Inverse Problems and Imaging
Volume 2, No. 1, 2008, 1–21

Web site: http://www.aimSciences.org

INVERSE PROBLEMS FOR QUANTUM TREES

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(Communicated by William Rundell)

Abstract. Three different inverse problems for the Schrödinger operator on a metric tree are considered, so far with standard boundary conditions at the vertices. These inverse problems are connected with the matrix Titchmarsh-Weyl function, response operator (dynamic Dirichlet-to-Neumann map) and scattering matrix. Our approach is based on the boundary control (BC) method and in particular on the study of the response operator. It is proven that the response operator determines the quantum tree completely, *i.e.* its connectivity, lengths of the edges and potentials on them. The same holds if the response operator is known for all but one boundary points, as well as for the Titchmarsh-Weyl function and scattering matrix. If the connectivity of the graph is known, then the lengths of the edges and the corresponding potentials are determined by just the diagonal terms of the data.

1. **Introduction.** Let Γ be a finite compact metric tree, *i.e.* a metric graph without cycles. Every edge $e_j = [a_{2j-1}, a_{2j}]$ is identified with an interval of the real line. The set of edges will be denoted by $E = \{e_j\}_{j=1}^N$. The edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{a_j\}$. The set of vertices will be denoted by $\{v_1, \dots, v_{N+1}\} = V$. For a rigorous definition of the metric graph suitable for our consideration see [23, 32, 31].

In what follows it will be convenient to consider loose end points (vertices having valency one) separately. Let us denote these vertices by $\{\gamma_1, \dots, \gamma_m\} = \partial\Gamma \subset V$. These vertices will play the role of the graphs boundary.

The graph Γ determines naturally the Hilbert space of square integrable functions $L_2(\Gamma)$. The (Dirichlet) Laplacian L in $L_2(\Gamma)$ is the second derivative operator

$$L = -\frac{d^2}{dx^2}, \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary: 35R30, 81U20, 93B05; Secondary: 35L05, 49J15.

Key words and phrases. Quantum graphs, inverse problems, Schrödinger equation, wave equation, controllability, boundary control.

S.A.'s research is supported in part by the National Science Foundation, grants OPP 0414128, ARC 0724860 and DMS 0648786; P.K.'s research is supported in part by the grants from Swedish Research Council and The Swedish Royal Academy of Sciences.

defined on the domain of functions ψ from the Sobolev space $W_2^2(\Gamma \setminus V)$ satisfying the so-called standard boundary conditions at all internal vertices $v_l \in V \setminus \partial\Gamma$

$$\left\{ \begin{array}{l} \displaystyle\sum_{a_j \in v_l} \partial_n \psi(a_j) = 0; \\ \psi \ \text{ is continuous at the vertex } v_l; \end{array} \right. \tag{1.2}$$

and the Dirichlet conditions at all boundary vertices

$$\psi(v_l) = 0, \ \ v_l \in \partial\Gamma. \tag{1.3}$$

Here ∂_n denotes the internal normal derivative taken along the corresponding edge.[1] Let $q \in L_1(\Gamma)$ be a real valued function. Then quadratic forms can be used to define the Schrödinger operator

$$H = L + q = -\frac{d^2}{dx^2} + q. \tag{1.4}$$

The domain of the quadratic form for H coincides then with the domain of the quadratic form of the Laplace operator L and consists of all functions from $W_2^1(\Gamma \setminus V)$ which are in addition continuous at the vertices. One can show that the domain of the operator H consists of all functions ψ, such that ψ, ψ' are absolutely continuous on $\Gamma \setminus V$ and satisfy boundary conditions (1.2), (1.3) at the vertices.

The Laplace operator L is uniquely determined by the geometric tree and reflects its connectivity. The Hilbert space $L_2(\Gamma)$ does not "feel" the way the edges are connected to each other. It is the boundary conditions (1.2) that "glue" different edges to each other. Instead of the standard boundary conditions (1.2) other symmetric boundary conditions at the vertices may be considered. Then the Schrödinger operator H is determined by both the geometric graph Γ, real potential q and boundary conditions at the vertices.

By **quantum graph** one means the metric graph Γ together with the Schrödinger operator H in $L_2(\Gamma)$. Hence the inverse problem for a quantum graph consists of reconstructing of

- geometric graph Γ,
- real potential q , and
- symmetric boundary conditions at the vertexes.

In the current article we restrict our consideration to the case of the standard boundary conditions (1.2) at internal and Dirichlet conditions (1.3) at boundary vertices. In this case reconstruction of the quantum graph consists of reconstruction of the metric graph and real potential.

In what follows we assume that the graph is clean, i.e., no vertex of valency 2 occurs. The reason is that the standard boundary conditions (1.2) for such vertex imply that the function and its first derivative are continuous and therefore the corresponding two edges may be substituted by one edge having the length equal to the sum of the lengths. Thus this assumption is not restrictive.

[1] Note that the standard Laplace operator on Γ is defined by the same differential operator of minus second derivative (1.1) on the domain of functions from $W_2^2(\Gamma \setminus V)$ satisfying boundary conditions (1.2) at **all** edges including the boundary edges of Γ.

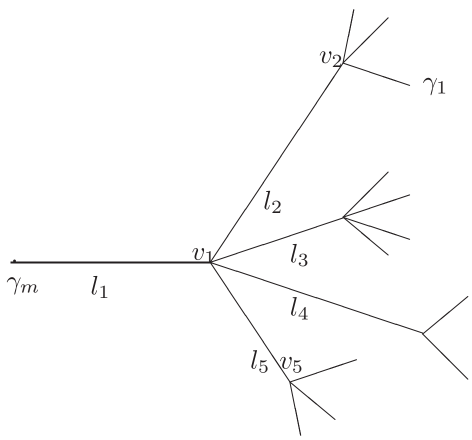

Fig. 1 Metric tree.

The spectrum of the operator H is determined by nontrivial solutions to the following differential equation on Γ

$$-\frac{d^2\psi}{dx^2} + q\,\psi = \lambda\psi \quad \text{in } \{\Gamma \setminus V\}; \tag{1.5}$$

subject to standard boundary conditions (1.2) at the internal vertices and Dirichlet conditions (1.3) on the boundary. The spectrum of H is pure discrete, since the operator H is a finite rank perturbation (in the resolvent sense) of the orthogonal sum of Sturm-Liouville operators on the edges with Dirichlet boundary conditions at all vertices. The spectrum of each of the Sturm-Liouville operators consists of an infinite sequence tending to $+\infty$ [38]. Therefore the spectrum of H is formed by eigenvalues of finite multiplicity: $\lambda_0 \leq \lambda_1 \leq ...$ increasing to $+\infty$.

The system of differential equations (1.5) with boundary conditions (1.2), (1.3) has no solution for $\lambda \notin \mathbb{R}$. Therefore this system of equations has a unique solution satisfying standard boundary conditions (1.2) at internal vertices and non-zero boundary conditions on the graph's boundary

$$\psi(\gamma_i) = 1\,, \text{ and } \psi = 0 \quad \text{on } \partial\Gamma \setminus \{\gamma_i\}. \tag{1.6}$$

Then the $m \times m$ matrix $\mathbf{M}(\lambda)$ with the entries

$$M_{\gamma_i,\gamma_j}(\lambda) = \partial_n \psi(\gamma_j),\ i,j = 1, ..., m, \tag{1.7}$$

will be called the Titchmarsh-Weyl $m-$matrix (T-W matrix).

