JOURNAL OF FUNCTIONAL ANALYSIS **148**, 152–169 (1997)
ARTICLE NO. FU963050

Rank One Perturbations, Approximations, and Selfadjoint Extensions

S. Albeverio

Department of Mathematics, Ruhr-University Bochum, 44780 Bochum, Germany; SFB 237 Essen-Bochum-Düsseldorf, Germany; BIBoS Research Center, D-33615 Bielefeld, Germany; and CERFIM, Locarno, Switzerland

and

P. Kurasov

Department of Mathematics, Ruhr-University Bochum, 44780 Bochum, Germany; Department of Mathematical and Computational Physics, St. Petersburg University, 198904 St. Petersburg, Russia; and Department of Mathematics, Luleå University, 97187 Luleå, Sweden

Received April 9, 1996; accepted August 25, 1996

Rank-one perturbations

$$A_\alpha = A + \alpha(\varphi, \cdot)\,\varphi$$

of a semibounded selfadjoint operator A are studied with the help of distribution theory. It is shown that such perturbations can be defined for finite values of α even if the element φ does not belong to $\mathscr{H}_{-1}(A)$. Approximations of the rank one perturbations are constructed in the strong operator topology. It is proven that rank one $\mathscr{H}_{-2}$ perturbations can be defined uniquely for the homogeneous operators. The results are applied to a Schrödinger operator with a delta interaction in dimension 3. © 1997 Academic Press

1. INTRODUCTION

Rank one perturbations of selfadjoint positive operators A of the form

$$A_\alpha = A + \alpha(\varphi, \cdot)\,\varphi \tag{1}$$

are studied in the present paper. Such perturbations were considered in a series of papers [9, 12, 17]. It was shown that the operator A_α is well defined only if the element φ belongs to the space $\mathscr{H}_{-1}(A)$ from the standard scale of Banach spaces for the nonnegative operator A and only

0022-1236/97 $25.00

Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.

if the coefficient α is finite. A natural generalization to the case of infinite values of α was constructed in the paper by Gesztesy and Simon [9]. There it was shown that rank one perturbations can also be studied in the case where $\varphi \in \mathscr{H}_{-2}(A)$, but the operator A_α is not defined uniquely in the latter case. In [13] a special definition for the perturbation of the operator has been picked out but the analysis of all possible choices was not carried through. The latter is presented in the very recent work [3], by a method different from that of the present paper. Similar problems have been considered by Albeverio *et al.* in [2] for the squares of selfadjoint operators. An approach based on the analytical properties of the Q-function associated with a symmetric operator and its selfadjoint extensions has been developed recently by S. Hassi and H. de Snoo [10]. This allowed them to study rank one perturbations of selfadjoint operators which are not necessarily semibounded. Their approach is based on the fact that the original operator A and any rank one perturbation of it are two different selfadjoint extensions of a certain symmetric operator with deficiency indices $(1, 1)$. The relations between two selfadjoint operators whose resolvents differ by a rank one operator have been also studied in [10]. However, Hassi and de Snoo do not discuss the question of which selfadjoint operator corresponds to the given formal expression (1).

A construction using "infinitesimal coupling" was suggested by Kiselev and Simon [12]. We are going to show here that the theory of rank one perturbations for $\varphi \in \mathscr{H}_{-2}(A)$ can be developed for any finite or infinite value of the coupling constant α. The main idea is to define the linear operator (1) first on a certain natural domain with the range being a subset of $\mathscr{H}_{-2}(A)$. The corresponding self-adjoint operators are defined by standard restriction of these linear operators to domains consisting of elements having their ranges in the Hilbert space $\mathscr{H}_0(A)$. This approach leads to natural approximations of the linear operators with rank one perturbations given by elements of $\mathscr{H}_{-2}(A)$ in the strong operator topology. The advantage of the suggested approach is that the singular perturbations thus constructed remain additive even when $\varphi \in \mathscr{H}_{-2}(A)$. Moreover, all basic formulas for the rank one $\mathscr{H}_{-1}$ perturbations can be extended to rank one $\mathscr{H}_{-2}$ perturbations. We show that if the original operator and the element φ are homogeneous with respect to a certain one parameter group, then the perturbed operator is uniquely defined.

A corresponding problem for nonsemibounded operators has been already studied in [4, 16]. Using special scaling properties the self-adjoint operator corresponding to the formal expression (1) can be defined even if the perturbation is not form bounded.

The abstract approach is developed in Section 2. In the third section we apply these results to homogeneous operators. Schrödinger operators with a delta (i.e. point) interaction in dimension 3 are studied in the last section.

2. $\mathscr{H}_{-2}$ CONSTRUCTION OF RANK ONE PERTURBATIONS

2.1. *$\mathscr{H}_{-2}$ Perturbations and Distribution Theory*

Let A be a selfadjoint positive operator in a separable Hilbert space $\mathscr{H}$ with domain $\mathrm{Dom}(A)$. Let $\mathscr{H}_p(A)$, $p \in \mathbf{Z}$ be the standard scale of Banach spaces associated with A; see, e.g., [17]. We consider in this section the basic construction of the rank one perturbations under the general condition $\varphi \in \mathscr{H}_{-2}(A)$. We normalize φ as follows:

$$\|\varphi\|_{\mathscr{H}_{-2}(A)} = \left\| \frac{1}{A+1} \varphi \right\|_{\mathscr{H}_0(A)} = 1. \tag{2}$$

