JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 297–323 (1996)
ARTICLE NO. 0256

Distribution Theory for Discontinuous Test Functions and Differential Operators with Generalized Coefficients

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Submitted by Joseph A. Ball

Received September 14, 1994

Investigation of the differential operators with the generalized coefficients having singular support on a disjoint set of points requires the consideration of the distribution theory with the set of discontinuous test functions. Such a distribution theory for test functions having discontinuity at one point is developed. A four-parameter family of Schrödinger operators, formed by the operators with singular potential, singular metrics and singular gauge field, is considered. It is proved that this family of singular interactions describes all possible selfadjoint extensions of the second derivative operator defined on the functions vanishing in a neighbourhood of the point. Approximation by operators with smooth coefficients is discussed. © 1996 Academic Press, Inc.

1. INTRODUCTION

Differential operators with coefficients equal to the generalized functions appear in different problems of applied mathematics and mathematical physics. These operators are closely related to exactly solvable problems in quantum mechanics, atomic physics, and acoustics [3, 11]. An important class of such operators is formed by the differential operators with the coefficients having singular support on a disjoint set of points. Such operators will be named "*operators with singular interactions*" in the future. Every such selfadjoint operator can be described as an extension of the symmetric operator defined by the same differential expression on the set of functions with the support disjoint from the singular support of the coefficients. All selfadjoint extensions of these operators will be named "*selfadjoint perturbations*".

0022-247X/96 $18.00

The selfadjoint perturbations of the differential operators can be studied with the help of the von Neumann extensions theory for symmetric operators or the Krein theory. The operator with singular interaction was introduced by Fermi. The first mathematically correct investigation of the operator with singular interaction was carried out by Berezin and Faddeev [6]. A Laplace operator in $L_2(\mathbf{R}^3)$ with singular interaction at one point has been considered. The most complete collection of solved problem involving singular interactions in dimensions one, two, and three can be found in the monograph by Albeverio *et al.* [3]. The monograph by Demkov and Ostrovskii [11] contains numerous applications of the described Hamiltonians to physics. Generalizations of the singular interactions involving additional space of interaction have been introduced by Pavlov [19]. The singular interactions with support on the low-dimensional manifolds have been studied in [20].

Differential operators in dimension one are investigated in this paper. The relations between the singular interactions and selfadjoint perturbations for the first and second derivative operators have been discussed [3−5, 9, 10, 12−14, 16−18, 21]. The set of the selfadjoint perturbations and singular interactions for the second order differential operator in dimension one is much wider than in the three dimensional space—it is described by four real parameters. The four parameter family of selfadjoint perturbations has been studied recently in [1, 2, 8, 9]. The aim of this paper is to clarify the relations between the singular interactions and selfadjoint perturbations for the second derivative operator. Four important questions arise in this context:

1. How does one describe the selfadjoint perturbation corresponding to the singular interaction?
2. Is it possible to describe all selfadjoint perturbations by the singular interactions?
3. How does one describe all singular interactions leading to the selfadjoint operators?
4. How does one approximate the operators with singular interactions by operators with smooth coefficients?

The answers to these questions require detailed analysis of the distribution theory [15]. Consider for example the following first order differential operator with the singular interaction $M_X = i(d/dx) + X\delta$, where δ is the Dirac delta function, $X \in \mathbf{R}$. This differential expression is well-defined on the functions $f \in W_2^1(\mathbf{R})$ because these functions are continuous at the origin. But the range of an operator defined in such a way is not contained in $L_2(\mathbf{R})$ and consequently the operator is not selfadjoint. The selfadjoint operator corresponding to this differential expression can be defined on the discontinuous functions only. The product of the delta function and discontinuous function is defined only in the framework of the distribution theory for the discontinuous test functions. Then the natural extension of the delta function to the set of discontinuous test functions ψ, $\delta(\psi) = (\psi(+0) + \psi(-0))/2$, can be used. Using this extension of the delta function, the operator M_X is defined as the operator of the first derivative with the domain of functions, satisfying the boundary condition at the origin $(2 - iX)\psi(+0) = (2 + iX)\psi(-0)$ (see [9, Section 6]). It is possible to consider the singular interactions with the coefficient X equal to ∞ using the projective space formalism (see for details [7, 17]).

A similar problem appears during the consideration of the following second derivative operator with the singular interaction

$$L_{X_1 X_2} = -\frac{d^2}{dx^2} + X_1 \delta + X_2 \delta^{(1)}, \qquad X_1, X_2 \in \mathbf{R}, \tag{1}$$

the Schrödinger operator with the generalized potential. The corresponding selfadjoint operator cannot be defined on a space consisting of functions which are continuously differentiable at the origin.

Two examples considered here show that the singular interactions for the first and second derivative operators cannot be defined in the framework of standard distribution theory (except the operator $-(d^2/dx^2) + X_1 \delta$). We present and prove here only the most important facts from the distribution theory for the discontinuous test functions (see [7, 17] for details). The operators with the singular interactions will be defined on the basis of the developed technique.

A positive answer to the second question, originally pointed out by Seba [21], can be given only if a wider family of singular interactions is considered. The family (1) is described by two real parameters, but the family of the corresponding selfadjoint perturbations is described by the unitary 2×2 matrix, which contains 4 real parameters [9, 21]. Two other families of singular interactions have been studied in [9, 21]. But these families of singular interactions do not cover all selfadjoint perturbations. We are going to present here the four parameter family which describes the whole class of selfadjoint perturbations (see also [1, 2]). This family is formed by the operator with the singular potential (1), singular metrics, and singular gauge field. We extend the family of operators by considering the parameters from the projective space $\mathbf{P}^4$. It is proved that the set of all selfadjoint perturbations of the second derivative operator coincides with the family of operators with the singular interactions.

We give an answer to the third question in the framework of the distribution theory developed here. It is proved that only the four parameter family of singular interactions considered leads to selfadjoint operators.

We show that every second order differential operator with a singular interaction can be approximated by a certain sequence of differential operators with the interaction defined by continuous short range coefficients.

2. DISTRIBUTION THEORY FOR DISCONTINUOUS TEST FUNCTIONS

2.1. *Test Functions and Distributions*

We introduce the set K of test functions with a possible discontinuity at the origin.

DEFINITION 1. The set of test functions K is the set of all functions with compact support on the line $(-\infty, +\infty)$ having uniformly bounded derivatives of any order outside the origin.

The support of these functions is not necessarily separated from the origin. Functions from K can be discontinuous at the origin, but the limits of the functions and all derivatives from the left and from the right of the point zero exist and are finite. Convergence in this space is defined as follows:

DEFINITION 2. A sequence $\{\varphi_n\}$ of functions in K is said to converge to a function $\varphi \in K$ if and only if

(1) There exists an interval outside which all the functions φ_n vanish;

(2) The sequence $\{\varphi_n^{(k)}\}$ of derivatives of order k converges uniformly outside the origin on this interval to $\varphi^{(k)}$ for every k.