Let $\psi^{\vec{f}}(x,k)$ be a solution to equation (1.5) satisfying (1.2) at all internal vertexes and boundary conditions

$$\psi^{\vec{f}}(\gamma_j, k) = f_j, \quad j = 1, 2, ..., m \tag{1.8}$$

at all boundary points. Then the Titchmarsh-Weyl function allows one to calculate the normal derivatives of the solution

$$\vec{\partial}_n \psi^{\vec{f}}|_{\partial\Gamma} = \mathbf{M}(\lambda)\vec{f}. \tag{1.9}$$

The main reason that inverse problems for quantum graphs are difficult is that the Cauchy problem is not uniquely solvable if the graph contains cycles. But for trees one may consider a sort of Cauchy problem: if the solution and its derivative are known at all but one boundary points, then it is possible to find the unique solution to the differential equation with prescribed initial data.

The main results of the paper are Theorems 4.1 and 4.2 (see Section 4 below). It is proven that quantum tree can be reconstructed from the reduced response operator associated with all except one boundary points. If the connectivity of the tree is known then the reconstruction can be carried out using the diagonal of this operator.

This problem is closely related to the reconstruction of the quantum tree from the Titchmarsh-Weyl function and from the scattering matrix. The relations between these three inverse problems are clarified.

Recent interest in quantum graphs and trees in particular is motivated by possible applications to nano-electronics and quantum waveguides [2, 28, 29]. Mathematically rigorous approach to differential operators on metric graphs was developed in the 80's by N.I. Gerasimenko, B.S. Pavlov, P. Exner, P.Šeba, and V. Adamyan [1, 16, 17, 18, 19, 20]. It is worth to mention recent articles by V. Kostrykin and R. Schrader (see e.g. [23, 24, 25, 26]) describing most general models of quantum graphs, as well as [32], where the connection between the geometric properties of graphs and their models is described. Spectral properties of differential operators on metric graphs have been studied by many authors and we refer to [12, 34] for complete reference list. Here we would like to mention works by K. Naimark, A. Sobolev and M. Solomyak [33, 36, 37, 39], where spectral properties of quantum trees have been studied in detail. It was discovered that the inverse spectral and inverse scattering problems in general do not have unique solutions [21, 27, 31]. For the inverse scattering problem this fact is usually connected with graphs having internal symmetries [11] and for inverse spectral problem - with the existence of isospectral graphs. It is well-known that one spectrum does not determine potential even in the case of classical Sturm-Liouville operator on one interval, but examples presented in [21] show that even the geometry of the graph cannot be recovered from one spectrum.[2] It has been shown recently that some information concerning the topological structure of the graph is uniquely determined by the spectrum of the corresponding Schrödinger or Laplace operator [35, 30], in particular the Euler characteristic of the graph can be calculated from one of these spectra. Therefore in the current article we concentrate our attention to the case of metric trees and study the corresponding inverse problems. It is important to have in mind that there exist isospectral trees and therefore the knowledge of just one spectrum is not enough to reconstruct even the underlying geometric graph.

The first question to be asked when one studies the inverse problem is to establish the uniqueness result, i.e. to characterize spectral data ensuring unique solution of the inverse problem. Such result for trees with a priori known connectivity and lengthes of edges was established independently by M. Brown and R. Weikard [12] and V. Yurko [41]. See also recent paper [13] containing generalizations for trees of Levinson and Marchenko results obtained originally for a compact intervals. These results are related to our studies in Section 4.2, where trees with arbitrary lengths of edges are considered. M. Belishev [9] and later M. Belishev and A. Vakulenko [10] considered the inverse spectral data (eigenvalues and derivatives of eigenfunctions at the boundary vertices) equivalent to the knowledge of the whole Titchmarsh-Weyl function. We prove in Theorem 1 that to solve the inverse problem it is enough

[2] See also [14] where the inverse spectral problem is considered in the special case of directed graphs.

to know just the reduced Titchmarsh-Weyl function associated with all except one boundary points. The proof of Yurko is 'spectral', it is based on the theory of the Titchmarsh-Weyl function; Belishev reduces the spectral problem to the time-domain problem and uses the Boundary Control (BC) method. Our approach uses the both spectral and time-domain techniques and also the connections between spectral and time domain data and the proofs are constructive: together with the uniqueness theorems we give the constructive procedures to solve the inverse problems. We believe that our approach is more transparent than that of Yurko and Belishev and can be used in numerical calculations.

Notice that the BC method is an important part of our approach. This method is based on deep connections between controllability and inverse (identification) problems [3, 4]. Controllability results for the wave equation on graphs relevant to our identification results can be found, for example, in [5, Chapter VII] and [15].

2. **Titchmarsh-Weyl function and dynamical response operator for a finite interval.** In this section we demonstrate the equivalence of two kinds of inverse boundary data: the Titchmarsh-Weyl function and the non-stationary Dirichlet-to-Neumann map (response operator). For clarity of the exposition we begin with the case of the Sturm-Liouville operator on a finite interval. Certainly in this case the equivalence of these two data is not a new result (at least if the potential is smooth); an interplay between spectral and time-domain data is widely used in inverse problems (see, e.g., [22] where the equivalence of several types of boundary inverse problems is discussed for smooth coefficients; notice, however, that we consider the case of a not necessarily smooth but are just L_1 potentials). Moreover, the boundary inverse problem on a finite interval can easily be solved using either the Titchmarsh-Weyl function or the dynamical response operator and there is no need to reduce one problem to another. However, for the graph (tree) the equivalence of these two inverse problems turns out to be very useful and significantly simplifies the proof of the uniqueness and provides an algorithm to solve the boundary inverse problem.

2.1. **Wave equation on a finite interval: regularity and controllability.** The application of the BC method to solve boundary inverse problems relies on the regularity and controllability results for a closely related boundary value problem for the wave equation, which we briefly discuss in this subsection.

2.1.1. *Control from the left end: small values of t.* Consider the one dimensional wave equation on the interval $x \in [a_1, a_2]$

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} + q(x)w(x,t) = 0, \quad x \in (a_1, a_2),\ t \in (0,T), \tag{2.1}$$

with the Dirichlet boundary control at a_1 and the homogeneous Dirichlet boundary condition at a_2

$$w(a_1,t) = f(t), \quad w(a_2,t) = 0,\ t \in (0,T), \tag{2.2}$$

and zero initial data

$$w(x,0) = \frac{\partial w}{\partial t}(x,0) = 0,\ x \in (a_1,a_2). \tag{2.3}$$

We denote the solution of (2.1)–(2.3) by $w^f(a_1; x, t)$. We consider w^f as a function of (x, t) depending on a_1 as a parameter. Here and below we assume all functions to be real.

The solution w^f can be obtained using the integral kernel $h(a_1; x, t)$ which is the unique solution to the Goursat problem [40] (for shortness we often omit the parameter a_1 in arguments of w^f and h when it leads to no confusion):

$$h_{tt} - h_{xx} + q(x)h = 0, \quad 0 < x - a_1 < t < T\,, \tag{2.4}$$

$$h(a_1, t) = 0\,, \quad h(x, x - a_1) = -\frac{1}{2}\int_{a_1}^{x} q(s)\, ds\,. \tag{2.5}$$

Using the standard successive approximation method, one can prove the following:

Proposition 1. *For $q \in L_1(a_1, a_2)$ the problem (2.4), (2.5) has a unique generalized solution h continuous in $\overline{\Omega^T}$, $\Omega^T := \{(x, t) : a_1 < x < t + a_1 < T + a_1 \leq a_2\}$.*

Similar proposition is proved e.g. in [40, Sec. II.4] for smooth q, but the method works for $q \in L_1(a_1, a_2)$ as well (see [6, 7, 8]).

