COMMENT

Comment on 'On the Coulomb potential in one dimension' by P Kurasov

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Abstract. The mathematical analysis in 1996 *J. Phys. A: Math. Gen.* **29** 1767, is not sufficient to decide whether in one dimension the singularities of the potentials $-\gamma/x$ and $-\gamma/|x|$ split the corresponding one-particle quantum systems at the origin into two completely decoupled subsystems. In fact, it is argued that this question cannot be answered by mathematical considerations alone.

In [1] Pavel Kurasov constructed quantum Hamiltonians as self-adjoint Schrödinger operators on the Hilbert space $\mathrm{L}^2(\mathbb{R})$ for a point particle moving along the real line $\mathbb{R}$ under the influence of the potential $-\gamma/x$ or $-\gamma/|x|$, where $\gamma, x \in \mathbb{R}$. Employing physical units in which twice the mass of the particle equals the square of Planck's constant, these operators are formally given by the differential expressions

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\gamma}{x} \tag{1}$$

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\gamma}{|x|}. \tag{2}$$

The question whether the non-integrable singularities of the potentials make the origin of the real line 'impenetrable,' that is, whether they split the corresponding quantum systems into two completely decoupled subsystems associated with the two half-lines, has been lively debated, see for example [2] and the references 11–24 in [3]. Theorem 1 in [1] establishes the self-adjointness of the operator H given by the action of (1) in the distributional sense with a principal-value prescription at the origin on the domain $\mathrm{Dom}(H) := \{\psi \in \mathrm{L}^2(\mathbb{R}) : \mathrm{PV}(-\mathrm{d}^2/\mathrm{d}x^2 - \gamma/x)\psi \in \mathrm{L}^2(\mathbb{R})\}$. This domain contains functions which yield a non-zero probability current density at the origin, and it is concluded that the quantum system with the odd potential $-\gamma/x$ is 'penetrable' at the origin. According to theorem A1 in [1] the operator H^{c} given by the distributional action of (2) on the domain $\mathrm{Dom}(H^{\mathrm{c}}) := \{\psi \in \mathrm{L}^2(\mathbb{R}) : (-\mathrm{d}^2/\mathrm{d}x^2 - \gamma/|x|)\psi \in \mathrm{L}^2(\mathbb{R})\}$ is only symmetric and has deficiency indices $(2, 2)$. By taking the Friedrichs extension $H^{\mathrm{c}}_{\mathrm{D}}$ of H^{c}, which amounts [4] to imposing a Dirichlet boundary condition at the origin, it is concluded in [1] that the quantum system with the even potential $-\gamma/|x|$ is 'impenetrable' at the origin.

The purpose of this comment is to prevent the reader from getting the wrong impression that the self-adjoint operators H and $H^{\mathrm{c}}_{\mathrm{D}}$, as correctly constructed in [1], are the only self-adjoint operators which can be associated with (1) and (2), respectively. Moreover, we

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would like to point out that the mere knowledge of the operators H and $H_{\mathrm{D}}^{\mathrm{c}}$ does not allow one to decide upon the 'penetrability' of a quantum-physical situation in the presence of a $1/x$- or $1/|x|$-potential.

In fact, rather than resorting to the theory of distributions as in [1], one can also apply von Neumann's extension theory to the operator H_0 given by (1) on the domain $\mathrm{Dom}(H_0) := \mathcal{C}_0^\infty(\mathbb{R}\backslash\{0\})$, respectively to the operator H_0^{c} given by (2) on the same domain $\mathrm{Dom}(H_0^{\mathrm{c}}) := \mathcal{C}_0^\infty(\mathbb{R}\backslash\{0\})$ of smooth functions compactly supported away from the origin. This yields, as noted by Kurasov [1] himself, for each case a four-parameter family of self-adjoint operators among which there are ones describing 'penetrable' quantum systems and others describing 'impenetrable' quantum systems. The family for the even potential has been studied explicitly in [3], see also the recent relativistic generalization [5]. The odd case can be treated analogously. Note that the above operators H and H_D^{c} are members of the respective four-parameter family.

Self-adjointness is the only property of a Hamiltonian required by the axioms of quantum mechanics. Therefore mathematics alone cannot tell which particular member of the four-parameter family of self-adjoint operators should be chosen to model a given experimental situation. Accordingly, there is no justification for claiming that the 'natural' self-adjoint extension of H_0 is the one which can also be constructed in the framework of the theory of distributions. Instead of personal mathematical preferences one needs additional physical information to serve as a guideline, since different self-adjoint extensions describe different physics [6].

The multitude of possible Hamiltonians offered by these four-parameter families are not taken into account in [1] because Kurasov considers them to model a $1/x$- or $1/|x|$-potential plus a point interaction at the origin. However, this interpretation has to be discarded because—unlike in three dimensions [7]—in one dimension a self-adjoint Schrödinger operator with a $1/x$- or $1/|x|$-potential cannot be defined without the specification of a boundary condition at the origin. The singularity of the potential itself necessarily demands it. It is only in the limit of vanishing coupling constant $\gamma \to 0$ that a certain point interaction emerges as a relic of the singularity. This effect goes under the name Klauder phenomenon [8].

To summarize, we have argued that von Neumann's extension theory provides a four-parameter family of Hamiltonians as candidates for modelling a one-dimensional quantum-physical situation with a $1/x$- or $1/|x|$-potential. In both cases the respective four-parameter family contains some members describing 'penetrable' quantum systems and other members describing 'impenetrable' quantum systems. The question, which member is the most adequate one, cannot be answered by mathematical considerations alone, but only along with experiment. This remains true, even if one or another member out of the family may also be obtained as the unique restriction of a suitably chosen distribution-valued differential operator [1] or—as favoured by some other authors [9]—through a resolvent limit of a sequence of Hamiltonians with regularized potentials [4, 9].

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