\lambda, \sqrt{\lambda^2 + 4\gamma^2}]$. Normalized eigenfunctions of the operator can be chosen with the real zero component

$$\psi_+(p,n) = \frac{e^{inp}}{2\sqrt{\pi E(p)}} \begin{pmatrix} \sqrt{E(p) + \lambda} \\ \mathrm{sign}(\cos p)\sqrt{E(p) - \lambda} \end{pmatrix}$$

$$\psi_-(p,n) = \frac{e^{inp}}{2\sqrt{\pi E(p)}} \begin{pmatrix} -\mathrm{sign}(\cos p)\sqrt{E(p) - \lambda} \\ \sqrt{E(p) + \lambda} \end{pmatrix}. \tag{41}$$

These eigenfunctions satisfy the following normalization conditions

$$\begin{aligned} &\ll \psi_+(p), \psi_+(p') \gg = \delta(p - p'); \\ &\ll \psi_-(p), \psi_-(p') \gg = \delta(p - p'); \\ &\ll \psi_+(p), \psi_-(p') \gg = 0. \end{aligned}$$

The eigenfunction $\psi_+(p)$ corresponds to the positive band of the spectrum and $\psi_-(p)$ - to the negative one.

Eigenfunctions $\phi(p,n), 0 \le p \le \pi/2$ of the perturbed stationary operator $\mathcal{A}_\alpha$ are equal to the linear combinations of the unperturbed Bloch waves outside the origin

$$\phi_\pm(p,n) = \begin{cases} a_1\psi_\pm(p,n) + b_3\psi_\pm(-p,n) + a_2\psi_\pm(\pi - p, n) + b_4\psi_\pm(p - \pi, n), \; n < 0 \\ a_3\psi_\pm(-p,n) + b_1\psi_\pm(p,n) + a_4\psi_\pm(p - \pi, n) + b_2\psi_\pm(\pi - p, n), \; n > 0 \end{cases} \tag{42}$$

with certain coefficients $a_j, b_j \in \mathbf{C}, j = 1,2,3,4$. Incoming eigenfunctions $\phi_{\pm}^{inc,m}, m = 1,2,3,4$ are characterized by the following conditions on the coefficients $a_j^{inc,m} = \delta_{j,m}, j = 1,2,3,4$. Similar outgoing eigenfunctions $\phi_{\pm}^{out,m}, m = 1,2,3,4$ are defined by the conditions $b_j^{out,m} = \delta_{j,m}, j = 1,2,3,4$. Incoming and outgoing eigenfunctions define two different spectral decompositions for the absolutely continuous subspace of the operator $\mathcal{A}_\alpha$. The stationary scattering matrix is the operator, which connects these spectral representations.

We are going to calculate as example incoming eigenfunction of the first type. Corresponding coefficients in the representation (42) will be denoted by $a_j^{inc,1}, b_j^{inc,1}$. Special choice of the perturbation defines no interaction between the eigenfunctions with different absolute value of the quasimomenta and coefficients $b_2^{inc,1}$, $b_4^{inc,1}$ can be chosen equal to zero. The form of the ansatz is chosen in such a way, that the eigenfunction equation is obviously satisfied for all values of the parameters at every point except $n = 0, \pm 1$. Consideration of the equation at these points allows us to calculate unknown coefficients

$$b_1^{inc,1} = b_2^{inc,2} = b_3^{inc,3} = b_4^{inc,4} \equiv T(p) = \frac{2i \sin p}{\alpha + 2i \sin p}$$

$$b_3^{inc,1} = b_4^{inc,2} = b_1^{inc,3} = b_2^{inc,4} \equiv R(p) = \frac{-\alpha}{\alpha + 2i \sin p}. \tag{43}$$

The transition $T(p)$ and reflection $R(p)$ coefficients posses the following property $T(-p+\pi) = T(p), R(-p+\pi) = R(p)$. These coefficients form the 4×4 stationary scattering matrix

$$S(E) = \begin{pmatrix} T(p) & R(p) & 0 & 0 \\ R(p) & T(p) & 0 & 0 \\ 0 & 0 & T(p) & R(p) \\ 0 & 0 & R(p) & T(p) \end{pmatrix} = S_0 \times \mathbf{I}, \tag{44}$$

$$S_0(E) = \begin{pmatrix} T(p) & R(p) \\ R(p) & T(p) \end{pmatrix}. \tag{45}$$

Here $\mathbf{I}$ denotes the unit 2×2 matrix.

The matrix $S_0(E)$ differs from the Lax-Phillips scattering matrix by the factor -1. It is due to the fact that incoming and outgoing bases have been chosen in such a way, that the Lax-Phillips scattering matrix for the unperturbed operator is equal to -1.

The stationary scattering matrix has singularities at the points $-2i \sin p = \alpha$ on the Riemann surface of the quasimomentum. These singularities correspond to the bound states. The stationary operator $\mathcal{A}_\alpha$ has two eigenvalues

$$E_\pm = \pm\sqrt{\lambda^2 + 4\gamma^2 + \alpha^2\gamma^2}, \tag{46}$$

which depend on the absolute value of the interaction parameter α only. These eigenvalues are situated outside the continuous spectrum. We have defined by b the real positive solution of the equation $e^b - e^{-b} = \alpha$. Then the eigenfunctions of the stationary operator have the form

$$\phi^{\pm}(n) = \phi^{\pm}(0)e^{-b|n|}.$$

The amplitudes $\phi^{\pm}(0)$ are equal to

$$\phi^{\pm}(0) = c_{\pm}\begin{pmatrix} 2\gamma\cosh b \\ E_{\pm} - \lambda \end{pmatrix}$$

with the normalizing coefficients $c_{\pm}$. Calculated eigenfunctions of the perturbed stationary operator are related to the eigenfunctions of the perturbed evolution group corresponding to the eigenvalues $z_{\pm} = e^{\mp b}$.

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