The next proposition can be proven by direct calculations.

Proposition 2. *If $f \in L_2(0, T) := \mathcal{F}^T$, the problem (2.1)–(2.3) has the unique generalized solution $w^f \in C([0, T]; L_2(a_1, a_2))$. For $t < l_1 = a_2 - a_1$,*

$$w^f(a_1; x, t) = \begin{cases} f(t - x + a_1) + \int_{x-a_1}^{t} h(a_1; x, \tau) f(t - \tau)\, d\tau, & x < t + a_1; \\ 0, & x \geq t + a_1. \end{cases} \tag{2.6}$$

Let

$$\mathcal{H} = L_2(a_1, a_2) \ \text{ and } \ \mathcal{H}^T := \{u \in \mathcal{H} : \ \operatorname{supp} u \subset [a_1, a_1 + T]\,\}.$$

Proposition 2 implies in particular that the **control operator** $\mathcal{W}_{a_1}^T$ associated with the left end point,

$$\mathcal{W}_{a_1}^T : \mathcal{F}^T \mapsto \mathcal{H}^T, \ \mathcal{W}_{a_1}^T f = w^f(a_1; \cdot, T),$$

is bounded. The next proposition claims that the operator $\mathcal{W}_{a_1}^T$ is boundedly invertible. This property is called **exact controllability** of system (2.1), (2.2).

Proposition 3. *Let potential $q \in L_1(a_1, a_2)$ be known and $T \in (0, l_1]$, then for any function $u \in \mathcal{H}^T$, there exists a unique control $f \in \mathcal{F}^T$ such that*

$$w^f(a_1; x, T) = u(x) \ \text{ in } \mathcal{H}^T. \tag{2.7}$$

Proof. According to (2.6), condition (2.7) is equivalent to the following integral Volterra equation of the second kind

$$u(x) = f(T + a_1 - x) + \int_{x-a_1}^{T} h(a_1; x, \tau) f(T - \tau)\, d\tau \quad x \in (a_1, a_1 + T)\,. \tag{2.8}$$

The kernel $h(a_1; x, t)$ is a continuous function and therefore equation (2.8) is solvable, that proves the proposition. ☐

2.1.2. *Dynamical boundary inverse problem for a finite interval.* With the solution of the initial boundary value problem (2.1) – (2.3) we associate the **response operator** (the dynamical Dirichlet-to-Neumann map) $R^T_{a_1a_1}:\ \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$\tag{2.9} \begin{aligned} &(R^T_{a_1a_1}f)(t) = \frac{\partial}{\partial x} w^f(a_1;x,t)\Big|_{x=a_1}, \\ &Dom\ R^T = \{f \in C^2[0,T]:\ f(0)=f'(0)=0\}. \end{aligned}$$

Formula (2.6) implies that

$$\tag{2.10} (R^T_{a_1a_1}f)(t) = -f'(t) + \int_0^t r(a_1;t-\tau)f(\tau)\,d\tau,$$

where $r(a_1;t) = \frac{\partial}{\partial x}h(a_1;x,t)\big|_{x=a_1}$. This shows that operator $R^T_{a_1a_1}$ is completely determined by the kernel $h(a_1;x,t),\ t\in[0,T]$. It is proved in [7] that $r\in L_1$ if $q\in L_1$. Note that the kernel of the response operator $R^T_{a_1a_1}$ may be written using distributions as

$$\tag{2.11} -\delta'(t-\tau) + r(a_1;t-\tau)\Theta(t-\tau).$$

Now we are going to show how to recover the potential function $q(x)$ from the known $R^T_{a_1a_1},\ T\geq 2\ell$. Introduce the **connecting operator** $\mathcal{C}^T:\mathcal{F}^T\to\mathcal{F}^T$ for the wave equation (2.1)–(2.3):

$$\left(\mathcal{C}^T f, g\right)_{\mathcal{F}^T} = \left(w^f(\cdot,T), w^g(\cdot,T)\right)_{\mathcal{H}}.$$

The operator $\mathcal{C}^T$ is bounded and boundedly invertible, since $\mathcal{C}^T = \left(\mathcal{W}^T\right)^*\mathcal{W}^T$. The operator $\mathcal{C}^T$ plays a central role in the BC method. The important fact is that it can explicitly be expressed through the response operator $R^{2T}_{a_1a_1}$ (see, e.g., [4, 3]):

$$\tag{2.12} (\mathcal{C}^T f)(t) = f(t) + \int_0^T [p(2T-t-s) - p(|t-s|)]f(s)\,ds,$$

where

$$p(t) = \frac{1}{2}\int_0^t r(s)\,ds.$$

Let $y(x)$ be a solution to the boundary value problem

$$\tag{2.13} y''(x) - q(x)y(x) = 0,\ y(a_1)=0,\ y'(a_1)=1, x\in(a_1,\,a_2)$$

and let us find a control $z^T\in\mathcal{F}^T$ such that

$$\tag{2.14} w^{z^T}(x,T) = \begin{cases} y(x), & x-a_1\leq T, \\ 0, & x-a_1>T. \end{cases}$$

Note that, since $q(x)$ is unknown, both $y(x)$ and $z^T(t)$ are unknown functions at this point. For any $g \in C_0^\infty[0,T]$, we have

$$\begin{aligned}
(\mathcal{C}^T z^T, g)_{\mathcal{F}^T} &= \left(w^{z^T}(\cdot,T), w^g(\cdot,T)\right)_{\mathcal{H}} \\
&= \int_{a_1}^{a_1+T} y(x) w^g(x,T)\, dx \\
&= \int_0^T (T-t)\left(\int_{a_1}^{a_1+T} y(x) w^g_{tt}(x,t)\, dx\right) dt \\
&= \int_0^T (T-t)\left(\int_{a_1}^{a_1+T} y(x)\left[w^g_{xx}(x,t) - q(x) w^g(x,t)\right]\, dx\right) dt \\
&= \int_0^T (T-t)[(y(x)w^g_x(x,t) - y'(x)w^g(x,t)]|_{x=a_1}^{a_1+T}\, dt \\
&= \int_0^T (T-t) g(t)\, dt
\end{aligned}$$

(we used that for $g \in C_0^\infty[0,T]$, the function w^g and its derivatives are equal to zero at $x = T$). Hence the function z^T satisfies the equation

$$\left(\mathcal{C}^T z^T\right)(t) = T - t, \quad t \in [0,T].$$

Since $\mathcal{C}^T$ is boundedly invertible this equation has a unique solution, $z^T \in \mathcal{F}^T$, for any $T \leq \ell$.