Distributions corresponding to these test functions can be defined in the standard way:

DEFINITION 3. A distribution f from K' is a linear form on K such that for every compact set $B \subset \mathbf{R}$ there exist constants C and n such that

$$|f(\varphi)| \leq C \sum_{\alpha \leq n} \sup_{x \neq 0} \left| \left(\frac{d}{dx} \right)^{\alpha} \varphi \right|, \qquad \varphi \in K, \mathrm{supp}(\varphi) \in B. \tag{2}$$

Standard methods of the theory of distributions can be applied to studying the set K'. We are going to compare these distributions with the distributions corresponding to the test space $D = C_0^{\infty}(\mathbf{R})$. The set of distributions for the test functions D is usually denoted by D'. The difference between the spaces K' and D' is ‘‘local’’ and related to the special behaviour of the test functions from K near the origin.

2.2. *Generalized Derivative*

The derivative of any distribution in K' will be defined using the formula, which is valid for any distribution defined by the function $f \in C_0^\infty(\mathbf{R} \setminus \{0\})$.

DEFINITION 4. Let $f \in K'$, φ be a test function from K, then the derivative $D_x f$ of the distribution f is defined by the equation

$$(D_x f)(\varphi) = -f\left(\frac{d}{dx}\varphi\right), \tag{3}$$

where the derivative of the test function $(d/dx)\varphi$ is calculated in the classical sense at every point x outside the origin.

We note that according to our definition the derivative of the test function $(d/dx)\varphi$ does not contain any delta-functional singularity at the origin, even if the test function is discontinuous there. This definition of the derivative allows us to calculate the derivative of any distribution from K'. The definition of the derivative involving the delta function would restrict the class of differentiable distributions.

The derivative of a distribution in K' does not coincide with the derivative defined in the classical sense. For example, the derivative of the constant distribution is not equal to zero. We shall use in future the notation $(d/dx)\psi$ for the classical derivative and $D_x\psi = \psi^{(1)}$- for the generalized derivative in K'.

LEMMA 1. *The derivative of the constant distribution c is equal to the distribution $c\beta$, where distribution β is defined by the formula*

$$\beta(\varphi) = \varphi(+0) - \varphi(-0). \tag{4}$$

Proof. Let $\varphi \in K$, then

$$\begin{aligned}(D_x c)(\varphi) &= -c\left(\frac{d}{dx}\varphi\right)\\ &= -c\left(\int_{-\infty}^{0} \frac{d}{dx}\varphi(x)\,dx + \int_{0}^{+\infty} \frac{d}{Dx}\varphi(x)\,dx\right)\\ &= c(\varphi(+0) - \varphi(-0)). \quad \blacksquare\end{aligned}$$

Higher derivatives of the constant distribution can be calculated in a similar way:

$$D_x^n c(\varphi) = (-1)^{n-1} c\left(\left(\frac{d}{dx}\right)^{n-1} \varphi(+0) - \left(\frac{d}{dx}\right)^{n-1} \varphi(-0)\right). \quad (5)$$

All derivatives calculated here are distributions vanishing on the set of test functions infinitely differentiable at the origin.

2.3. *Delta Function and Its Derivatives*

We are going to discuss the definition of the delta function with support at the origin. The delta function is defined usually as a functional on the set of C_0^∞ functions by the following formula:

$$\delta(\varphi) = \varphi(0). \quad (6)$$

It is obvious that this linear functional can be extended to the set of all functions continuous at the origin using the same formula (6). But this formula cannot be used for the delta function in K' since the value of a test function from K at the origin is not defined.

The delta function is an even distribution. We can use this property to calculate the delta function on the discontinuous test functions. If the distribution f_{even} is even, then the following formula is valid for every test function φ:

$$\begin{aligned} f_{\text{even}}(\varphi) &= f_{\text{even}}\left(\frac{\varphi(x) - \varphi(-x)}{2}\right) + f_{\text{even}}\left(\frac{\varphi(x) + \varphi(-x)}{2}\right) \\ &= f_{\text{even}}\left(\frac{\varphi(x) + \varphi(-x)}{2}\right). \end{aligned}$$

Every even function from K is continuous at the origin and formula (6) can be used to calculate the value of the delta function. We shall use the following definition in the future:

DEFINITION 5. The delta function in K' with support at the origin is a linear functional on K defined by the formula:

$$\delta(\varphi) = \frac{\varphi(+0) + \varphi(-0)}{2}.$$

The approximative delta-function sequence can be defined by any even function $V(x) \in C_0^\infty$, $\int_{-\infty}^{+\infty} V(x)\, dx = 1$. The sequence of functionals corresponding to the functions $V^{\epsilon}(x) = (1/\epsilon)V(x/\epsilon)$ converges to the delta-function in the space K' when $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} V^{\epsilon}(\varphi) = \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{1}{\epsilon}V\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx = \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}V(x)\varphi(\epsilon x)\,dx$$
$$= \int_{-\infty}^{0}V(x)\varphi(-0)\,dx + \int_{0}^{+\infty}V(x)\varphi(+0)\,dx$$
$$= \frac{\varphi(-0)+\varphi(+0)}{2}.$$

The derivatives of the delta function can be easily calculated using Definition 4:

$$D_x^n\delta(\varphi) = (-1)^n\frac{(d/dx)^n\varphi(+0) + (d/dx)^n\varphi(-0)}{2}. \tag{7}$$

The delta function and its corresponding derivatives so defined possess the same properties as the standard delta function with respect to the inversion and scaling transformations. The inversion and scaling transformations are defined on the test functions as follows

$$(\mathbf{I}\varphi)(x) = \varphi(-x); \qquad (\mathbf{S}_c\varphi)(x) = \varphi(cx).$$

Similar transformations for the distributions are defined in the standard way

$$(\mathbf{I}f)(\varphi) = f(\mathbf{I}\varphi); \qquad (\mathbf{S}_c f)(\varphi) = f\left(\frac{1}{c}\mathbf{S}_{1/c}\varphi\right). \tag{8}$$

LEMMA 2. *The nth derivative of the delta function is a homogeneous distribution of order* $-(n+1)$: $\mathbf{S}_c D_x^n\delta = (1/c)^{n+1}D_x^n\delta$. *The nth derivative of the delta function is an even distribution if n is an even number and an odd distribution if n is an odd number*: $\mathbf{I}D_x^n\delta = (-1)^n D_x^n\delta$.

Proof. The proof can be carried out by direct calculations.

We note that the delta function in D' possesses the same properties with respect to the inversion and scaling transformations. Moreover, the following lemma can be proved [17].

LEMMA 3. *Let the distribution f from* K'

1. *be equal to* $D_x^n\delta$ *for the test functions from* $C_0^n(\mathbf{R})$;

2. *be a homogeneous distribution*;

3. *be an even distribution if n is an even number and an odd distribution if n is an odd number*;

then this distribution coincides with the nth derivative of the delta function on K': $f = D_x^n \delta$.

