Inverse scattering problem on the half line and positon solutions of the KdV equation. [1]

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Abstract. The inverse scattering problem for the Schrödinger operator on the half line is studied for potentials of positon type with long range oscillating tails at infinity. The inverse problem can be solved for the scattering matrices with arbitrary finite phase shift. Solution of the inverse problem is unique if the following scattering data are given: scattering matrix, energies of the bound states and corresponding normalizing constants, zeroes of the spectral density on the real line.

1.Introduction

The Schrödinger operator $Hu = -\frac{d^2u}{dx^2} + V(x)u$ defined on the functions on the half axis satisfying the Dirichlet boundary condition at the origin $u(0) = 0$ is investigated in the present paper. We study the inverse scattering problem for potentials with long range oscillating tails at infinity violating the Faddeev condition $\int_0^\infty (1+x)|V(x)|dx < \infty$. Such potentials were studied first in relation with the positive bound states [1,3,9]. Corresponding solutions of the KdV equation were named positons [11,12]. Uniqueness results for the inverse problem can be established using the inverse spectral problem originaly studied by I.M.Gelfand, B.M.Levitan, V.A.Marchenko and L.D.Faddeev [4,2,13] and recently by F.Gesztesy and B.Simon [5]. The existence of the potential has been proven only for the scattering matrices $S(k)$ with the negative phase shift on the real axis: $arg\ S(k)|_{-\infty}^{\infty} \leq 0$, which is related to the number of the bound states through the Levinson theorem. We prove that the inverse problem can be solved for the scattering matrices with any finite phase shift on the real line (not only negative one). It is related to the fact that the spectral density coresponding to long range oscillating potentials can have zeroes on the real axis. See [6,7] where examples of such potentials are considered. We note, that potentials considered have only negative bound states. We continue here investigation carried out in [6-8].

2.Scattering matrices and Jost functions.

Def 1. *The set $\mathcal{S}$ of scattering matrices consists of all continuous functions on the real line with the following properties:* $|S(k)| = S(\infty) = 1$;

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$S(-k)=\overline{S(k)}$; $S(k)-1=\int_{-\infty}^{\infty}\sigma(t)e^{ikt}dt$, $\int_{-\infty}^{\infty}|\sigma(t)|dt<\infty$.

The set of Jost functions for potentials with the finite first momentum will be denoted by $\mathcal{F}_F$. Let $F(k)\in\mathcal{F}_F$ then

1. $F(k)$ is analytical in the open upper halfplane $\Im k>0$;
2. $F(k)$ is continuous in the closed upper half plane $\Im k\geq 0$;
3. $\lim_{k\to\infty}F(k)=1; F(k)=\overline{F(-\bar{k})}$;
4. $F(k)$ has representation $F(k)=1+\int_0^{\infty}f(t)e^{ikt}dt$; $\int_0^{\infty}|f(t)|dt<\infty$.

Def 2. *The set $\mathcal{F}$ is the set of functions $F(k)$ which possess the following representation*

$$F(k)=\frac{k+ia_0}{k}\left(\prod_{j=1}^{N_{sing}}\frac{k+ia_j}{k-b_j}\frac{k+i\bar{a}_j}{k+b_j}\right)F_F(k), \tag{1}$$

where function $F_F(k)\in\mathcal{F}_F$, $a_0\geq 0$, $\Re a_j>0$. If $F_F(0)=0$, then $a_0=0$.

The spectral density $\rho(k)=\frac{2}{\pi}\frac{k^2}{|F(k)|^2}$ corresponding to the Jost functions from $\mathcal{F}$ vanishes at the points $\pm b_j$.