Moreover, it can be proved that $z^T \in H^1(0,T)$. Indeed, by (2.12), the operator $\mathcal{C}^T$ can be represented in the form

$$\mathcal{C}^T = I^T + \mathcal{C}_0^T\,,$$

where I^T is the identity operator in $\mathcal{F}^T$ and $\mathcal{C}_0^T$ is bounded from $\mathcal{F}^T$ to $H^1(0,T)$. Hence,

$$z^T = -\mathcal{C}_0^T z^T + T - t \in H^1(0,T)\,.$$

Formula (2.6) yields

$$w^f(t-0, a_1+t) = f(+0)$$

provided the limit $f(+0)$ exists. Applying the propagation of singularities property to $f = z^T$, we obtain

$$w^{z^T}(T-0, a_1+T) = z^T(+0).$$

Denote $z^T(+0)$ by $\mu(T)$. From (2.14), $w^{z^T}(T-0, a_1+T) = y(a_1+T)$, and (2.13) gives

$$q(T) = \frac{y''(T)}{y(T)} = \frac{\mu''(T)}{\mu(T)}.$$

By varying T in $(0,\ell)$, we obtain $q(\cdot)$ in that interval. Since the function $y(T)$ may have only a finite number of zeroes in $(0,\ell)$, this completes the solution of the identification problem.

Let us notice an important property of this approach – its local character: operator R^T determines potential $q(x)$ for $x \in [0, T/2]$. Thus to recover the potential on the whole interval $[a_1, a_2]$ we have to know $R^{2l_1}_{a_1 a_1}$. Note that for $T > l_1$ the solution is not given by formula (2.6) anymore – it contains a certain additional wave reflected from the right end point (to be discussed in subsection 2.1.3). However, this wave could be ignored during the calculation of $R^T_{a_1a_1}$ for $T < 2l_1$, since it does not have enough time to reach the left end point.

2.1.3. *Solution to the wave equation on the interval.* In the present section we construct the solution to the equation (2.1) with boundary and initial conditions (2.2), (2.3) for $t \in (0, 3l_1)$, $l_1 = a_2 - a_1$. First we observe that the solution $w^f(a_1; x, t)$ admits the Duhamel representation

$$w^f(a_1; x, t) = w^\delta(a_1; x, t) * f(t), \tag{2.15}$$

where $*$ stands for the convolution with respect to t and $w^\delta(a_1; x, t)$ is the solution to initial boundary value problem (2.1)–(2.3) with $f(t) = \delta(t)$. The representation formula (2.15) implies that for the construction of $w^f(a_1, x, t)$ it is sufficient to construct $w^\delta(a_1; x, t)$.

In the sequel we use also $w^\delta(a_2; x, t)$, the solution to (2.1), (2.3) with the boundary conditions

$$w^\delta(a_2; a_1, t) = 0, \quad w^\delta(a_2; a_2, t) = \delta(t). \tag{2.16}$$

For $t \in (0, l_1)$, the representations hold:

$$w^\delta(a_1; x, t) = \delta(t - x + a_1) + h(a_1; x, t), \tag{2.17}$$

$$w^\delta(a_2; x, t) = \delta(t + x - a_2) + h(a_2, ; x, t). \tag{2.18}$$

The function $h(a_1; x, t)$ is a solution to the Goursat problem (2.4), (2.5), and $h(a_2; x, t)$ has the similar properties.

For $t \in (l_1, 2l_1)$ the function given by formula (2.17) solves wave equation, satisfies zero initial data and the boundary condition at $x = a_1$ but not at $x = a_2$. The solution satisfying Dirichlet condition at $x = a_2$ can be obtained by adding to (2.17) the term

$$-w^{\delta(t-l)}(a_2; x, t) - w^{h(a_1;a_2,t)}(a_2; x, t). \tag{2.19}$$

Then the solution $u^\delta(a_1, x, t)$ for $l_1 < t < 2l_1$ is given by

$$\begin{aligned} &w^\delta(a_1; x, t) = \delta(t - x + a_1) + h(a_1; x, t) - \delta(t + x - a_2 - l_1) + H(x, t), \\ &H(x, t) = -h(a_2; x, t - l_1) - w^{\delta(t-l_1)}(a_2; x, t) * h(a_1; a_2, t). \end{aligned} \tag{2.20}$$

Consider now the time interval $2l_1 < t < 3l_1$. One needs to take into account the reflection from the left end point and to add to solution (2.20) the "correcting" term

$$w^{\delta(t-2l)}(a_1; x, t) - u^{H(a_1,t)}(a_1; x, t), \tag{2.21}$$

which leads to the following representation:

$$\begin{aligned} w^\delta(a_1; x, t) &= \delta(t - x + a_1) + h(a_1; x, t) - \delta(t + x - a_2 - l_1) + H(x, t) \\ &+\delta(t - x - 2l_1 + a_1) + h(a_1; x, t - 2l_1) - w^\delta(a_1; x, t) * H(a_1, t). \end{aligned} \tag{2.22}$$

Using this formula we can write down the representation for the response function $r(t) = u^\delta_x(a_1, a_1, t)$ and response operator $R^T_{a_1a_1}$ for $2l < t < 3l$:

$$r(t) = -\delta'(t) - 2\delta'(t - 2l) + G(a_1, a_2, t), \tag{2.23}$$

$$(R^T_{a_1a_1} f)(t) = -f'(t) - 2f'(t - 2l) + \int_0^t G(a_1, a_2, t) f(t - s)\, ds, \tag{2.24}$$

with some $G(a_1, a_2, \cdot) \in L_1(0, 3l)$.

This method can be continued in order to get representations for the solution of the initial boundary value problem for any t. It is clear that the solution will contain several nonintegral terms corresponding to the reflections from the boundaries. We conclude that the knowledge of the response operator $R^T_{a_1a_1}$ for some $T > 2l_1$ allows one to reconstruct not only the potential q but also the length l_1 of the interval.

2.2. **Connection between spectral and dynamical data.** Let $f \in C_0^\infty(0,\infty)$ and

$$F(k) := \int_0^\infty f(t)\, e^{ikt}\, dt$$

be its inverse Fourier transform. $F(k)$ is well defined for $k \in \mathbb{C}$ and, if $\Im k > 0$,

$$|F(k)| \leq C_\alpha (1+|k|)^{-\alpha} \tag{2.25}$$

for any $\alpha > 0$.

Let ψ be a solution of the equation

$$-\frac{d^2}{dx^2}\psi(x,k) + q(x)\psi(x,k) = k^2\psi(k,x)\,, \quad a_1 < x < a_2\,, \tag{2.26}$$

with boundary conditions

$$\psi(a_1,k) = F(k)\,, \quad \psi(a_2,k) = 0\,.$$

Estimate (2.25) implies that $|\psi(x,k)|$ decreases rapidly when $|k| \to \infty$, $\Im k \geq \epsilon > 0$. The values of the function and its first derivative at $x = a_1$, $\psi(a_1,k)$ and $\psi_x(a_1,k)$ are related through the Titchmarsh-Weyl m-function

$$\psi_x(a_1,k) = M(k^2)F(k)\,. \tag{2.27}$$

The Titchmarsh-Weyl function is a Nevanlinna function (analytic in the upper half plane, having positive imaginary part there) which is usually defined as $M(k^2) = \phi_x(a_1,k)$ where $\phi(x,k)$ is a solution of the equation (2.26) with boundary conditions

$$\phi(a_1,k) = 1\,, \quad \phi(a_2,k) = 0\,.$$

The Fourier transform

$$w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,\kappa+i\nu)\, e^{-i(\kappa+i\nu)t}\, d\kappa\,, \ \nu > 0\,,$$

defines the function which solves the initial boundary value problem (2.1)–(2.3). Using (2.27) we get

$$\left(R^T_{a_1a_1} f\right)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M\left((\kappa+i\nu)^2\right)\, F(\kappa+i\nu)\, e^{-i(\kappa+i\nu)t}\, d\kappa\,, \ t \in [0,T]\,. \tag{2.28}$$

Thus the Titchmarsh-Weyl function $M(\lambda)$ and the response operator $R^T_{a_1a_1}$ taken for all $T > 0$ are in one-to-one correspondence, i.e. the response operator $R^T_{a_1a_1}$ determines the Titchmarsh-Weyl's m-function and the m-function determines the response operator for all T. In fact it is enough to know the response operator for $T = 2l$ only in order to reconstruct the potential and therefore in order to calculate the m-function. On the other hand, the knowledge of the m-function is sufficient to reconstruct the potential and therefore to determine the response operator for all $T > 0$.