We are going to concentrate our attention on the case of perturbations which do not belong to $\mathscr{H}_{-1}(A)$ (the case where $\varphi \in \mathscr{H}_{-1}(A)$ has already been considered in detail in [17]). It was shown in [12] that every rank one perturbation A_α coincides with one of the selfadjoint extensions of the symmetric operator $A^0 = A\,|_{D_\varphi}$, where $D_\varphi = \{\psi \in \mathrm{Dom}(A) \,|\, (\varphi, \psi) = 0\}$. The condition $(\varphi, \psi) = 0$ is well defined for $\psi \in \mathrm{Dom}(A) = \mathscr{H}_{+2}(A)$, since $\varphi \in \mathscr{H}_{-2}(A)$. The deficiency indices of the operator A^0 are equal to one and corresponding deficiency elements for every $\Im \lambda \neq 0$ are equal to $g_\lambda = (A - \lambda)^{-1} \varphi$. All selfadjoint extensions of A^0 can be constructed using Birman–Krein–Vishik theory (see [5, 8, 14, 18]). The adjoint operator A^{0*} is defined on the domain $\mathrm{Dom}(A^{0*}) = \mathrm{Dom}(A) + \{[1/(A+1)]\, \varphi\}$. We remark that the element $[1/(A+1)]\, \varphi$ belongs to the Hilbert space $\mathscr{H}_0$. Every element $\psi \in \mathrm{Dom}(A^{0*})$ possesses the representation

$$\psi = \tilde{\psi} + a(\psi)\, \frac{1}{A+1}\, \varphi, \tag{3}$$

where $\tilde{\psi} \in \mathrm{Dom}(A)$, $a(\psi) \in \mathbf{C}$. A^{0*} acts as follows:

$$A^{0*} \left(\tilde{\psi} + a(\psi)\, \frac{1}{A+1}\, \varphi \right) = A\tilde{\psi} - a(\psi)\, \frac{1}{A+1}\, \varphi. \tag{4}$$

The boundary form of A^{0*} is equal to

$$\begin{aligned}
&(A^{0*}\psi, \eta) - (\psi, A^{0*}\eta) \\
&\quad = \left((A+1)\, \tilde{\psi}, \frac{1}{A+1}\, \varphi \right) a(\eta) - \overline{a(\psi)} \left(\frac{1}{A+1}\, \varphi, (A+1)\, \tilde{\eta} \right) \\
&\quad = \overline{(\varphi, \tilde{\psi})}\, a(\eta) - \overline{a(\psi)} (\varphi, \tilde{\eta}).
\end{aligned}$$

All selfadjoint extensions of the operator A^0 are parameterized by a one parameter $\gamma \in \mathbf{R} \cup \{\infty\}$. For $\gamma \in \mathbf{R}$, every such extension A_γ is defined as the restriction of the adjoint operator A^{0*} to the domain $\mathrm{Dom}(A_\gamma) = \{\psi \in \mathrm{Dom}(A^{0*}) \mid a(\psi) = -\gamma(\varphi, \tilde{\psi})\}$. The extension corresponding to $\gamma = 0$ coincides with the original operator A. If the parameter γ is equal to ∞ then the corresponding extension is defined by the boundary condition $(\varphi, \tilde{\psi}) = 0$. The corresponding operator will be denoted by A_∞.

We are going to discuss the dependence of the parameter γ on the coupling constant α entering (1). This relation is not straightforward. The reason is that the operator A_α is not defined on each domain $\mathrm{Dom}(A_\gamma)$. The operator A is well defined as a linear operator on the domain $\mathrm{Dom}(A^{0*})$, but the corresponding projector is not defined there as φ does not belong to $\mathscr{H}_{-1}(A)$. We can look at this problem from the point of view of the theory of distributions. The distribution φ is defined only on the domain $\mathscr{H}_{+2}(A)$, but the element $[1/(A+1)]\,\varphi$ does not belong to this domain. It is necessary to extend the distribution φ as a bounded linear functional to the set of test functions from $\mathscr{H}_{+2}(A)$. This extension is not unique. If the distribution φ has special symmetry and/or scaling properties then the extension possessing the same properties may be unique. We consider here the general case first. We are going to extend the linear functional φ to the whole domain $\mathrm{Dom}(A^{0*})$. One can define the norm on this linear space to be equal to

$$\|g\|_{\mathrm{Dom}(A^{0*})} = \|\tilde{g}\|_{\mathscr{H}_{+2}(A)} + |a(g)|, \tag{5}$$

where $g = \tilde{g} + a(g)[1/(A+1)]\,\varphi$. To extend the functional φ from $\mathrm{Dom}(A)$ to $\mathrm{Dom}(A^{0*})$ it is enough to define it on the element $[1/(A+1)]\,\varphi$. All possible extensions φ_c are parametrized by a constant c such that

$$\left(\varphi_c, \frac{1}{A+1}\,\varphi\right) = c. \tag{6}$$

This parameter c can be fixed only by choosing some additional conditions. (In the paper [12] it was assumed that $c = \infty$). We suppose that $c \in \mathbf{R}$. This assumption guarantees that the quadratic form of the resolvent $[1/(A+1)]$ is real. The distribution φ_c is defined as a linear bounded functional on the space $\mathrm{Dom}(A^{0*}) = \mathscr{H}_{+2}(A) \dotplus \{[1/(A+1)]\,\varphi\} \ni \tilde{g} + a(g)[1/(A+1)]\,\varphi = g$ by

$$(\varphi_c, g) = (\varphi, \tilde{g}) + c a(g). \tag{7}$$

We remark that by Lemma 2.1 below every such functional φ_c can be approximated by a certain sequence of distributions from $\mathscr{H}_0(A)$.

THEOREM 2.1. *The domain of the selfadjoint operator* $A_\alpha = A + \alpha(\varphi_c, \cdot)\,\varphi$ *coincides with the set*

$$\mathrm{Dom}(A_\alpha) = \left\{ \psi \in \mathrm{Dom}(A^{0*}) \,|\, a(\psi) = -\frac{\alpha}{1+\alpha c}\,(\varphi, \tilde{\psi}) \right\}. \tag{8}$$

A_α *is a selfadjoint extension of* A^0. *For* $\alpha = 0$ *we have* $A_0 = A$.

Proof. The operator A_α is well defined as a linear operator on $\mathrm{Dom}(A^{0*})$ and we have:

$$\begin{aligned} A_\alpha \psi &= A_\alpha \left(\tilde{\psi} + a(\psi)\,\frac{1}{A+1}\,\varphi \right) \\ &= A\tilde{\psi} + \alpha(\varphi_c, \tilde{\psi})\,\varphi - a(\psi)\,\frac{1}{A+1}\,\varphi + a(\psi)\,\varphi + a(\psi)\,\alpha \left(\varphi_c, \frac{1}{A+1}\,\varphi \right) \varphi \\ &= A\tilde{\psi} - a(\psi)\,\frac{1}{A+1}\,\varphi + a(\psi)\,\varphi + \alpha(\varphi, \tilde{\psi})\,\varphi + \alpha a(\psi)\,c\varphi. \end{aligned}$$