2.4. *Generalized and Classical Derivatives*

We define by K_{loc} the set of all bounded functions which are infinitely differentiable outside the origin with possibly a jump discontinuity at the origin. We suppose that the limits of all derivatives from both sides of the origin are finite. Distributions β and δ have unique extension to this class of test functions. Two different derivatives are defined such functions:

the derivative calculated as an ordinary function at every point outside the origin—the classical derivative $(d/dx)\psi$;

the derivative calculated as a distribution—the generalized derivative $D_x\psi = \psi^{(1)}$.

The difference between these two derivatives is illustrated by the following.

LEMMA 4. *The generalized derivative* $D_x\psi$ *and the classical derivative* $(d/dx)\psi$ *of an arbitrary function* $\psi \in K_{\mathrm{loc}}$ *are related as*

$$D_x\psi = \frac{d}{dx}\psi + \beta(\psi)\delta + \delta(\psi)\beta, \tag{9}$$

where δ *is the delta function and* β *is the derivative of the unit distribution.*

Proof. The generalized derivative for any distribution $\psi \in K_{\mathrm{loc}}$ acting on an arbitrary test function $\varphi \in K$ is equal to

$$\begin{aligned} D_x\psi(\varphi) &= -\psi\left(\frac{d}{dx}\varphi\right) = -\int_{-\infty}^{0}\psi(x)\frac{d}{dx}\varphi(x)\,dx - \int_{0}^{+\infty}\psi(x)\frac{d}{dx}\varphi(x)\,dx \\ &= -\psi(-0)\varphi(-0) + \int_{-\infty}^{0}\frac{d}{dx}\psi(x)\varphi(x)\,dx + \psi(+0)\varphi(+0) \\ &\quad + \int_{0}^{\infty}\frac{d}{dx}\psi(x)\varphi(x)\,dx \\ &= \frac{d}{dx}\psi(\varphi) + \beta(\psi)\delta(\varphi) + \delta(\psi)\beta(\varphi). \quad \blacksquare \end{aligned}$$

The last lemma shows another time that the derivative of the distribution in K' does not coincide with the derivative in D'. The difference vanishes on the test functions from D.

Similar results can be proved for the second derivative:

LEMMA 5. *The second generalized derivative $D_x^2\psi$ and the second classical derivative $(d^2/dx^2)\psi$ of arbitrary function $\psi \in K_{\text{loc}}$ are related as follows*:

$$D_x^2\psi = \frac{d^2}{dx^2}\psi + \delta(\psi)D_x\beta - D_x\delta(\psi)\beta + \beta(\psi)D_x\delta - D_x\beta(\psi)\delta. \quad (10)$$

2.5. *Product of Distributions*

The product of two distributions can be defined if one of these distributions is a function from K_{loc}.

DEFINITION 6. The product of any distribution $f \in K'$ and any function $\psi \in K_{\text{loc}}$ is defined as

$$f\psi(\varphi) = \psi f(\varphi) = f(\psi\varphi),$$

where $\varphi \in K$ is an arbitrary test function.

This definition is correct because the product of $\psi \in K_{\text{loc}}$ and any test function from K is a function from K again. We have in particular for the delta function and any $\psi \in K_{\text{loc}}, \varphi \in K$,

$$\psi\delta(\varphi) = \delta(\psi\varphi) = \frac{\psi(+0)\varphi(+0) + \psi(-0)\varphi(-0)}{2}$$
$$= \delta(\psi)\delta(\varphi) + \frac{\beta(\psi)}{4}\beta(\varphi) \Rightarrow \psi\delta = \delta(\psi)\delta + \frac{\beta(\psi)}{4}\beta. \quad (11)$$

We used in the last formula a natural extension of the definition of the distributions δ and β to the set K_{loc}. The formula for the derivative of the product of two distributions involves their classical and generalized derivatives.

LEMMA 6. *Let $\psi \in K_{\text{loc}}$, $f \in K'$, then*

$$D_x(\psi f) = \psi D_x f + f\frac{d}{dx}\psi. \quad (12)$$

Proof. We have for any $\varphi \in K$

$$D_x(\psi f)(\varphi) = -\psi f\left(\frac{d}{dx}\varphi\right) = -f\left(\psi\frac{d}{dx}\varphi\right) = -f\left(\frac{d}{dx}(\psi\varphi)\right) + f\left(\varphi\frac{d}{dx}\psi\right)$$

$$= D_x f(\psi\varphi) + \left(f\frac{d}{dx}\psi\right)(\varphi) = \psi D_x f(\varphi) + \left(f\frac{d}{dx}\psi\right)(\varphi). \quad \blacksquare$$

The product of any function $\psi \in K_{\mathrm{loc}}$ and the derivative of an arbitrary distribution $f \in K'$ can be calculated in accordance with the formula

$$\psi D_x f = D_x(\psi f) - \left(\frac{d}{dx}\psi\right) f. \tag{13}$$

The following formula can be derived using Eqs. (11) and (13) for the delta function and arbitrary $\psi \in K_{\mathrm{loc}}$:

$$\psi D_x \delta = \delta(\psi) D_x \delta + D_x \delta(\psi)\delta + \frac{D_x \beta(\psi)}{4}\beta + \frac{\beta(\psi)}{4} D_x \beta. \tag{14}$$

3. SECOND-ORDER DIFFERENTIAL OPERATOR WITH SINGULAR INTERACTION

3.1. *Selfadjoint Perturbations*

We are going to study now the selfadjoint perturbations of the second derivative operator $-D_x^2$ in dimension one. A selfadjoint perturbation at the origin for this operator is a selfadjoint extension of the symmetric operator

$$L^{00} = -D_x^2, \qquad \mathrm{Dom}(L^{00}) = \left\{\psi \in W_2^2(\mathbf{R}),\ \psi(0) = \psi'(0) = 0\right\}. \tag{15}$$

The adjoint operator $L^{00*} = -D_x^2$ is defined on the domain $\mathrm{Dom}(L^{00*}) = \{\psi \in W_2^2(\mathbf{R}\setminus\{0\})\}$.

LEMMA 7. *Every selfadjoint extension of the operator L^{00} coincides with the operator L^{00*}, restricted to the set of functions, satisfying the boundary conditions at the origin of one of the types*

(1)
$$\begin{pmatrix}\psi(+0)\\ \psi'(+0)\end{pmatrix} = J\begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix}, \qquad J = e^{i\varphi}\begin{pmatrix} a & b\\ c & d\end{pmatrix} \tag{16}$$

with the real parameters $\varphi \in [-\pi/2, \pi/2]$, $a, b, c, d \in \mathbf{R}$ *such that* $ad - bc = 1$;

(2)
$$\begin{cases} h_0^+ f'(+0) = h_1^+ f(+0) \\ h_0^- f'(-0) = h_1^- f(-0) \end{cases}, \tag{17}$$

with the parameters $\mathbf{h}^{\pm} = (h_0^{\pm}, h_1^{\pm})$ *from the projective space* $\mathbf{P}^1$.