3. **Dynamical boundary inverse problem for star graph.** In this section we consider the solution to the boundary inverse problem for the simplest graph $\Gamma_{\rm star}$ formed by m intervals $[a_{2j-1}, a_{2j}],\ j = 1, 2, ..., m$ having lengths $l_j = a_{2j} - a_{2j-1}$ joined together at one vertex – the star graph. It follows that the vertex has valence m. Without loss of generality we can suppose that the common vertex joins together the right end points, $v_0 = \{a_2, a_4, ..., a_{2m}\}$. Then the left end points form the boundary of the graph $\partial\Gamma_{\rm star} = \{a_1, a_3, ..., a_{2m-1}\}$. This graph can be considered as elementary building block for more sophisticated graphs.

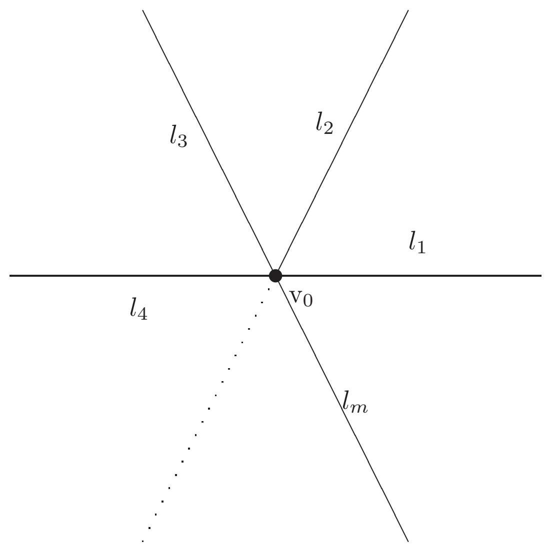

Fig. 2 Star-like graph.

In the current section we are going to study the boundary control problem for the wave evolution associated with the operator $H = L + q$

$$-\frac{\partial^2}{\partial t^2} w(x,t) = Hw(x,t).$$

The control will be first introduced using one of the boundary points only. This control is not enough for exact controllability of the system: it is impossible to transfer the system to an arbitrary function from $L_2(\Gamma_{\rm star})$. Therefore in order to recover the potential it is necessary to use the response operator associated with all but one boundary points.

3.1. **Wave equation on star graph: partial controllability/reconstruction from one boundary point.** In this subsection we are going to study the possibility to reconstruct the potential on one of the edges using the response operator corresponding to the graph's boundary point situated on this edge. It will be shown that the response operator allows one to reconstruct not only the potential, but the length of the edge and the number of edges in the star graph. Without loss of generality we may assume that the graph is controlled by introducing the boundary control at the point a_1.

The problem we are going to solve is

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} + q(x)w(x,t) = 0, \quad x \in \Gamma_{\rm star} \setminus v_0, t \in (0,T), \tag{3.1}$$

with standard boundary conditions at the internal vertex

$$\left\{ \begin{array}{l} \sum_{j=1}^m \partial_n w(a_{2j},t) \equiv - \sum_{j=1}^m \frac{\partial}{\partial x} w(a_{2j},t) = 0; \\ w(x,t) \text{is continuous on } \Gamma_{\rm star}, \end{array} \right. \tag{3.2}$$

boundary control at a_1

$$w(a_1,t) = f(t), \quad w(a_3,t) = w(a_5,t) = \cdots = w(a_{2m-1},t) = 0 \tag{3.3}$$

and zero initial data

$$w(x,0) = \frac{\partial}{\partial t} w(x,0) = 0. \tag{3.4}$$

We use the Duhamel representation:

$$w^f(x,t) = w^\delta(a_1;x,t) * f(t), \tag{3.5}$$

where $w^\delta(a_1;x,t)$ is a solution to (3.2), (3.4) with delta function boundary control at the vertex a_1:

$$\begin{array}{l} w^\delta(a_1,a_1,t) = \delta(t), \\ w^\delta(a_1;a_3,t) = w^\delta(a_1;a_5,t) = \cdots = w^\delta(a_1,;a_{2m-1},t) = 0. \end{array} \tag{3.6}$$

It is clear that for $t \le l_1 \equiv a_2 - a_1$ the solution to the problem (3.1), (3.2), (3.4), (3.6), on the edge $[a_1,a_2]$ is given by formula (2.17) , since for so small values of t the solution does not reach the internal vertex v_0 and it is not influenced by the boundary condition (3.2) imposed at that point. The solution is equal to zero on all other edges. Let us calculate now w^δ for $t \in (l_1, l_1 + \min\{l_j\}_{j=1}^m)$. The upper bound is necessary in order to ensure that the wave does not have time to reach the other boundary points $a_3, a_5, \ldots, a_{2m-1}$. If we were deal with one string $[a_1,a_2]$, the solution on the time interval $l_1 < t < 2l_1$ would be given by the sum (2.17) and (2.19):

$$w^\delta(a_1;x,t) = \delta(t-x+a_1) + h(a_1;x,t) - u^{\delta(t-l_1)}(a_2;x,t) - u^{h(a_1;a_2,t)}(a_2;x,t). \tag{3.7}$$

Let us set up the boundary value problem on each edge $[a_{2j-1},a_{2j}]$, $j = 1,\ldots,m$, for $t \in (l_1, l_1 + \min\{l_j\}_{j=1}^m)$:

$$\begin{array}{l} {u_j}_{tt} - {u_j}_{xx} + qu_j = 0, \\ u_j(x,0) = {u_j}_t(x,0) = 0, \\ u_j(a_{2j},t) = \alpha\delta(t-l_1) + g(t). \end{array} \tag{3.8}$$

The constant α and function $g \in W_1^1(l_1, l_1 + \min\{l_j\}_{j=1}^m)$ should be determined. On the edges $x \in [a_{2j-1},a_{2j}]$, $j = 2,3,...,m$, the solution $w^\delta(a_1;x,t)$ is equal to the solution of (3.8), $w_j(a_{2j};x,t)$:

$$w_j^\delta(a_{2j};x,t) = \alpha\delta(t-l_1+x-a_{2j}) + \alpha h(a_{2j};x,t-l_1) + w^{\delta(t-l_1)}(a_{2j};x,t) * g(t), \tag{3.9}$$

and on the edge $x \in [a_1,a_2]$ it is given by:

$$\begin{array}{c} w^\delta(a_1,x,t) = \delta(t-x+a_1) - \delta(t+x-a_2-l_1) + \alpha\delta(t-l_1+x-a_2) \\ + w^{\delta(t-l_1)}(a_2,x,t) * g(t) + \alpha h(a_2,x,t-l_1) + u(x,t) \\ u(x,t) := h(a_1,x,t) - h(a_2,x,t-l_1) - u^{\delta(t-l_1)}(a_2,x,t) * h(a_1,a_2,t) \end{array}$$