The range of the linear operator A_α is contained in $\mathscr{H}_{-2}(A)$. The selfadjoint operator A_α can be defined as usual on the domain

$$\mathrm{Dom}(A_\alpha) = \{ \psi \in \mathrm{Dom}(A^{0*}) \,|\, A_\alpha \psi \in \mathscr{H}_0(A) \}. \tag{9}$$

The element $A_\alpha \psi$ belongs to $\mathscr{H}_0(A)$ if and only if the following condition is satisfied:

$$a(\psi) = -\frac{\alpha}{1+\alpha c}\,(\varphi, \tilde{\psi}). \tag{10}$$

The operator A_α with the domain $\mathrm{Dom}(A_\alpha)$ is symmetric. The operators A_α and A^{0*} coincide on the domain $\mathrm{Dom}(A_\alpha)$. It follows that A_α is a selfadjoint extension of the operator A^0. For $\alpha = 0$ we have by construction $A_0 = A$. ▮

Remark. If $\alpha = -1/c$, then the extension A_α of A^0 is defined by the condition $(\varphi, \tilde{\psi}) = 0$ and coincides with the operator A_∞. It follows that the parameter c can be defined if one knows which value of α corresponds to the extension A_∞.

2.2. *Resolvent Formulas*

We are going to discuss here the generalization of the Krein's formula for the resolvent of two different selfadjoint extensions of a given symmetric operator. Let the operators A and A_α be two selfadjoint extensions of the operator A^0. Let $f \in \mathscr{H}_0(A)$, $\psi \in \mathrm{Dom}(A_\alpha)$, and

$$(A_\alpha - z)^{-1} f = \psi$$

(z is in the resolvent set of A_α). The resolvent of the operator A_α can be calculated as follows. One applies first the operator $A_\alpha - z$ to the latter equality,

$$\begin{aligned} f = (A_\alpha - z)\,\psi &= (A + \alpha(\varphi, \cdot)\,\varphi - z)\left(\tilde{\psi} + a(\psi)\,\frac{1}{A+1}\,\varphi\right) \\ &= (A - z)\,\tilde{\psi} - (z+1)\,a(\psi)\,\frac{1}{A+1}\,\varphi. \end{aligned}$$

By applying then the resolvent $(A-z)^{-1}$ of the unperturbed operator A to f we obtain

$$\frac{1}{A-z}\,f = \tilde{\psi} - (z+1)\,a(\psi)\,\frac{1}{A-z}\,\frac{1}{A+1}\,\varphi.$$

Projection on φ then gives the equation

$$\begin{aligned} \left(\varphi, \frac{1}{A-z}\,f\right) &= (\varphi, \tilde{\psi}) - (z+1)\,a(\psi)\left(\varphi, \frac{1}{A-z}\,\frac{1}{A+1}\,\varphi\right) \\ &= (\varphi, \tilde{\psi}) - (z+1)\,a(\psi)\left(\frac{1}{A-\bar{z}}\,\varphi, \frac{1}{A+1}\,\varphi\right). \end{aligned}$$

It follows that

$$(\varphi, \tilde{\psi}) = \frac{\left(\varphi, \frac{1}{A-z}\,f\right)}{1 + \frac{\alpha}{1+\alpha c}\,(z+1)\left(\frac{1}{A-\bar{z}}\,\varphi, \frac{1}{A+1}\,\varphi\right)}$$

and

$$\tilde{\psi} = \frac{1}{A-z}\,f - \frac{\alpha(z+1)\left(\varphi, \frac{1}{A-z}\,f\right)}{1 + \alpha c + \alpha(z+1)\left(\frac{1}{A-\bar{z}}\,\varphi, \frac{1}{A+1}\,\varphi\right)}\,\frac{1}{A-z}\,\frac{1}{A+1}\,\varphi.$$

The resolvent of the perturbed operator is equal to

$$\frac{1}{A_\alpha - z} = \frac{1}{A-z} - \frac{\alpha}{1+\alpha\left[c+(z+1)\left(\frac{1}{A-\bar{z}}\varphi, \frac{1}{A+1}\varphi\right)\right]}\left(\frac{1}{A-\bar{z}}\varphi, \cdot\right)\frac{1}{A-z}\varphi. \tag{11}$$

Define (for $\Im z \neq 0$)

$$F_\alpha(z) = \left(\varphi_c, \frac{1}{A_\alpha - z}\varphi\right) \tag{12}$$

The function $F(z) = F_0(z)$ can be calculated using the extended definition of the distribution φ,

$$\begin{aligned} F(z) &= \left(\varphi_c, \frac{1}{A-z}\varphi\right) \\ &= \left(\varphi_c, \frac{1}{A+1}\varphi\right) + (z+1)\left(\varphi_c, \frac{1}{A-z}\frac{1}{A+1}\varphi\right). \end{aligned}$$

The first term in the latter expression is equal to c due to our assumption. We can drop the subscript c in the last term since $[1/(A-z)][1/(A+1]\,\varphi$ is an element from $\mathscr{H}_{+2}(A)$; hence

$$F(z) = c + (z+1)\left(\frac{1}{A+1}\varphi, \frac{1}{A-z}\varphi\right). \tag{13}$$

Using this all five crucial formulas for the rank one perturbation [17] can be written in the same form as in the case of $\mathscr{H}_{-1}(A)$ perturbations:

$$F_\alpha(z) = \frac{F(z)}{1+\alpha F(z)}; \tag{14}$$

$$\frac{1}{A_\alpha - z}\varphi = \frac{1}{1+\alpha F(z)}\frac{1}{A-z}\varphi; \tag{15}$$

$$\frac{1}{A_\alpha - z} = \frac{1}{A-z} - \frac{\alpha}{1+\alpha F(z)}\left(\frac{1}{A-\bar{z}}\varphi, \cdot\right)\frac{1}{A-z}\varphi; \tag{16}$$

$$\operatorname{Tr}\left[\frac{1}{A-z} - \frac{1}{A_\alpha - z}\right] = \frac{d}{dz}\ln(1+\alpha F(z)); \tag{17}$$

$$\int_{-\infty}^{+\infty} [d\mu_\alpha(E)]\, d\alpha = dE, \tag{18}$$

where μ_α is the spectral measure corresponding to the operator A_α and the elements φ_c, φ. (See [17] for the correct interpretation of the latter formula.)