This lemma has been proved in [9, 21]. Selfadjoint operators, described by the boundary conditions of the first type, will be named "*connected*" because these conditions connect the boundary values of the function on the left and right halflines. Selfadjoint operators of the second type will be named "*separated*." These operators are equal to the orthogonal sum of the second derivative operators defined on the halflines.

3.2. *Singular Interactions*: *Four Parameter Family*

We are going to study now the four parameter family of the second derivative operators with singular interactions. The selfadjoint extensions, corresponding to the family, will be calculated. See Section 3.6 for the physical interpretation of the parameters.

THEOREM 1. *The second order differential operator with the singular interaction at the origin*

$$L_X = -D_x^2(1 + X_4\delta) + iD_x\left(2X_3\delta - iX_4\delta^{(1)}\right) + X_1\delta + (X_2 - iX_3)\delta^{(1)}, \tag{18}$$

$X = (X_1, X_2, X_3, X_4) \in \mathbf{R}^4$, *coincides with the second derivative operator* $-D_x^2$ *defined on the domain of functions* $\psi \in W_2^2(\mathbf{R} \setminus \{0\})$, *satisfying the following boundary condition at the origin*:

1.
$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} \dfrac{(2+X_2)^2 - X_1X_4 + X_3^2}{(2-iX_3)^2 + X_1X_4 - X_2^2} & \dfrac{-4X_4}{(2-iX_3)^2 + X_1X_4 - X_2^2} \\ \dfrac{4X_1}{(2-iX_3)^2 + X_1X_4 - X_2^2} & \dfrac{(2-X_2)^2 - X_1X_4 + X_3^2}{(2-iX_3)^2 + X_1X_4 - X_2^2} \end{pmatrix} \times \begin{pmatrix} \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix}, \tag{19}$$

if $(2 - iX_3)^2 + X_1X_4 - X_2^2 \neq 0$;

2. $$\begin{pmatrix} \frac{d}{dx}\psi(+0) \\ \frac{d}{dx}\psi(-0) \end{pmatrix} = \frac{1}{X_4}\begin{pmatrix} X_2 - 2 & 0 \\ 0 & X_2 + 2 \end{pmatrix}\begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix} \tag{20}$$

if $4 + X_1X_4 - X_2^2 = 0,\ X_3 = 0,\ X_4 \neq 0$;

3. $$\begin{cases} \frac{d}{dx}\psi(+0) = \frac{X_1}{4}\psi(+0) \\ \psi(-0) = 0 \end{cases} \tag{21}$$

if $X_2 = 2,\ X_3 = 0,\ X_4 = 0$;

4. $$\begin{cases} \psi(+0) = 0 \\ \frac{d}{dx}\psi(-0) = -\frac{X_1}{4}\psi(-0) \end{cases} \tag{22}$$

if $X_2 = -2,\ X_3 = 0,\ X_4 = 0$.

Proof. The domain of the operator L_X coincides with the set of functions $\psi \in L_2(\mathbf{R})$, which are solutions of the equation $L_X\psi = f$ for some function $f \in L_2(\mathbf{R})$. We consider the last equation in the generalized sense with the set of the test functions D.[1] Considering this equation for the test functions with the support separated from the origin we deduce that $\psi \in W_2^2(\mathbf{R} \setminus \{0\})$. The functions from this Sobolev space are continuous outside the origin and have continuous bounded first derivative there. The differential expression (18) is defined on such functions. The distributions δ and β and their first derivatives can be defined on the functions ψ from $W_2^2(\mathbf{R} \setminus \{0\})$ as follows:

$$\delta(\psi) = \frac{\psi(+0) + \psi(-0)}{2}; \qquad \delta^{(1)}(\psi) = -\frac{\psi'(+0) + \psi'(-0)}{2};$$

$$\beta(\psi) = \psi(+0) - \psi(-0); \qquad \beta^{(1)}(\psi) = -(\psi'(+0) - \psi'(-0)).$$

[1]The set of test functions is chosen equal to D because it forms a dense subset of the domain of the second derivative operator $-D_x^2$. Different choice of the test space would lead to a nonselfadjoint operator or an operator with the point interaction even if all coefficients X_1, X_2, X_3, X_4 are equal to zero.

Formulas (11), (14) define the product of the delta function or its derivative and any function from $W_2^2(\mathbf{R} \setminus \{0\})$. The distribution $L_X \psi \in K'$, $\psi \in W_2^2(\mathbf{R} \setminus \{0\})$ can have singular support only at the origin. The singular term is equal to the linear combination of the distributions β and δ and their first derivatives. The distributions β and $\beta^{(1)}$ vanish on the test functions from D. Then the distribution $L_X \psi$ is equivalent to some function from $L_2(\mathbf{R})$ if and only if the coefficients in front of the delta function and its derivative are equal to zero. We get the following linear system:

$$\delta\colon \beta^{(1)}(\psi) + X_1 \delta(\psi) + (X_2 - iX_3)\delta^{(1)}(\psi) = 0$$

$$\delta^{(1)}\colon -\beta(\psi) + (X_2 + iX_3)\delta(\psi) + X_4 \delta^{(1)}(\psi) = 0$$

$$\Rightarrow \begin{pmatrix} X_1 & -2 - X_2 + iX_3 & X_1 & 2 - X_2 + iX_3 \\ -2 + X_2 + iX_3 & -X_4 & 2 + X_2 + iX_3 & -X_4 \end{pmatrix}$$

$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \\ \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix} = 0. \tag{23}$$

The rank of the matrix in the last equation is equal to 2 and it defines the two dimensional subspace in the four dimensional space of the boundary values $(\psi(+0), (d/dx)\psi(+0), \psi(-0), (d/dx)\psi(-0))$. We write conditions (23) in the form:

$$\begin{pmatrix} -\dfrac{X_1}{2} & 1 + \dfrac{X_2 - iX_3}{2} \\ 1 - \dfrac{X_2 + iX_3}{2} & \dfrac{X_4}{2} \end{pmatrix} \begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{X_1}{2} & 1 - \dfrac{X_2 - iX_3}{2} \\ 1 + \dfrac{X_2 + iX_3}{2} & -\dfrac{X_4}{2} \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}. \tag{24}$$

The determinant of the matrix in the left-hand side of the last equation is equal to

$$\Delta = \frac{-1}{4}\left((2-iX_3)^2 + X_1X_4 - X_2^2\right).$$

If $\Delta \neq 0$, then the matrix is invertible and these boundary conditions can be written in the form (19).