Substituting these solutions into the first boundary condition (3.2) (the second continuity condition is trivially satisfied) and equating the singular part, we get the equation on α:

$$2\delta'(t-l_1) - m\alpha\delta'(t-l_1) = 0, \tag{3.10}$$

which gives that $\alpha = \frac{2}{m}$. Equating the regular parts, we obtain the integral equation on g:

$$g'(t-l_1) + \int_0^t G(s)g(t-s)\,ds = F(t), \tag{3.11}$$

where

$$G(t) = \frac{1}{m}\sum_{j=1}^{m} h_x(a_{2j}, a_{2j}, t-l_1),$$

$$F(t) = -\frac{1}{m}\left[u_x(a_2,t) + \alpha\sum_{j=1}^{m}\frac{2}{m}h_x(a_{2j}, a_{2j}, t-l_1)\right].$$

We can suppose that $g(0) = 0$ and reduce equation (3.11) to the second kind Volterra type equation with continuous kernel with respect to g'.

Thus the solution $w^\delta(a_1,x,t)$ on the interval $x \in [a_1,a_2]$, $l_1 < t < l_1 + \min\{l_j\}_{j=1}^m$ is given by

$$u^\delta(a_1,x,t) = \delta(t-x+a_1) - \frac{m-2}{m}\delta(t-l_1+x-a_2) + H(x,t) \text{ if } x \in [a_1,a_2] \tag{3.12}$$

where H is a function whose derivatives $H_x(x,\cdot), H_x(\cdot,t), H_t(x,\cdot), H_t(\cdot,t)$ belongs to $L_1(a_1,a_2)$. The coefficient $\frac{m-2}{m}$ appearing in front of $\delta(t-l_1+x-a_2)$ is the reflection coefficient from the vertex v_0.

Hence the response function $r(a_1,t) = u^\delta(a_1,x,t)$ for $t \in (2l_1, 2l_1 + 2\min\{l_j\}_{j=1}^m)$ associated with vertex a_1 and response operator $R^T_{a_1a_1}$, $2l_1 < T < 2l_1 + 2\min\{l_j\}_{j=1}^m$, corresponding to the graph Γ_{star} have the form (compare with (2.20), (2.22), (2.23), (2.24))

$$r(a_1,t) = -\delta'(t) - 2\tfrac{m-2}{m}\delta'(t-2l_1) + \widetilde{H}_x(a_1,t), \tag{3.13}$$

$$(R^T_{a_1a_1}f)(t) = -f'(t) - 2\tfrac{m-2}{m}f'(t-2l_1) + \textstyle\int_0^t \widetilde{H}_x(a_1,s)f(t-s)\,ds. \tag{3.14}$$

$$t \in [2l_1, 2l_1 + \min\{l_j\}_{j=1}^m].$$

An extra coefficient 2 in front of the retarded wave appears due to the reflection from the point a_1 (compare with formulas (2.23), (2.24)).

It is important to understand that the singular part of the response operator is just the same as for the Laplace operator on Γ_{star}. The integral part appears due to the "soft" reflection from the potential q.

We are ready to prove the following Lemma

Lemma 3.1. *Let Γ_{star} be a star graph with the central vertex of valence $m > 2$ and let $H = L + q$ be a Schrödinger operator in $L_2(\Gamma_{\text{star}})$. Then the knowledge of the corresponding response operator $R^T_{a_1a_1}$ for $0 \le t \le T$, $T > 2l_1$ allows one to reconstruct the length l_1 of the controlling edge $[a_1,a_2]$, the valence m of the central vertex and the potential q on the controlling interval $[a_1,a_2]$.*

Proof. Formula (3.12) shows that the solution to the wave equation on the controlled edge contains the retarded wave $-2\frac{m-2}{m}f(t+x-2a_2+a_1)$ corresponding to the reflection from the vertex. This wave causes δ'-singularity in the response operator given by (3.14). This means that by measuring the delay time of this wave one may calculate the length of the controlled edge. The amplitude $2\frac{m-2}{m}$ of this wave gives the valence m of the central vertex. The potential q may be reconstructed from the response operator for $0 \le t \le 2l_1$ using standard methods, since for $t < 2l_1$ the response operator is determined entirely by the Schrödinger equation on the interval $[a_1, a_2]$. The rest of the graph has no influence on the response operator for sufficiently small values of t. $\square$

Remark. The condition $m > 2$ is not restrictive, since the standard boundary conditions for vertex of valence 2 mean that the function and its first derivative are continuous along the vertex (see the comment in the Introduction).

In fact the response operator associated with one of the boundary points allows one to reconstruct the whole star graph.

Lemma 3.2. *Let $\Gamma_{\rm star}$ be a star graph with the central vertex of valence $m > 2$ and let $H = L + q$ be a Schrödinger operator in $L_2(\Gamma_{\rm star})$. Then the knowledge of the response operator $R^T_{a_1a_1}$ for $0 \le t \le T,\ T \ge 2l_1 + 2\max\{l_j\}_{j=2}^m$ allows one to reconstruct the graph, i.e. the total number of edges and their lengths.*

Proof. Let us continue to apply the same method as in the proof of Lemma 3.1 and take into account more and more reflections from the other boundary points and multiple reflections from the central vertex. Then for sufficiently large values of T the response operator has the form

$$(R^T_{a_1a_1}f)(t) = -f'(t) + 2\sum_{p\in\mathcal{P}_{a_1}(\Gamma_{\rm star})} S_p f'(t-l(p)) + \int_0^t k(t,\tau)f(\tau)d\tau\,, \tag{3.15}$$

where:

- $k(t,\cdot) \in L_1(0,T)$;
- $\mathcal{P}_{a_1}(\Gamma_{\rm star})$ is the set of all paths on $\Gamma_{\rm star}$ starting and ending at a_1 and turning only at the vertices of the graph;
- S_p is the product of scattering coefficients along the path p (see the precise definition below);
- $l(p)$ is the length of the path p.

Each path from $\mathcal{P}_{a_1}(\Gamma_{\rm star})$ is uniquely determined by the sequence of the vertexes that it comes across. For example the path $\tilde{p} = (a_1, v_0, a_3, v_0, a_3, v_0, a_1)$ starts at the point a_1 and goes along the interval $[a_1, a_2]$. It continues then along the interval $[a_4, a_3]$ reflects from the end point a_3 and returns back to v_0 along $[a_3, a_4]$. Then it reflects from v_0 and goes again along $[a_4, a_3]$ and $[a_3, a_4]$. Then it returns back to a_1 along $[a_2, a_1]$. Here $[a_4, a_3]$ indicates the interval $[a_3, a_4]$ crossed in the negative direction. It is clear that the set of paths is infinite but countable. The length $l(p)$ is equal to the sum of the lengths of the intervals the path comes across, for example the length of $\tilde{p}$ is equal to $l(\tilde{p}) = l_1 + l_2 + l_2 + l_2 + l_2 + l_1 = 2l_1 + 4l_2$. With the vertex v_0 we associate reflection $r(v_0) = -\frac{m-2}{m}$ and transition $t(v_0) = \frac{2}{m}$ coefficients. With the boundary vertexes we associate just reflection coefficients $r(a_{2j-1}) = -1$. Then the coefficient S_p is just the product of all scattering coefficients corresponding to the path p. For example $S_{\tilde{p}} = t(v_0)r(a_3)r(v_0)r(a_3)t(v_0) = \frac{2}{m}(-1)(-\frac{m-2}{m})(-1)\frac{2}{m} = -\frac{4(m-2)}{m^3}$.