Def 3. *The set $\mathcal{D}$ of scattering data consists of the following data:*
1. scattering matrix $S(k)$ from $\mathcal{S}$;
2. natural numbers $N_{bs},N_{sing}\in\mathbf{N}$ and $\alpha=-1,0,1$ such, that arg $S(k)|_{-\infty}^{\infty}=-2\pi(2N_{bs}-2N_{sing}+\alpha)$;
3. the energies of the bound states and corresponding normalizing constants $E_j<0, s_j>0, j=1,2,...,N_{bs}$;
4. positions of the zeroes of the spectral density on the real line $b_j>0, j=1,2,...,N_{sing}$.

Lemma 1. *Let $F(k)$ be any function from $\mathcal{F}$, then the quotient*

$$S(k)=\frac{F(-k)}{F(k)} \tag{2}$$

is a function from $\mathcal{S}$. Let $S(k)$ be any function from $\mathcal{S}$, then there exists $F(k)\in\mathcal{F}$, such that (2) holds. The function $F(k)$ from $\mathcal{F}$ is defined uniquelly by the scattering data from $\mathcal{D}$.

Lemma 2. *The scattering data from $\mathcal{D}$ and the spectral measures corresponding to the Jost functions from $\mathcal{F}$ are in one-to-one correspondence.*

2.The inverse problem

The following class of potentials will be considered

Def 5. *The set $\mathcal{V}$ of potentials is the set of all locally integrable potentials $V(x)$ on the half axis having the following representation*

$$V(x)=\frac{\sum_{j=1}^{N}c_j\sin 2(b_jx+\beta_j)}{x+1}+V_2(x)+V_F(x),$$

where $V_2(x) = O(\frac{1}{(1+x)^2})$; $\int_0^\infty (1+x)|V_F|dx < \infty$ *and constants* c_j, b_j, β_j *are real.*

Lemma 3. *Let us assume that potential* $\tilde{V} \in \mathcal{V}_F$ *and function* $S_1(k) = \prod_{j=1}^{N_{sing}} \frac{k-ia_j}{k+ia_j} \frac{k-i\bar{a}_j}{k+i\bar{a}_j}$ *are given, then there exists potential* $V_1(x) \in \mathcal{V}$, *such that* $S_1(k)$ *is a scattering matrix for the pair of operators* $\tilde{H} = -\frac{d^2}{dx^2} + \tilde{V}$, $\quad H_1 = -\frac{d^2}{dx^2} + \tilde{V} + V_1$. **Proof.** Solution of the problem is given by the following formula

$$V_1(x) = -2\frac{\partial^2}{\partial x^2} \log \det \mathbf{Q}(x) = -2\frac{(\det \mathbf{Q})'' \det \mathbf{Q} - (\det \mathbf{Q}')^2}{(\det \mathbf{Q})^2},$$

where $\mathbf{Q}(x)$ is $2N_{sing} \times 2N_{sing}$ matrix with the following elements

$$\begin{array}{rcl} Q_{2l-1,2j-1}(x) & = & \frac{2b_j i}{(a_l^2+b_j^2)^2} W[f(ia_l,x), \varphi(b_j,x)] - \frac{i}{a_l^2+b_j^2} W[f(ia_l,x), \frac{\partial \varphi}{\partial k}(b_j,x)]; \\ Q_{2l-1,2j}(x) & = & \frac{1}{a_l^2+b_j^2} W[f(ia_l,x), \varphi(b_j,x)]; \end{array}$$

$$Q_{2l,2j-1}(x) = -\overline{Q_{2l-1,2j-1}(x)}; \;\; Q_{2l,2j}(x) = \overline{Q_{2l,2j-1}(x)}.$$

The regular solution for potential $\tilde{V} + V_1$ is equal to

$$\varphi_1(k,x) = \frac{\det \begin{vmatrix} \mathbf{Q}(x) & f(x) \\ \beta_1(k,x) & \varphi(k,x) \end{vmatrix}}{\det \mathbf{Q}(x)} \tag{3}$$

with the following vectors

$$\beta_{1,2j-1} = -i \int_0^x \left(\frac{\partial}{\partial k} \varphi(b_j,t) \right) \varphi(k,t) dt, \;\; \beta_{1,2j} = \int_0^x \varphi(b_j,t) \varphi(k,t) dt$$

$$(f(x))_{2l-1} = f(ia_l,x), \;\; (f(x))_{2l} = f(i\bar{a}_l,x).$$

Functions $f(k,x)$ and $\varphi(k,x)$ denote the Jost and regular solutions for the potential $\tilde{V}$. The scattering matrix is defined by the asymptotics of the regular solution.