Consider the case $\Delta = 0$. It follows that the coefficient X_3 is equal to zero. The boundary conditions (23) can be written as

$$\begin{pmatrix} 1+\dfrac{X_2}{2} & -1+\dfrac{X_2}{2} \\ \dfrac{X_4}{2} & \dfrac{X_4}{2} \end{pmatrix} \begin{pmatrix} \dfrac{d}{dx}\psi(+0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix} = \begin{pmatrix} \dfrac{X_1}{2} & \dfrac{X_1}{2} \\ -1+\dfrac{X_2}{2} & 1+\dfrac{X_2}{2} \end{pmatrix} \begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix}.$$

The determinant of the matrix in the left-hand side of the last equation is equal to X_4. If $X_4 \neq 0$, then the inverse matrix can be calculated and the boundary conditions have the form (20).

Consider now the case $\Delta = 0$, $X_4 = 0$. It follows that $X_2 = \pm 2$. Boundary conditions, defined by $X_2 = 2$, $X_4 = 0$ and $X_2 = -2$, $X_4 = 0$ are equal to (21) and (22) correspondingly. All possible values of the coefficients $X_1, X_2, X_3, X_4 \in \mathbf{R}$ have been considered.

Moreover, the image of every function $\psi \in W_2^2(\mathbf{R} \setminus \{0\})$ satisfying these boundary conditions is equivalent to a certain function from $L_2(\mathbf{R})$ on the set of test functions from D. This completes the proof of the Theorem. ▮

The boundary conditions (19)–(22) can be considered for infinite values of the parameters X_1, X_2, X_3, X_4. A good parametrization for this case can be done by using the formalism of projective space. We are going to parameterize all singular interactions by $\mathbf{X} \in \mathbf{P}^4$. We get the boundary conditions for all elements from the projective space with the nonzero component X_0 with the help of the standard embedding of the space $\mathbf{R}^4$ into the space $\mathbf{P}^4$: $(X_1, X_2, X_3, X_4) \to (1, X_1, X_2, X_3, X_4)$. The boundary conditions corresponding to the other elements from the projective space will be defined using the homogenized analog of the linear system (23)

$$\begin{pmatrix} X_1 & -2X_0 - X_2 + iX_3 & X_1 & 2X_0 - X_2 + iX_3 \\ -2X_0 + X_2 + iX_3 & -X_4 & 2X_0 + X_2 + iX_3 & -X_4 \end{pmatrix}$$

$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \\ \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix} = 0. \tag{25}$$

We shall use the following definition in the remainder.

DEFINITION 7. The algebraic set **W** is the set of elements from the projective space $\mathbf{P}^4$, satisfying the following three algebraic equations simultaneously:

$$X_0 = 0; \tag{26}$$

$$X_3 = 0; \tag{27}$$

$$(X_0 - iX_3)^2 + X_1 X_4 - X_2^2 = 0. \tag{28}$$

THEOREM 2. *Every element* **X** *from the projective space* $\mathbf{P}^4$, *which does not belong to the algebraic set W, determines a unique selfadjoint extension* $L_{\mathbf{X}}$ *of the operator* L^{00}, *described by the following boundary conditions*:

1.

$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{(2X_0 + X_2)^2 - X_1X_4 + X_3^2}{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2} & \dfrac{-4X_0X_4}{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2} \\ \dfrac{4X_0X_1}{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2} & \dfrac{(2X_0 - X_2)^2 - X_1X_4 + X_3^2}{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2} \end{pmatrix}$$

$$\times \begin{pmatrix} \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix}, \tag{29}$$

if $\mathbf{X} \in \mathbf{G}_1 = \{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2 \neq 0\}$;

2. $$\begin{pmatrix} \frac{d}{dx}\psi(+0) \\ \frac{d}{dx}\psi(-0) \end{pmatrix} = \frac{1}{X_4}\begin{pmatrix} X_2 - 2X_0 & 0 \\ 0 & X_2 + 2X_0 \end{pmatrix}\begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix} \quad (30)$$

if $\mathbf{X} \in \mathbf{G}_2 = \{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2 = 0,\ X_0 \neq 0,\ X_3 = 0,\ X_4 \neq 0\}$;

3. $$\begin{cases} 4X_0\frac{d}{dx}\psi(+0) = X_1\psi(+0) \\ \psi(-0) = 0 \end{cases} \quad (31)$$

if $\mathbf{X} \in \mathbf{G}_3 = \{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2 = 0,\ X_2 = 2X_0,\ X_0 \neq 0,\ X_4 = 0,\ X_3 = 0\}$;

4. $$\left\{\begin{array}{l} \psi(+0) = 0 \\ 4X_0\frac{d}{dx}\psi(-0) = -X_1\psi(-0) \end{array}\right\} \quad (32)$$

if $\mathbf{X} \in \mathbf{G}_4 = \{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2 = 0,\ X_2 = -2X_0,\ X_0 \neq 0,\ X_3 = 0,\ X_4 = 0\}$;

5. $$\begin{cases} \psi(+0) = -\psi(-0) \\ \frac{d}{dx}\psi(+0) = -\frac{d}{dx}\psi(-0) \end{cases}$$

if $\mathbf{X} \in \mathbf{G}_5 = \{(2X_0 - iX_3)^2 + X_1X_4 - X_2^2 = 0,\ X_3 \neq 0\}$.

Proof. The rank of the matrix in the linear system (25) is equal to 1 if and only if $X_0 = 0$ and $(X_0 - iX_3)^2 + X_1X_4 - X_2^2 = 0$. If the rank of the matrix is equal to 2, then the homogeneous linear system defines two different boundary conditions as was shown during the proof of the Theorem 1. Corresponding boundary conditions are the homogenized analogs of the boundary conditions (19)–(22) and cover the cases 1–4 of the present theorem.

Consider the case when the rank of the matrix is equal to 1, i.e., when conditions (26) and (28) are satisfied. The unique boundary condition defines a certain linear subset Q in the domain of the adjoint operator L^{00*}. The operator L^{00*} restricted on this subset is not symmetric. We are going to prove that if the additional condition $X_3 \neq 0$ is satisfied, then there exists only one selfadjoint extension of the operator L^{00} with domain equal to a subset of this linear set Q.

The unique boundary condition, defined by the system (25), is equal to

$$(X_2 + iX_3)(\psi(+0) + \psi(-0)) - X_4(\psi'(+0) + \psi'(-0)) = 0. \quad (33)$$

Every separate selfadjoint perturbation in this case should be described by the Dirichlet boundary conditions. It is possible only if $X_4 = 0$. The last equality together with the conditions (26), (28) leads to the equation $-X_3^2 - X_2^2 = 0$, which has only trivial solution $X_3 = 0$. Thus, no separated selfadjoint perturbations correspond to such an element **X**.