2.3. *Approximations of $\mathscr{H}_{-2}$ Perturbations*

We discuss in this section the approximations of the constructed linear operators by usual $\mathscr{H}_0$ perturbations of the operator A. We concentrate our attention on the $\mathscr{H}_{-2}$ perturbations.

LEMMA 2.1. *Let f be an element from $\mathscr{H}_0(A)\backslash\mathscr{H}_{+2}(A)$ and φ be an element from $\mathscr{H}_{-2}(A)$; then for any c there exists a sequence φ_n of elements from $\mathscr{H}_0(A)$ converging to φ in $\mathscr{H}_{-2}(A)$ norm such that (f, φ_n) converges to c.*

Proof. The subspace $\mathscr{H}_0(A)$ is dense in $\mathscr{H}_{-2}$. It follows that there exists a sequence $\tilde{\varphi}_n$ of elements from $\mathscr{H}_0$ converging in $\mathscr{H}_{-2}$ norm to φ. If the sequence $(f, \tilde{\varphi}_n) = a_n$ converges to c, then the lemma is proven. If this is not so, consider a sequence $\psi_n \in \mathscr{H}_0(A)$ with unit $\mathscr{H}_{-2}$ norm $\|\psi_n\|_{\mathscr{H}_{-2}} = 1$ such that $|(f, \psi_n)|$ diverges to ∞. Such a sequence exists because $f \notin \mathscr{H}_{+2}(A)$. We can then choose a subsequence such that $(c - a_n)/(f, \psi_n) \to 0$. We keep the same notation for the chosen subsequence. Consider the sequence

$$\varphi_n = \tilde{\varphi}_n + \frac{c - a_n}{(f, \psi_n)} \psi_n.$$

The following estimates are valid:

$$\|\varphi_n - \varphi\|_{\mathscr{H}_{-2}} \leqslant \|\tilde{\varphi}_n - \varphi\|_{\mathscr{H}_{-2}} + \left|\frac{c - a_n}{(f, \psi_n)}\right|.$$

It follows that φ_n converge to φ in $\mathscr{H}_{-2}$ norm. At the same time the sequence $(f, \tilde{\varphi}_n) = a_n + c - a_n = c$ obviously converges to c; hence the lemma is proven. ∎

Remark. The convergence in $\mathscr{H}_{-2}(A)$ was crucial for the proof of the lemma. For example, if $f \in \mathscr{H}_{+1}(A)$ and $\varphi \in \mathscr{H}_{-1}(A)$ then for every sequence φ_n weakly converging in $\mathscr{H}_{-1}(A)$ to φ one has $(f, \varphi_n) \to_{n \to \infty} (f, \varphi)$ and the constant c cannot be chosen arbitrarily. This corresponds to the case of a $\mathscr{H}_{-1}$ perturbation.

We are going to consider the linear operators $A_\alpha^n = A + \alpha(\varphi_n, \cdot)\, \varphi_n$ defined on the common domain $\mathrm{Dom}(A^{0*})$. The range of the linear operators A_α^n, A_α belongs to the space $\mathscr{H}_{-2}(A)$ with the standard norm.

THEOREM 2.2. *Let the sequence* $\varphi_n \in \mathscr{H}_0(A)$ *converge to* $\varphi \in \mathscr{H}_{-2}(A) \setminus \mathscr{H}_{-1}(A)$ *in* $\mathscr{H}_{-2}$ *norm and* $(\varphi_n, [1/(A+1)]\, \varphi)$ *converge to* c, *then the sequence of linear operators*

$$A_\alpha^n = A + \alpha(\varphi_n, \cdot)\, \varphi_n \tag{19}$$

defined on the domain $\mathrm{Dom}(A^{0*})$ *converges in the operator norm to the operator* A_α.

Proof. The linear operators A_α^n and A_α are defined on the elements from $\mathrm{Dom}(A^{0*})$ and the range belongs to the space $\mathscr{H}_{-2}(A)$. Consider an arbitrary element g from $\mathrm{Dom}(A^{0*})$, then the following estimates are valid

$$\begin{aligned}
&\|(A_\alpha^n - A_\alpha)\, g\|_{\mathscr{H}_{-2}}\\
&\quad = \|\alpha(\varphi_n, g)\, \varphi_n - \alpha(\varphi_c, g)\, \varphi\|_{\mathscr{H}_{-2}}\\
&\quad = |\alpha| \left\| (\varphi_n, \tilde{g})\, \varphi_n + a(g) \left(\varphi_n, \frac{1}{A+1}\, \varphi \right) \varphi_n \right.\\
&\qquad \left. - (\varphi, \tilde{g})\, \varphi - a(g) \left(\varphi_c, \frac{1}{A+1}\, \varphi \right) \varphi \right\|_{\mathscr{H}_{-2}}\\
&\quad \leqslant |\alpha| \Bigg(|(\varphi_n, \tilde{g}) - (\varphi, \tilde{g})|\, \|\varphi_n\|_{\mathscr{H}_{-2}} + |(\varphi, \tilde{g})|\, \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}}\\
&\qquad + |a(g)| \left| \left(\varphi_n, \frac{1}{A+1}\, \varphi \right) - c \right| \|\varphi_n\|_{\mathscr{H}_{-2}} + |a(g)|\, |c|\, \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}} \Bigg)\\
&\quad \leqslant |\alpha| \Bigg(\|\varphi_n\|_{\mathscr{H}_{-2}} \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}} \|\tilde{g}\|_{\mathscr{H}_{+2}} + \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}} \|\varphi\|_{\mathscr{H}_{-2}} \|\tilde{g}\|_{\mathscr{H}_{+2}}\\
&\qquad + \left| \left(\varphi_n, \frac{1}{A+1}\, \varphi \right) - c \right| \|\varphi_n\|_{\mathscr{H}_{-2}} |a(g)| + |c|\, \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}} |a(g)| \Bigg)\\
&\quad \leqslant |\alpha| \Bigg\{ (\|\varphi_n\|_{\mathscr{H}_{-2}} + \|\varphi\|_{\mathscr{H}_{-2}} + |c|)\, \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}}\\
&\qquad + \|\varphi_n\|_{\mathscr{H}_{-2}} \left| \left(\varphi_n, \frac{1}{A+1}\, \varphi \right) - c \right| \Bigg\} \|g\|.
\end{aligned}$$