Lemma 4. *Let potential* V *be from the set* $\mathcal{V}$, *such that the Jost function* $F_V(k)$ *does not vanish at the origin* $F_V(0) \neq 0$ *and the operator does not have zero energy bound state. Then for any positive* $0 < a_0 < \sqrt{-E_1}$ *(* E_1 *is the energy of the highest bound state) there exists potential* $V_0 \in \mathcal{V}_C$ *such, that function* $S_0(k) = \frac{k-ia_0}{k+ia_0}$ *is a scattering matrix for the pair of operators* $H_1 = -\frac{d^2}{dx^2} + V$, $\quad H_0 = -\frac{d^2}{dx^2} + V + V_0$.

Proof. Solution of the inverse problem is given by the following formula

$$V_0(x) = 2a^2 \frac{f_1^2(ia_0,x)\varphi_1'^2(0,x) - f_1'^2(ia_0,x)\varphi_1^2(0,x) + a^2 f_1^2(ia_0,x)\varphi_1^2(0,x)}{(f_1(ia_0,x)\varphi_1'(0,x) - f_1'(ia_0,x)\varphi_1(0,x))^2},$$

where $\varphi_1(k,x), f_1(k,x)$ are regular and Jost solutions for the Schrödinger operator H_1. The asymptotics of the potential is equal to $V_0(x) \sim_{x\to\infty} \frac{2}{(1+x)^2}$ and it is bounded on the real axis. It follows, that $V_0 \in \mathcal{V}$. The scattering matrix can be calculated using the regular solution

$$\varphi_0(k,x) = \varphi_1(k,x) + \frac{a_0^2}{k^2}\frac{W[\varphi_1(0,x),\varphi_1(k,x)]}{W[\varphi_1(0,x), f_1(ia_0,x)]} f_1(ia_0,x).$$

Theorem 1. *Let the scattering data*

$$S(k);\ N_{bs}, N_{sing}, \alpha;\ E_j, s_j, j = 1,2,...,N_{bs};\ b_j, j = 1,2,...,N_{sing}$$

from $\mathcal{D}$ bee given, then there exists potential $V \in \mathcal{V}$, corresponding to these scattering data.

Proof. Any function $S(k) \in \mathcal{S}$ can be presented by a quotient of two Jost functions from $\mathcal{F}$. The constant a_0 in the representation can be chosen in such a way, that $0 < a_0 < \sqrt{-E_1}$, where E_1 is the highest bound state of the operator.

The theorem can be proven in three steps. One can construct first potential $V_F \in \mathcal{V}_F$ corresponding to the scattering matrix $S_F(k)$ and having N_{bs} bound states with the energies E_j and normalizing constants $s_j,\ j = 1,2,...,N_{bs}$ [4,2]. On the second step the inverse problem should be solved for the scattering matrix $S_1(k) = \prod_{j=1}^{N_{sing}} \frac{k-ia_j}{k+ia_j}\frac{k-i\bar{a}_j}{k+i\bar{a}_j}$, considering potential V_F as a background potential. Solution of this problem - potential $V_1 \in \mathcal{V}_C$ - is given by Lemma 3. In the last step if it is necessary the zero energy singularity has to be introduced. We note, that the constant a_0 is not equal to zero only if the function $S_F(k)$ is equal to one at the origin. Thus Lemma 4 can be applied to find potential $V \in \mathcal{V}_C$, which solves the inverse problem for the scattering matrix $S_0(k) = \frac{k-ia_0}{k+ia_0}$ with the background potential V_1.

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