Consider the connected selfadjoint perturbations. Substitution of the boundary condition (16) into the equation (33) leads to the equation

$$(X_2 + iX_3)(a\psi(-0) + b\psi'(-0)) - X_4(c\psi(-0) + d\psi'(-0))$$
$$+(X_2 + iX_3)e^{-i\varphi}\psi(-0) - X_4 e^{-i\varphi}\psi'(-0) = 0,$$

which should be satisfied for all values of $\psi(-0)$ and $\psi'(-0)$. It follows that the real coefficients a, b, c, d are solutions to the following linear system with the real coefficients:

$$\begin{pmatrix} X_2 & 0 & -X_4 & 0 \\ X_3 & 0 & 0 & 0 \\ 0 & X_2 & 0 & -X_4 \\ 0 & X_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -X_2 \cos\varphi - X_3 \sin\varphi \\ X_2 \sin\varphi - X_3 \cos\varphi \\ X_4 \cos\varphi \\ -X_4 \sin\varphi \end{pmatrix}.$$

Coefficients a, b, c, d can be calculated if $X_3 \neq 0$:

$$a = \frac{X_2}{X_3}\sin\varphi - \cos\varphi; \qquad b = -\frac{X_4}{X_3}\sin\varphi;$$

$$c = \frac{X_2^2 - X_3^2}{X_4 X_3}\sin\varphi; \qquad d = -\cos\varphi - \frac{X_2}{X_3}\sin\varphi.$$

Then $ad - bc = 1 - 2(\sin\varphi)^2$ and it is equal to one if and only if $\varphi = 0$. The matrix $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ determines the unique selfadjoint operator. It accomplishes the proof of the theorem. ∎

THEOREM 3. *Every element* $\mathbf{X} \in \mathbf{W}$ *determines the families of selfadjoint extensions of the operator* L^{00}, *described by the following boundary conditions*:

1. *if* $X_1 \neq 0,\ X_4 \neq 0$ *then*

$$\begin{pmatrix} \psi(+0) \\ \frac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} a & -\frac{X_2}{X_1}(a+1) \\ \frac{X_1}{X_2}(a+1) & -2-a \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \frac{d}{dx}\psi(-0) \end{pmatrix},$$

$$a \in \mathbf{R}, \quad (34)$$

or

$$\begin{cases} X_2 \frac{d}{dx}\psi(+0) = X_1 \psi(+0) \\ X_2 \frac{d}{dx}\psi(-0) = X_1 \psi(-0) \end{cases}; \quad (35)$$

2. *if* $X_1 = 0$, *then*

$$\begin{pmatrix} \psi(+0) \\ \frac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \frac{d}{dx}\psi(-0) \end{pmatrix}, \qquad b \in \mathbf{R}, \quad (36)$$

or

$$\begin{cases} \frac{d}{dx}\psi(+0) = 0 \\ \frac{d}{dx}\psi(-0) = 0 \end{cases}; \quad (37)$$

3. *if* $X_4 = 0$, *then*

$$\begin{pmatrix} \psi(+0) \\ \frac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \frac{d}{dx}\psi(-0) \end{pmatrix}, \qquad c \in \mathbf{R}, \quad (38)$$

or

$$\begin{cases} \psi(+0) = 0 \\ \psi(-0) = 0 \end{cases}. \quad (39)$$

Proof. We are going to consider the three different cases separately.

1. Suppose that $X_1 \neq 0$, $X_4 \neq 0$, $\mathbf{X} \in \mathbf{W}$. It follows that $X_2 \neq 0$. The linear system (25) defines the unique condition

$$X_1(\psi(+0) + \psi(-0)) - X_2\left(\frac{d}{dx}\psi(+0) + \frac{d}{dx}\psi(-0)\right) = 0.$$

Every connected selfadjoint perturbation, corresponding to $\mathbf{X}$, is defined by the matrix $J = e^{i\varphi}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This matrix should be real ($\varphi = 0$) and the coefficients should satisfy the following liner system:

$$\begin{cases} aX_1 - cX_2 = -X_1 \\ bX_1 - dX_2 = X_2 \end{cases} \Rightarrow \begin{cases} c = \dfrac{X_1}{X_2}(a+1) \\ b = \dfrac{X_2}{X_1}(d+1) \end{cases}.$$

The condition $ad - bc = 1$ leads to the equation $a + d = -2$. Then the matrix J, corresponding to the element $\mathbf{X}$, should be of the form (34). Every such matrix defines the selfadjoint perturbation. Every separated selfadjoint perturbation corresponding to the element $\mathbf{X}$ is defined by the boundary conditions (35).

2. Suppose that $\mathbf{X} \in \mathbf{W}$, $X_1 = 0$, then $X_2 = 0$. The unique boundary condition, defined by $\mathbf{X}$, is equal to

$$\frac{d}{dx}\psi(+0) + \frac{d}{dx}\psi(-0) = 0.$$

This boundary condition leads to the matrices J of the following type:

$$J = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, \qquad b \in \mathbf{R}.$$

Corresponding separated selfadjoint perturbations are defined by the Neumann boundary conditions.

3. The case $\mathbf{X} \in \mathbf{W}$, $X_4 = 0$ can be considered in a similar way.

Theorem 3 is proved. ∎

Thus the elements from $\mathbf{W}$ do not determine the selfadjoint perturbation uniquely. Any operator from the corresponding family can be used to describe the singular interaction.

Theorems 2 and 3 cover all possible values of $\mathbf{X}$ from the projective space.

LEMMA 8. *The sets* $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4, \mathbf{G}_5$, *and* $\mathbf{W}$ *cover the projective space* $\mathbf{P}^4$.

We note that the element $\mathbf{X}$ cannot be uniquely defined by the domain of the operator. For example, elements with $X_0 = 0$, $X_2^2 - X_1X_4 + X_3^2 \neq 0$ correspond to the same selfadjoint operator, defined by the boundary conditions $\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = -\begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}$.

3.3. *Classification of the Selfadjoint Perturbations*

We are ready now to prove our main result.

THEOREM 4. *The set of all selfadjoint perturbations of the second derivative operator in* $L_2(\mathbf{R})$ *coincides with the family of operators with the singular interactions* $\{L_{\mathbf{X}}, \mathbf{X} \in \mathbf{P}^4\}$.

Proof. Every operator $L_{\mathbf{X}}$ is defined as the restriction of the second derivative operator in $W_2^2(\mathbf{R} \setminus \{0\})$ on a certain linear set. The boundary conditions defining the operators $L_{\mathbf{X}}$ are of the type (16) or (17). It follows that every operator $L_{\mathbf{X}}$ is a selfadjoint extension of the operator L^{00}. We have to prove only that every such extension can be described by certain singular interaction.