Now the operator norm can be estimated as follows:

$$\begin{aligned}
&\|A_\alpha^n - A_\alpha\|_{\mathrm{Dom}(A^{0*}) \Rightarrow \mathscr{H}_{-2}(A)}\\
&\qquad \leqslant |\alpha| \Bigg\{ (\|\varphi_n\|_{\mathscr{H}_{-2}} + \|\varphi\|_{\mathscr{H}_{-2}} + |c|)\, \|\varphi_n - \varphi\|_{\mathscr{H}_{-2}}\\
&\qquad\quad + \|\varphi_n\|_{\mathscr{H}_{-2}} \left| \left(\varphi_n, \frac{1}{A+1}\, \varphi \right) - c \right| \Bigg\}.
\end{aligned}$$

The sequence φ_n converges in $\mathscr{H}_{-2}(A)$ norm to φ, the sequence $\|\varphi_n\|_{\mathscr{H}_{-2}}$ is bounded and the sequence $(\varphi_n, [1/(A+1)]\,\varphi)$ converges to c. It follows that

$$\lim_{n\to\infty} \|A_\alpha^n - A_\alpha\|_{\mathrm{Dom}(A^{0*}) \Rightarrow \mathscr{H}_{-2}(A)} = 0. \quad \blacksquare$$

The following more general fact can be proven:

THEOREM 2.3. *Let $\varphi \in \mathscr{H}_{-2}(A)\backslash\mathscr{H}_{-1}(A)$, then there exists a sequence $\varphi_n \in \mathscr{H}_0(A)$ converging to φ in $\mathscr{H}_{-2}$ norm such that the sequence of linear operators* (19) *defined on the domain* $\mathrm{Dom}(A^{0*})$ *converges in the operator norm to the operator $A_\alpha = A + \alpha(\varphi_c, \cdot)\,\varphi$.*

Proof. If $\varphi \in \mathscr{H}_{-2}(A)\backslash\mathscr{H}_{-1}(A)$ then $[1/(A+1)]\,\varphi \in \mathscr{H}_0(A)\backslash\mathscr{H}_{+1}(A)$ and it follows from Lemma 2.1 that there exists a sequence $\varphi_n \in \mathscr{H}_0(A)$ converging to φ in $H_{-2}(A)$ norm and such that $(\varphi_n, [1/(A+1)]\,\varphi)$ converges to c. It follows from Theorem 2.2 that the operators (19) converge to A_α in the operator norm. $\blacksquare$

To study the strong resolvent convergence we prove first the following.

LEMMA 2.2. *Let $\mathscr{H}$ be a separable Hilbert space and C be positive bounded self-adjoint operator. If the sequence $C\psi_n$, $\psi_n \in \mathscr{H}$ converges weakly in $\mathscr{H}$ to $f \notin \mathrm{Dom}(C^{-1})$, then $\|\psi_n\|$ diverges to ∞.*

Proof. Suppose that $\|\psi_n\|$ does not diverge to ∞. This means that there exists an infinite bounded subsequence ψ_{n_j}. This subsequence converges weakly on $\mathrm{Dom}(C^{-1})$ which is dense in $\mathscr{H}$. It follows from the Banach–Steinhaus theorem, that such a subsequence converges weakly on the whole $\mathscr{H}$. Let $g \in \mathscr{H}$ then

$$\lim_{n\to\infty} (g, \psi_{n_j}) \equiv F(g)$$

exists. This implies that there exists $h \in \mathscr{H}$ such that

$$\lim_{n\to\infty} (g, \psi_{n_j}) = (g, h).$$

It follows that $\mathrm{Ch} = f$ due to the uniqueness of the weak limit and $f \in \mathrm{Dom}(C^{-1})$. We get a contradiction, which proves the lemma. $\blacksquare$

COROLLARY 2.1. *Let $\varphi_n \in \mathscr{H}_0(A)$ be a sequence converging weakly in $\mathscr{H}_{-m}(A)$, $m \geqslant 2$ to $\varphi \in \mathscr{H}_{-m}(A)\backslash\mathscr{H}_{-l}(A)$, $m > l \geqslant 0$. Then $\|\varphi_n\|_{\mathscr{H}_{-l}}$ diverges to ∞.*

Proof. Consider the sequence $\psi_n = (A+1)^{-l/2} \varphi_n$ and the positive bounded selfadjoint operator $C = (A+1)^{-(m-l)/2}$. The sequence $C\psi_n$ converges weakly in the Hilbert space $\mathscr{H}_0$ to $f = C\psi = (A+1)^{-m/2} \varphi \notin \mathscr{H}_{m-l}(A) = \mathrm{Dom}(C^{-1})$. It follows from the previous lemma that

$$\|(A+1)^{-l/2} \varphi_n\|_{\mathscr{H}_0} = \|\varphi_n\|_{\mathscr{H}_{-l}} = \|\psi_n\|_{\mathscr{H}_0}$$

diverges to ∞. ∎

It follows that Theorem 5.1 in [12] can be generalized to include any sequence φ_n converging weakly to an element $\varphi \in \mathscr{H}_{-\infty}$. (Only a special sequence φ_n has been considered in [12].) We shall need the following lemma.

LEMMA 2.3. *Let $A \geqslant 0$ and $\varphi \in \mathscr{H}_{-m}(A) \backslash \mathscr{H}_{-1}(A)$, $m \geqslant 2$. Let φ_n be any sequence weakly converging to φ in $\mathscr{H}_{-m}(A)$. Then the sequence of selfadjoint operators*

$$A_n = A + \alpha(\varphi_n, \cdot)\, \varphi_n$$

converges to A in the strong resolvent sense.