In order to ensure that the sum in (3.15) converges let us consider $f \in C_0^\infty(\mathbb{R}_+)$. Consider the sum of singular terms only

$$\sum_{p \in \mathcal{P}_{a_1}(\Gamma_{\mathrm{star}})} S_p f'(t - l(p)). \tag{3.16}$$

As we have already seen during the proof of Lemma 3.1 the first retarded wave gives us the length l_1 and the number of edges. Let us substract from the sum (3.16) the contribution from all orbits containing only the edge $[a_1, a_2]$. Then the first retarded wave which is left corresponds to the orbit with the length $2l_1 + 2\min\{l_j\}_{j=2}^m$ and its amplitude gives the number of edges among $l_2, l_3, ..., l_m$ having the minimal length. Suppose that such edges are $l_2, ..., l_k$. Then let us substract from the sum (3.16) the contribution from all paths containing only the edges $[a_1, a_2], ..., [a_{2k-1}, a_{2k}]$. Examining the rest of the sum we can reconstruct the second shortest length, say l_{k+1} and the number of edges having exactly this length. Continuing this procedure the whole star graph will be reconstructed. □

These two lemmas will be very important in applications to arbitrary graphs. Let us summarize our results: The knowledge of one diagonal element of the response operator for a star graph allows one to reconstruct the graph and potential on the edge directly connected to the control point. In general it is impossible to reconstruct the potentials on the other edges. The reason will be clear after the following subsection where the connection between the response operator and Titchmarsh-Weyl function is investigated.

3.2. **Spectral, dynamical and scattering data for star graph.** The Titchmarsh-Weyl m-matrix $\mathbf{M}(\lambda), \lambda = k^2$ associated with the graph Γ has already been introduced in the Introduction.

In order to define the matrix response operator $\mathbf{R}^T$ let us consider the solution $w^{\vec{f}}(x,t),\ x \in \Gamma, t \in [0,T]$ to the wave equation

$$-\frac{\partial^2}{\partial t^2} w^{\vec{f}} = (-\frac{\partial^2}{\partial x^2} + q) w^{\vec{f}} \tag{3.17}$$

on the graph, standard boundary conditions (3.2) at all internal vertexes and the Dirichlet control conditions

$$w^{\vec{f}}|_{\partial\Gamma} = \vec{f}(t) \tag{3.18}$$

on the graph's boundary. Then the response operator $\mathbf{R}^T = \{R^T_{\gamma_j\gamma_k}\}_{j,k=1}^m$ is defined by

$$\left(\mathbf{R}^T \vec{f}\right)(t) = \partial_n w^{\vec{f}}(x,t)|_{\partial\Gamma}, \tag{3.19}$$

where ∂_n denotes the normal derivative (taken along the normal pointing inside the graph).

The connection between the Titchmarsh-Weyl matrix and the response operator can be established in the same way as in the case of one interval (see section 2.2). Let $\vec{f} \in (C_0^\infty(0,\infty))^m$ be a vector valued function having m components, and let $\vec{F}(k)$ be its inverse Fourier transform

$$\vec{F}(k) = \int_0^\infty \vec{f}(t) e^{ikt} dt,$$

well-defined at least for any complex k. For $\Im k \geq 0$ the function possesses the estimate

$$|\vec{F}(k)| \leq C_\alpha (1+|k|)^{-\alpha} \tag{3.20}$$

for any $\alpha > 0$. Then the following formula similar to (2.28) is valid

$$\left(\mathbf{R}^T \vec{f}\right)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{M}((k+i\nu)^2) \vec{F}(k+i\nu) e^{-i(k+i\nu)t} dk, \quad t \in [0,T]. \tag{3.21}$$

This formula implies that the knowledge of the response operator allows one to reconstruct the Titchmarsh-Weyl matrix and vice versa. Note that this relation is valid for arbitrary quantum graphs, not only for star graphs or trees.

Let us establish now the relation between the Titchmarsh-Weyl function $\mathbf{M}(\lambda)$ and the scattering matrix. In order to introduce the scattering matrix, let us attach to each boundary vertex γ_j interval $[\gamma_j, \pm\infty)$ with $q_j = 0$ there. The sign $\pm$ depends on whether γ_j is left $(-)$ or right $(+)$ end point of the corresponding edge. Then every solution ψ to the equation (1.5) on the whole graph (including new attached infinite branches) satisfying standard boundary conditions (1.2) at all vertexes (including $\gamma_j, j = 1, 2, ..., m$) when restricted to new branches is a combination of plane waves

$$\psi|_{[\gamma_j, \pm\infty)} = a_j e^{ik|x-\gamma_j|} + b_j e^{-ik|x-\gamma_j|}, \tag{3.22}$$

Consider the restriction of the function ψ and its normal derivative to the boundary of the original graph Γ

$$\vec{\psi}|_{\partial\Gamma} = \vec{a} + \vec{b}, \quad \vec{\partial_n \psi}|_{\partial\Gamma} = -ik\vec{a} + ik\vec{b}.$$

Taking into account that these boundary values are connected via Titchmarsh-Weyl function (1.9) we get the following formula for the scattering matrix

$$\vec{b} = \mathbf{S}(k)\vec{a} \Rightarrow \mathbf{S}(k) = \frac{ik + \mathbf{M}(\lambda)}{ik - \mathbf{M}(\lambda)}, \quad \lambda = k^2\,. \tag{3.23}$$

Similarly the knowledge of the scattering matrix allows one to reconstruct the Titchmarsh-Weyl matrix

$$\mathbf{M}(\lambda) = ik \frac{\mathbf{S}(k) - I}{\mathbf{S}(k) + I}. \tag{3.24}$$

Using the scattering matrix one may reformulate Lemmas 3.1 and 3.2 as follows:

Lemma 3.3. *Let $\Gamma_{\rm star}$ be a star graph with the central vertex of valence $m > 2$ and let $H = L + q$ be a Schrödinger operator in $L_2(\Gamma_{\rm star})$. Then the knowledge of one back scattering coefficient allows one to reconstruct the graph, i.e. the total number of edges and their lengths and the potential q on the corresponding interval.*

3.3. **Exact controllability and complete reconstruction of the star graph.** Lemma 3.1 implies that the knowledge of the response operator $\mathbf{R}^T$ for sufficiently large T allows one to reconstruct the star graph and potentials on the edges. In fact it is clear that one needs to know the diagonal part of the response operator only, since from any of $R_{a_{2j-1}a_{2j-1}}$ one may reconstruct the graph (Lemma 3.2) and then the knowledge of each $R_{a_{2j-1}a_{2j-1}}$ allows one to calculate the potential on the corresponding interval $[a_{2j-1}, a_{2j}]$. It is clear that it is possible to reduce further the amount of information necessary to reconstruct the potential. For example in the case of just one interval, it is enough to know the Titchmarsh-Weyl function associated with one of the end points. We are going to prove now that the knowledge of $R_{a_{2j-1}a_{2j-1}}, j = 1,2,...,m-1$ – all except one diagonal elements of the response operator – is enough.