Consider first the arbitrary connected perturbation, defined by a certain matrix J (16). We are going to use the homogenized analog of the conditions (24). If the element $\mathbf{X}$ defines the boundary conditions (16), then the following equation is fulfilled:

$$\begin{pmatrix} -X_1 & 2X_0 + X_2 - iX_3 \\ 2X_0 - X_2 - iX_3 & X_4 \end{pmatrix} e^{i\varphi} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} X_1 & 2X_0 - X_2 + iX_3 \\ 2X_0 + X_2 + iX_3 & -X_4 \end{pmatrix}.$$

The last equation can be written as a 4×5 homogeneous linear system:

$$\begin{pmatrix} 2e^{i\varphi}c & -e^{i\varphi}a - 1 & e^{i\varphi}c & -ie^{i\varphi}c & 0 \\ -2 + 2e^{i\varphi}d & -e^{i\varphi}b & e^{i\varphi}d + 1 & -ie^{i\varphi}d - i & 0 \\ -2 + 2e^{i\varphi}a & 0 & -e^{i\varphi}a - 1 & -ie^{i\varphi}a - i & e^{i\varphi}c \\ 2e^{i\varphi}b & 0 & -e^{i\varphi}b & -ie^{i\varphi}b & e^{i\varphi}d + 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$
$$= 0. \tag{40}$$

Let us denote by Δ_j, $j = 0, 1, 2, 3, 4$, the determinants of the 4×4 matrices obtained from the 4×5 matrix by erasing the jth column. These determinants are equal to

$$\begin{aligned}
\Delta_0 &= 2ie^{2i\varphi}(2\cos\varphi + a + d)^2;\\
\Delta_1 &= -8ie^{2i\varphi}c(2\cos\varphi + a + d);\\
\Delta_2 &= 4ie^{2i\varphi}(a - d)(2\cos\varphi + a + d);\\
\Delta_3 &= -8i\sin\varphi e^{2i\varphi}(2\cos\varphi + a + d);\\
\Delta_4 &= -8ie^{2i\varphi}b(2\cos\varphi + a + d).
\end{aligned}$$

All the determinants are equal to the product of the phase factor $ie^{i\varphi}$ and a certain real factor.

If $\Delta_0 \neq 0$, then the rank of the 4×5 matrix is equal to 4. The solution of the system (40) is equal to $(-ie^{-i\varphi}\Delta_0, ie^{-i\varphi}\Delta_1, ie^{-i\varphi}\Delta_2, ie^{-i\varphi}\Delta_3, ie^{-i\varphi}\Delta_4, ie^{-i\varphi}\Delta_5) \in \mathbf{P}^4$. This element does not belong to $\mathbf{W}$ and it defines the selfadjoint perturbation uniquely. This perturbation necessarily coincides with the one defined by the matrix J.

If $\Delta_0 = 0$, then the element $(0, c, a + \cos\varphi, \sin\varphi, -b) \in \mathbf{P}^4$ is a solution of the linear system. If $\sin\varphi \neq 0$, then this element does not belong to $\mathbf{W}$ and consequently defines the matrix of boundary conditions J.

Consider the case $\varphi = 0$, $a + d = -2$. The set of all such matrices is covered by the families

$$\left\{ \begin{pmatrix} a & b \\ -\dfrac{(a+1)^2}{b} & -2 - a \end{pmatrix}, a, b \in \mathbf{R}, b \neq 0 \right\}$$

or

$$\left\{ \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}, c \in \mathbf{R} \right\}.$$

These matrices can be described by the singular interactions $\mathbf{X}$ from the algebraic set $\mathbf{W}$. Both families are covered by the boundary conditions (34), (36), and (38). It is proved that every connected selfadjoint perturbation is defined by a certain singular interaction.

Consider now the separated perturbations defined by the boundary conditions (17). Suppose that both zero components of the elements $\mathbf{h}^{\pm}$

are not equal to zero, $h_0^+ \neq 0$, $h_0^- \neq 0$. The coordinates of the element $\mathbf{X}$ can be calculated

$$\begin{cases} 2X_2 = X_4\left(\dfrac{h_1^-}{h_0^-} + \dfrac{h_1^+}{h_0^+}\right) \\ 4X_0 = X_3\left(\dfrac{h_1^-}{h_0^-} - \dfrac{h_1^-}{h_0^+}\right) \end{cases}.$$

If $\mathbf{h}^- \neq \mathbf{h}^+$ as elements of $\mathbf{P}^1$, then the elements $\mathbf{X}$ from $\mathbf{G}_2$ will define the boundary conditions. The coordinate X_3 can be chosen equal to zero. The first coordinate should be calculated from the condition $4X_0^2 + X_1X_4 - X_2^2 = 0 \Rightarrow X_1 = (-4X_0^2 + X_2^2)/X_4$.

The case $\mathbf{h}^- = \mathbf{h}^+$ is described by the elements of $\mathbf{W}$. The boundary conditions (35), (37), (39) cover all conditions of this type.

If $h_0^- = 0$, then the element $\mathbf{X} = (h_0^+, 4h_1^+, 2h_0^+, 0, 0)$ defines such separated boundary conditions. If $h_0^+ = 0$, then the boundary conditions are defined by the element $\mathbf{X} = (h_0^-, -4h_1^-, -2h_0^-, 0, 0)$. The theorem is proved. ∎

3.4. *Complete Description of the Singular Interactions*

We are going to prove here that only the considered four parameter family of singular interactions can be described by the selfadjoint operators in the framework of the developed approach. Every second derivative operator with singular interaction having support at the origin has the following form:

$$L = -D_x^2\left(1 + \sum_{n=0}^{N_2} a_2^n \delta^{(n)}\right) + iD_x\left(\sum_{n=0}^{N_1} a_1^n \delta^{(n)}\right) + \sum_{n=0}^{N_0} a_0^n \delta^{(n)}.$$

This differential expression is defined on the functions from $W_2^2(\mathbf{R} \setminus \{0\})$ only if the coefficients a_2^n, a_1^n, a_0^n, $n = 2, 3, 4, \dots,$ are equal to zero. Consider the formal operator

$$L = -D_x^2\left(1 + a_2^0 \delta + a_2^1 \delta^{(1)}\right) + iD_x\left(a_1^0 \delta + a_1^1 \delta^{(1)}\right) + a_0^0 \delta + a_0^1 \delta^{(1)}.$$

If $\psi \in W_2^2(\mathbf{R} \setminus \{0\})$, then the singular part of the distribution $L\psi$ is equal to the linear combination of the distributions δ and β and their first three derivatives. This distribution is equivalent to a function from $L_2(\mathbf{R})$ on the set of test functions D only if the coefficients in front of the δ function and its derivatives are equal to zero. We get the following linear system:

$$\begin{pmatrix} -a_2^1 & 0 & 0 & 0 \\ -a_2^0 + ia_1^1 & -a_2^1 & 0 & 0 \\ ia_1^0 + a_0^1 & ia_1^1 & -1 & 0 \\ a_0^0 & a_0^1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta(\psi) \\ \delta^{(1)}(\psi) \\ \beta(\psi) \\ \beta^{(1)}(\psi) \end{pmatrix} = 0.$$

This linear system defines a selfadjoint operator only if its rank is equal to 2. Thus the following conditions should be satisfied

$$a_2^1 = 0, \qquad -a_2^0 + ia_1^1 = 0. \tag{41}$$

The boundary conditions defined by the linear system can be written as

$$\begin{pmatrix} \beta(\psi) \\ \beta^{(1)}(\psi) \end{pmatrix} = \begin{pmatrix} ia_1^0 + a_0^1 & ia_1^1 \\ -a_0^0 & -a_0^1 \end{pmatrix} \begin{pmatrix} \delta(\psi) \\ \delta^{(1)}(\psi) \end{pmatrix}.$$

These boundary conditions define a symmetric operator if and only if the coefficients a_n^k satisfy the following homogeneous linear system:

$$\begin{cases} a_0^0 - \bar{a}_0^0 = 0, \\ a_0^1 + i\bar{a}_1^0 - \bar{a}_0^1 = 0, \\ i\bar{a}_1^1 + ia_1^1 = 0. \end{cases}$$

These equations together with the equations (41) lead to the following conditions on the coefficients

$$a_0^0, a_2^0, a_1^0 \in \mathbf{R}: \qquad a_2^1 = 0;\ a_1^0 = -2\Im a_0^1.$$

Such coefficients describe the four parmeter family of singular interactions considered in Section 3.2. The following theorem is proven.