Proof. The resolvent of the operator A_n is equal to

$$\frac{1}{A_n - z} = \frac{1}{A-z} - \frac{\alpha}{1 + \alpha\left(\varphi_n, \dfrac{1}{A-z}\varphi_n\right)} \left(\frac{1}{A-\bar{z}}\varphi_n, \cdot\right) \frac{1}{A-z}\varphi_n. \quad (20)$$

We are going to prove that the resolvent of the perturbed operator converges to the resolvent of the unperturbed operator in the weak topology. It will follow then that the resolvents converge in the norm also. We have for arbitrary $\psi_1, \psi_2 \in \mathscr{H}_0$:

$$\left|\left(\psi_1, \left(\frac{1}{A_n - z} - \frac{1}{A-z}\right)\psi_2\right)\right|$$
$$= \left|\frac{\alpha}{1 + \alpha\left(\varphi_n, \dfrac{1}{A-z}\varphi_n\right)} \left(\psi_1, \frac{1}{A-z}\varphi_n\right)\left(\frac{1}{A-\bar{z}}\varphi_n, \psi\right)\right|.$$

The sequence $(\psi_1, [1/(A-z)]\, \varphi_n) = (\psi_1, [1/(A+1)]\, \varphi_n) + (z+1)([1/(A-z)]\psi_1, [1/(A+1)]\, \varphi_n)$ converges due to the weak convergence of φ_n

in $\mathscr{H}_{-2}$. This sequence is uniformly bounded. Similarly the sequence $([1/(A+z)]\,\varphi_n, \psi_2)$ is also uniformly bounded. The sequence

$$\frac{\alpha}{1+\alpha\left(\varphi_n, \dfrac{1}{A-z}\,\varphi_n\right)}$$

converges to zero because $\varphi \notin \mathscr{H}_{-1}$. ∎

Thus the operators A_n for a specially chosen sequence φ_n converge to the operator A_α in the operator norm of linear operators defined on $\mathrm{Dom}(A^{0*})$. If the sequence converges to an element from $\mathscr{H}_{-2}(A)$ then the resolvents of the corresponding selfadjoint operators converge strongly to the resolvent of the unperturbed operator. To construct approximations of the $\mathscr{H}_{-2}$ perturbations in the strong resolvent sense infinitesimal couplings should be considered. This approach has been developed in [12].

3. RANK ONE PERTURBATIONS FOR HOMOGENEOUS OPERATORS

In this section we study the rank one perturbations in the case where the original operator and the element φ are homogeneous with respect to a certain one parameter group of unitary transformations of the Hilbert space $\mathscr{H}$. Then the homogeneous extension of the functional φ to the set of all functions from the domain $\mathrm{Dom}(A^{0*})$ is defined uniquely under certain conditions. The following lemma is valid:

LEMMA 3.1. *Let the positive selfadjoint operator A and the vector $\varphi \in \mathscr{H}_{-2}(A)$ be homogeneous with respect to a certain one parameter (multiplicative) unitary group $G(t)$, in the sense that $G(t_1)\,G(t_2) = G(t_1 t_2)$, $t_1, t_2 \in \mathbf{R}$ and there exist real constants β, γ such that*

$$G(t)\,A = t^{-\beta} A G(t), \tag{21}$$

$$(G(t)\,\varphi, \psi) = (\varphi, G(1/t)\,\psi) = t^{\gamma}(\varphi, \psi) \tag{22}$$

for every $\psi \in \mathscr{H}_2(A)$. Then φ can be extended as a homogeneous linear bounded functional to the domain $\mathrm{Dom}(A^{0*})$ *if and only if*

$$f(t) = -\frac{1-t^{-\beta}}{1-t^{\beta-2\gamma}}\left(\frac{1}{(1+A)(1+t^{-\beta})}\,\varphi, \varphi\right) \tag{23}$$

does not depend on $t \neq 1$.

Proof. Every extension of φ to the domain $\mathrm{Dom}(A_0^*)$ is defined by its value on the element $[1/(A+1)]\,\varphi$. Consider an arbitrary extension φ_c defined by the equality (6). Suppose that it satisfies the homogeneity property (22). It follows that

$$
\begin{aligned}
f(t) &= -\frac{1}{1-t^{\beta-2\gamma}}\left(\left(\varphi_c, \frac{1}{A+t^{-\beta}}\varphi\right) - \left(\varphi_c, \frac{1}{A+1}\varphi\right)\right) \\
&= -\frac{1}{1-t^{\beta-2\gamma}}\left(t^{-\gamma}\left(\varphi_c, \frac{1}{A+t^{-\beta}}G(t)\,\varphi\right) - \left(\varphi_c, \frac{1}{A+1}\varphi\right)\right) \\
&= -\frac{1}{1-t^{\beta-2\gamma}}\left(t^{\beta-\gamma}\left(G(1/t)\,\varphi_c, \frac{1}{A+1}\varphi\right) - \left(\varphi_c, \frac{1}{A+1}\varphi\right)\right) \\
&= \left(\varphi_c, \frac{1}{A+1}\varphi\right).
\end{aligned}
$$

This implies that for any extension φ_c the function $f(t)$ is constant, i.e., independent of t, and this constant defines the unique extension φ_c.

Suppose now that the function $f(t)$ is constant $f(t)=c$. Let us define the extension φ_c of φ using the equality (6). We have to show that this extension is homogeneous with respect to the symmetry group $G(t)$. In fact it is enough to show that $(G(t)\,\varphi_c, [1/(A+1)]\,\varphi) = t^{\gamma}(\varphi_c, [1/(A+1)]\,\varphi)$. By calculations similar to those above we get then

$$
\begin{aligned}
&\left(G(1/t)\,\varphi_c, \frac{1}{A+1}\varphi\right) \\
&\quad = \left(\varphi_c, G(t)\frac{1}{A+1}\varphi\right) \\
&\quad = t^{\gamma-\beta}\left(\varphi_c, \frac{1}{A+t^{-\beta}}\varphi\right) \\
&\quad = t^{\gamma-\beta}\left(\left(\varphi_c, \frac{1}{A+1}\varphi\right) + (1-t^{-\beta})\left(\varphi_c, \frac{1}{A+1}\frac{1}{A+t^{-\beta}}\varphi\right)\right) \\
&\quad = t^{\gamma-\beta}\left(\left(\varphi_c, \frac{1}{A+1}\varphi\right) + (t^{\beta-2\gamma}-1)\left(\varphi_c, \frac{1}{A+1}\varphi\right)\right) \\
&\quad = t^{-\gamma}\left(\varphi_c, \frac{1}{A+1}\varphi\right),
\end{aligned}
$$

and the lemma is proven. ∎

It has been shown that the homogeneity properties allow one to define uniquely the rank one $\mathscr{H}_{-2}$-perturbations. The most important examples of homogeneous rank one perturbations are

- $-d^2/dx^2 + \alpha\delta$ in $L_2(\mathbf{R})$;
- $-d^2/dx^2 + \alpha(\delta', \cdot)\delta'$ in $L_2(\mathbf{R})$;
- $-\Delta + \alpha\delta$ in $L_2(\mathbf{R}^3)$.