THEOREM 5. *The set of selfadjiont second derivative operators with singular interaction of finite strength coincides with the four parmeter family of operators* $\{L_X,\ X \in \mathbf{R}^4\}$.

3.5. *Approximation by Operators with Smooth Coefficients*

Every second order differential operator with singular interaction L_X, $X = (X_1, X_2, X_3, X_4) \in \mathbf{R}^4$, can be approximated by the second order differential operators with smooth coefficients. The approximative operator can be chosen in the form

$$L_\varepsilon = -D_x^2\big(1 + X_4 V_4^\varepsilon(x)\big) + iD_x\big(2X_3 V_3^\varepsilon(x) - iX_4 V_4^{\epsilon(1)}(x)\big) \\ + X_1 V_1^\epsilon(x) + X_2 V_2^{\varepsilon(1)}(x) - iX_3 V_3^{\varepsilon(1)}(x),$$

where $V_i^\epsilon(x)$, $i = 1, 2, \ldots, 4$, are even continuously differentiable delta-functional sequences (may be different) constructed in Section 2.3: $V_i(x) \in C_0^\infty(\mathbf{R}) \Rightarrow V_i^\epsilon = (1/\epsilon)V(x/\epsilon)$. The linear operators L_ε are defined on the domain $\mathrm{Dom}(L_\epsilon) = W_2^2(\mathbf{R} \setminus \{0\})$. The sequence of the linear operators converges in the weak operator topology to the operator

$$\tilde{L}_X = -D_x^2(1 + X_4\delta) + iD_x\big(2X_3\delta - iX_4\delta^{(1)}\big) + X_1\delta + (X_2 - iX_3)\delta^{(1)}$$

with the domain $\mathrm{Dom}(\tilde{L}_X) = W_2^2(\mathbf{R} \setminus \{0\})$. It is enough to show that $D_x^{(k)} V^{\epsilon(n)}(x)$ converges to $D_x^{(k)}\delta^{(n)}(x)$ in K' for all $k, n \in \mathbf{N}$. Let $\psi \in W_2^2(\mathbf{R} \setminus \{0\})$, $\varphi \in K$, then

$$\big(D_x^{(k)} V^{\epsilon(n)}\psi\big)(\varphi) = (-1)^{k+n} V^\varepsilon\Big(\big(\psi\varphi^{(k)}\big)^{(n)}\Big) \to_{\varepsilon\to 0} (-1)^{k+n}\delta\Big(\big(\psi\varphi^{(k)}\big)^{(n)}\Big) \\ = \big(D_x^{(k)}\delta^{(n)}\psi\big)(\varphi).$$

The operator $\tilde{L}_X$ on the domain $\mathrm{Dom}(\tilde{L}_X) = W_2^2(\mathbf{R} \setminus \{0\})$ is not selfadjoint. This operator is an extension of the selfadjoint operator L_X.

Thus every operator with the singular interaction can be approximated by a sequence of selfadjoint operators (every operator L_ϵ, $\epsilon \neq 0$, is selfadjoint on the domain $W_2^2(\mathbf{R})$). The convergence should be considered in the sense of linear operators.

3.6. *Several Examples*

We are going to discuss the interpretation of the parameters X_1, X_2, X_3, X_4, defining the four parameter family of singular interactions. Three different subfamilies of the operators, which appear in different problems of mathematical physics, will be considered.

Let us study first the two dimensional subfamily of the Schrödinger operators with the generalized potentials

$$L_{X_1X_2}\psi = -D_x^2\psi + \big(X_1\delta + X_2\delta^{(1)}\big)\psi. \tag{42}$$

Every such operator coincides with the second derivative operator defined on the domain of functions from $W_2^2(\mathbf{R} \setminus \{0\})$ satisfying the boundary conditions

$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} \dfrac{2+X_2}{2-X_2} & 0 \\ \dfrac{4X_1}{4-X_2^2} & \dfrac{2-X_2}{2+X_2} \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix}. \tag{43}$$

The regularized Schrödinger operator with the singular Gauge field $(iD_x + X_3\delta)^2 - (X_3\delta)^2$ is the operator

$$L_{X_3} = -D_x^2 + iX_3\big(2D_x\delta - \delta^{(1)}\big). \tag{44}$$

It is defined by the boundary conditions

$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} \dfrac{2+iX_3}{2-iX_3} & 0 \\ 0 & \dfrac{2+iX_3}{2-iX_3} \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix}. \tag{45}$$

The Schrödinger operator with the singular density $-D_x(1+X_4\delta)D_x$ is the heuristic operator

$$L_{X_4} = -D_x^2(1+X_4\delta) + X_4 D_x \delta^{(1)}. \tag{46}$$

It is equal to the second derivative operator with the domain of functions from $W_2^2(\mathbf{R}\setminus\{0\})$ satisfying the boundary conditions

$$\begin{pmatrix} \psi(+0) \\ \dfrac{d}{dx}\psi(+0) \end{pmatrix} = \begin{pmatrix} 1 & -X_4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \dfrac{d}{dx}\psi(-0) \end{pmatrix}. \tag{47}$$

It follows that the coefficients X_1 and X_2 in the four parameter family of singular interactions (18) can be interpreted as the coefficients in front of the δ and δ' potentials. The coefficient X_3 defines the strength of the Gauge field with singularity at the origin. The coefficient X_4 corresponds to the singular density.

CONCLUSIONS

The results of the present paper can be easily generalized to the case of a general second order differential operator with the singular support of the coefficients on the disjoint set of points. An infinite number of points can be investigated [3, 22]. The methods developed have been applied already to different problems in atomic and computational physics [10, 18].

ACKNOWLEDGMENTS

The author thanks Professor J. Boman for his close collaboration and for suggesting proofs of several of the lemmas. The author thanks Professor H. Holden for a stimulating discussion and Professors B. Pavlov and S. Albeverio for their continuous interest in the work. The author is grateful to A. Rivero for pointing out Ref. [8]. The author also thanks the referee for an extremely careful reading of the manuscript and several remarks which helped to improve it. The author is grateful to Alexander von Humboldt Foundation for the financial support.

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