The first two operators has been considered in detail in [7, 15] where the distribution theory for discontinuous test functions has been developed. The first operator is defined uniquely since δ is an element from the space $\mathscr{H}_{-1}$ for the second derivative operator in one dimension. The second and third operators can be defined using the groups of scaling transformations in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}^3)$ correspondingly. A detail analysis of the third operator is presented in the next section.

4. POINT INTERACTIONS IN DIMENSION THREE

We study the Schrödinger operator in dimension three defined by the heuristic expression

$$L_\alpha = -\Delta + \alpha\delta, \tag{24}$$

where Δ is the Laplace operator, α is a coupling constant in $\mathbf{R}$ and δ is a Dirac delta function in dimension three. The first correct mathematical definition of such a "point interaction Hamiltonian" has been given by F. A. Beresin and L. D. Faddeev [6] in the framework of the extension theory for symmetric operators. Important question concerning approximations of the point interaction Hamiltonian by rank one perturbations of the Laplace operator has been considered in a series of papers by S. Albeverio and R. Høegh-Krohn (see [1] for an extensive review of the problem).

The operator L_α to be defined in $L^2(\mathbf{R}^3)$ can be considered as a rank one perturbation of the Laplace operator because $\delta\varphi = \varphi(0)\,\delta = (\varphi, \delta)\,\delta$ and the generalized function δ is an element from $\mathscr{H}_{-2}(-\Delta)$. Consider the group $S(t)$, $t > 0$ of the scaling transformations of $L_2(\mathbf{R}^3)$ defined as follows for every function ψ and distribution f:

$$(S(t)\,\psi)(x) = t^{3/2}\psi(tx);$$

$$(S(t)\,f)(\psi) = f(S(1/t)\,\psi).$$

The Laplace operator and the delta function are homogeneous with respect to the group $S(t)$

$$S(t)\,\Delta = t^2 \Delta S(t);$$

$$S(t)\,\delta = t^{-3/2}\delta.$$

The perturbed operator coincides with one of the selfadjoint extensions of the symmetric Laplace operator $-\Delta_0$ defined on smooth functions vanishing in a neighborhood of the origin. The domain of the adjoint operator $(-\Delta_0)^*$ coincides with the space $W_2^2(\mathbf{R}^3\backslash\{0\})$. The distribution δ possesses a unique extension to the set $W_2^2(\mathbf{R}^3\backslash\{0\})$. The parameter c which defines the extension is equal to

$$\begin{aligned} c &= -\frac{1-t^2}{1-t}\left(\frac{1}{-\Delta+1}\delta, \frac{1}{-\Delta+t}\delta\right) \\ &= -\frac{1-t^2}{1-t}\int_{\mathbf{R}^3}\frac{e^{-|x|}}{4\pi\,|x|}\frac{e^{-t\,|x|}}{4\pi\,|x|}\,d^3x \\ &= -\frac{1}{4\pi}\equiv\left(\delta, \frac{1}{-\Delta+1}\delta\right). \end{aligned}$$

It follows from Theorem 2.1 that the selfadjoint operator L_α coincides with the operator $-\Delta_0$ restricted to the domain

$$\begin{aligned} \mathrm{Dom}(L_\alpha) = \Bigg\{ \psi = \tilde{\psi} + a(\psi)\left(\frac{e^{-|x|}}{4\pi\,|x|}\right) &: \tilde{\psi}\in W_2^2(\mathbf{R}^3), \\ a(\psi)\in\mathbf{C},\ a(\psi) &= -\frac{\alpha}{1-\alpha/4\pi}\tilde{\psi}(0)\Bigg\}. \end{aligned}$$

In the case $\alpha=0$, the selfadjoint operator L_0 coincides with the Friedrichs extension of $-\Delta_0$.

Approximations of the operator L_α can be constructed explicitly. Let ω be a $C_0^\infty(\mathbf{R}_+)$ function with compact support and vanishing at the origin, normalized such that $\int_0^\infty \omega(x)\,dx = 1$. We choose a special (but "standard") delta functional sequence equal to

$$V_\varepsilon(x) = \frac{-1}{4\pi}\frac{1}{\varepsilon^2\,|x|}\,\omega'\left(\frac{|x|}{\varepsilon}\right).$$

V_ε has compact support and it is easily verified that

$$\int_{\mathbf{R}^3} V_1(x)\, d^3x = -\int_0^\infty r v_1(r)\, dr$$

$$= -\int_0^\infty r\omega'(r)\, dr = -r\omega(r)\,|_0^\infty + \int_0^\infty \omega(r)\, dr = 1.$$

Moreover V_ε has the usual scaling properties: $V_\varepsilon(x) = (1/\varepsilon^3)\, V_1(x/\varepsilon)$.

LEMMA 4.1. *Let* $\psi = \tilde{\psi} + a(\psi)(e^{-|x|}/4\pi\,|x|)$ *be any function from* $\mathrm{Dom}(-\Delta_0^*)$, *then the following limit holds*:

$$\lim_{\varepsilon \to 0}\, (V_\varepsilon, \psi) = -a(\psi)/4\pi + \tilde{\psi}(0).$$

Proof. Every function $\psi \in E$ possesses the following representation

$$\psi(x) = \left(\frac{a(\psi)}{4\pi\,|x|} + \psi_0\right)\chi(x) + \hat{\psi}(x),$$

where $\psi_0 = -a(\psi)/4\pi + \tilde{\psi}(0)$, the function χ has compact support and is equal to one in a neighborhood of the origin and the function $\tilde{\psi}$ satisfies the asymptotic representation

$$\hat{\psi}(x) = o(1), \qquad x \to 0.$$

The following limits exist:

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^3} d^3x V_\varepsilon(x)\, \hat{\psi}(x) = 0;$$

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^3} d^3x V_\varepsilon(x)\, \chi(x)\, \psi_0 = \psi_0;$$

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^3} d^3x V_\varepsilon(x)\, \frac{\chi(x)}{4\pi\,|x|}\, \psi_- = 0.$$

The latter limit follows from the orthogonality of the functions $V_\varepsilon(x)$ and $1/|x|$ in $L^2(\mathbf{R}^3)$. The lemma is proven. ∎

Consider now the sequence of linear operators defined in the generalized sense

$$L_{\alpha,\varepsilon} = -\Delta + \alpha V_\varepsilon(x)(V_\varepsilon(x), \cdot).$$

This sequence of linear operators $L_{\alpha,\varepsilon}$ converges as $\varepsilon \searrow 0$ to the operator L_α in the weak operator topology. We prove now that the sequence of linear operators $L_{\alpha,\varepsilon}$ converges to the operator L_α in the operator norm. All these operators are defined on the domain $\mathrm{Dom}(-\Delta_0^*)$ and their ranges belong to $\mathscr{H}_{-2}(-\Delta)$. The norms are defined by Eqs. (5) and (2) correspondingly.

LEMMA 4.2. *Let ω be an infinitely differentiable function with compact support on the positive half axis and assume $\omega(0)=0$ and $\int_0^\infty \omega(r)\,dr=1$. Then*

$$V_\varepsilon(x)=\left(\frac{-1}{4\pi r}\frac{\partial}{\partial r}\frac{1}{\varepsilon}\,\omega\left(\frac{r}{\varepsilon}\right)\right)\Bigg|_{r=|x|}, \qquad x\in \mathbf{R}^3$$

converges to δ in $\mathscr{H}_{-2}(-\Delta)$ when $\varepsilon\searrow 0$.

Proof. We have to prove that

$$\lim_{\varepsilon\to 0}\left\|\frac{1}{-\Delta+1}(\delta-V_\varepsilon)\right\|_{L_2(\mathbf{R}^3)}=0. \tag{25}$$

The Fourier transform of the function V_ε depends only on the absolute value of the vector $\mathbf{p}\in\mathbf{R}^3$:

$$\begin{aligned}\hat{V}_\varepsilon &= \int_0^\infty dr\, r^2\int_0^\pi d\theta\,\sin\theta e^{irp\cos\theta}\,\frac{-2\pi}{4\pi r}\,\frac{\partial}{\partial r}\frac{1}{\varepsilon}\,\omega\left(\frac{r}{\varepsilon}\right)\\ &=\int_0^\infty \cos rp\,\frac{1}{\varepsilon}\,\omega\left(\frac{r}{\varepsilon}\right)dr.\end{aligned}$$

The function

$$\hat{V}_\varepsilon(p)-1=\int_0^\infty(\cos rp-1)\,\frac{1}{\varepsilon}\,\omega\left(\frac{r}{\varepsilon}\right)dr$$

is uniformly bounded and tends to zero uniformly on every compact domain $D\subset\mathbf{R}^3$. It follows that, with $g_\varepsilon(p)\equiv[1/(p^2+1)](\hat{V}_\varepsilon(p)-1)$,

$$\|g_\varepsilon\|_{L_2(\mathbf{R}^3)}\to_{\varepsilon\to 0} 0,$$

and the limit (25) holds. ∎

THEOREM 4.1. *The sequence of linear operators $L_{\alpha,\varepsilon}$ converges in the operator norm to the linear operator L_α on $W_2^2(\mathbf{R}^3\backslash\{0\})$.*

Proof. This follows easily from Lemma 4.2 and Theorem 2.2 ∎

ACKNOWLEDGMENTS

The authors are grateful to Dr. A. Kiselev for many fruitful discussions and to Professor V. Koshmanenko and Professor B. Pavlov for their interest in this work. The financial support of the Alexander von Humboldt Foundation for the second author is gratefully acknowledged.

REFERENCES

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, "Solvable Models in Quantum Mechanics," Springer-Verlag, Berlin/New York, 1988. [Russian Transl. Nauka, Moscow, 1990]
2. S. Albeverio, W. Karwowski, and V. Koshmanenko, Square power of singularly perturbed operators, *Math. Nachr.* **173** (1995), 5–24.
3. S. Albeverio and V. Koshmanenko, Some remarks on the Gesztesy–Simon version of rank one perturbations, preprint, Kiev, 1996.
4. S. Albeverio and P. Kurasov, Rank one perturbations of not semibounded operators, *J. Integral Equations Oper. Theory* **27** (1997), 379–400.
5. A. Alonso and B. Simon, The Birman–Krein–Vishik theory of self-adjoint extensions of semibounded operators, *J. Operator Theory* **4** (1980), 251–270.
6. F. A. Beresin and L. D. Faddeev, A remark on Schrödinger equation with a singular potential, *Sov. Math. Dokl.* **137**, No. 5 (1961), 1011–1014.
7. J. Boman and P. Kurasov, Finite rank singular perturbations and distributions with discontinuous test functions, in preparation.
8. M. S. Birman, On the self-adjoint extensions of positive definite operators, *Math. Sb.* **38** (1956), 431–456. [In Russian]
9. F. Gesztesy and B. Simon, Rank one perturbations at infinite coupling, *J. Funct. Anal.* **128** (1995), 245–252.
10. H. Hassi and H. de Snoo, On rank one perturbations of selfadjoint operators, in preparation.
11. L. Hörmander, "The Analysis of Linear Partial Differential Operators, I," Springer-Verlag, Berlin, 1983.
12. A. Kiselev and B. Simon, Rank one perturbations with infinitesimal coupling, *J. Funct. Anal.* **130** (1995), 345–356.
13. V. D. Koshmanenko, Towards the rank one singular perturbations theory of selfadjoint operators, *Ukrainian Math. J.* **43**, No. 11 (1991), 1559–1566.
14. M. Krein, The theory of self-adjoint extensions of semibounded hermitian transformations and its applications, I, *Rec. Math. (Math. Sb.)* **20** (1947), 431–495.
15. P. Kurasov, Distribution theory for the discontinuous test functions and differential operators with the generalized coefficients, preprint, 1994; *J. Math. Anal. Appl.*, to appear.
16. P. Kurasov, "Rank-One perturbations of Nonsemibounded Operators," Research Report 1995-15, Luleå University of Technology, Luleå, Sweden, 1995.
17. B. Simon, Spectral analysis of rank one perturbations and applications, *in* "CRM Proceedings and Lecture Notes," Vol. 8, pp. 109–149, 1995.
18. M. Vishik, On general boundary condition for elliptic differential equation, *Trud. Moskov. Mat. Obshch.* **1** (1952), 187–246. [Russian]