Lemma 3.4. *Let $\Gamma_{\rm star}$ be a star graph and let $H = L+q$ be a Schrödinger operator in $L_2(\Gamma_{\rm star})$. Then the graph and potential q are uniquely determined by the diagonal minus one element of the response operator $\mathbf{R}^T$ for sufficiently large values of T.*

Proof. Let us suppose without loss of generality that $R^T_{a_{2j-1}a_{2j-1}}, j = 1,2,...,m-1$, are known. It follows that the graph $\Gamma_{\rm star}$ can be reconstructed (Lemma 3.1) and as well as potential on the intervals $[a_{2j-1}, a_{2j}], j = 1,2,...,m-1$. It remains to show that the potential on the interval $[a_{2m-1}, a_{2m}]$ may be reconstructed. Formula (3.21) implies that the knowledge of $R^T_{a_{2j-1}a_{2j-1}}, j = 1,2,...,m-1$ is equivalent to the knowledge of the Titchmarsh-Weyl coefficients $M_{a_{2j-1}a_{2j-1}}, j = 1,2,...,m-1$. Consider the solution $\psi(x,k)$ to the equation (3.1) satisfying standard boundary condition (1.2) at v_0 and the following conditions at the boundary

$$\psi(a_1,k) = 1, \;\; \partial_n \psi(a_1,k) = M_{a_1a_1}(k^2), \;\; \Im k^2 > 0$$

at a_1 and the homogeneous Dirichlet conditions at all other boundary points. It is clear that this problem has a unique solution. This solution is uniquely determined on the interval $[a_1, a_2]$ (the potential on this interval is already reconstructed). Therefore the value $\psi(a_2,k)$ of the solution at the vertex v_0 is determined. Note that it is different from zero, since $\Im\lambda \neq 0$. Consider the Titchmarsh-Weyl coefficients $M_j(\lambda)$ associated with the right end points of the intervals $[a_{2j-1}, a_{2j}]$ and Dirichlet conditions at the left points:

$$\begin{cases} -\dfrac{d^2}{dx^2} f(x,k) + q(x) f(x,k) = k^2 f(x,k), \\ f(a_{2j},k) = 1, \;\; f(a_{2j-1},k) = 0, \;\; x \in [a_{1j-1}, a_{2j}]; \end{cases} \Rightarrow M_j(\lambda) = -\frac{d}{dx} f(a_{2j},k).$$

These coefficients are uniquely determined for $j = 2,3,...,m-1$ by the potential on the corresponding intervals (already known). Taking into account that $\psi(x,k)$ satisfies standard boundary conditions (1.2) we conclude that

$$\psi(a_{2m},k) = \psi(a_2,k) \;\;\text{and}\;\; \frac{\partial}{\partial x}\psi(a_{2m},k) = -\frac{\partial}{\partial x}\psi(a_2,k) + \sum_{j=2}^{m-1} M_j(\lambda)\psi(a_2,k)$$

$$\Rightarrow M_m(\lambda) = \frac{\frac{\partial}{\partial x}\psi(a_2,k)}{\psi(a_2,k)} - \sum_{j=2}^{m-1} M_j(\lambda). \tag{3.25}$$

Hence the Titchmarsh-Weyl coefficient for the last interval $[a_{2m-1}, a_{2m}]$ is uniquely determined and therefore the potential on the whole graph $\Gamma_{\rm star}$ may be reconstructed. □

This proof shows that in general it is impossible to reconstruct the potential from the knowledge of less than $m-1$ diagonal elements of the response operator. Assume that only part of these elements is given, say $R^T_{a_{2j-1}a_{2j-1}}, \; j = 1,2,...,m_0-1; \; m_0 < m$. Then it is possible to calculate only the sum $\sum_{j=m_0}^{m} M_j(\lambda)$ of Titchmarsh-Weyl coefficients associated with the intervals, which are not controlled directly.

Lemma 3.5. *Let $\Gamma_{\rm star}$ be a star graph and let $H = L+q$ be a Schrödinger operator in $L_2(\Gamma_{\rm star})$. Then the graph and potential q are uniquely determined by all except one back scattering coefficients.*

4. **Titchmarsh-Weyl matrix function and the response operator for a tree.** In this section we are going to discuss the possibility to reconstruct the tree and potential from the response operator $\mathbf{R}^T$. Using methods developed it is easy to see that the knowledge of the whole response operator is enough. It is interesting to find necessary conditions. Our experience from star graphs tells, that it is necessary to know at least all minus one diagonal elements of the response operator.

4.1. **Reconstruction of the tree from the reduced response operator.** Let Γ be a tree with m boundary points $\gamma_1, \gamma_2, ..., \gamma_m$. Then the corresponding Titchmarsh-Weyl function is an $m \times m$ energy dependent matrix. Any of the boundary edges can be considered as a root of the tree. Without loss of generality suppose that the corresponding boundary point is γ_m. Consider the Titchmarsh-Weyl coefficients associated with all other boundary points

$$\mathbf{M}_{m-1}(\lambda) = \{M_{\gamma_i \gamma_j}(\lambda)\}_{i,j=1}^{m-1}. \tag{4.1}$$

This is a $(m-1)\times(m-1)$ matrix function. It appears that the tree and the potential may reconstructed using only the reduced Titchmarsh-Weyl matrix $\mathbf{M}_{m-1}(\lambda)$.

The reduced scattering matrix and response operator associated with all except one boundary points can be defined similarly

$$\mathbf{S}_{m-1}(k) = \{S_{\gamma_i \gamma_j}(k)\}_{i,j=1}^{m-1}; \mathbf{R}_{m-1}^T = \{R_{\gamma_i \gamma_j}^T\}_{i,j=1}^{m-1}. \tag{4.2}$$

Theorem 4.1. *Let Γ be a tree with m boundary points and $H = L + q$ be the corresponding Schrödinger operator, where $q \in L_1(\Gamma)$ is a real valued function. Then the connectivity of the tree, the lengths of edges and the potential q are uniquely determined by one of the following sets of data:*

- *reduced $(m-1) \times (m-1)$ Titchmarsh-Weyl matrix $\mathbf{M}_{m-1}(\lambda)$ (defined by (4.1));*
- *reduced $(m-1) \times (m-1)$ Scattering matrix $\mathbf{S}_{m-1}(k)$ (defined by (4.2));*
- *reduced $(m-1) \times (m-1)$ operator $\mathbf{R}_{m-1}^T$ for sufficiently large T (greater or equal to the double the distance from the root γ_m to the most remote other boundary vertex).*

Proof. The diagonal elements of $\mathbf{M}_{m-1}(\lambda)$ allows one to calculate the lengths of the corresponding boundary edges and the potential on these edges (via the corresponding response operator). Two boundary edges, say $[\gamma_1, b_1]$ and $[\gamma_2, b_2]$ have one common end point if and only if

$$R_{\gamma_1 \gamma_2}^T = \begin{cases} = 0, & \text{for } T < (b_1 - \gamma_1) + (b_2 - \gamma_2); \\ \neq 0, & \text{for } T > (b_1 - \gamma_1) + (b_2 - \gamma_2); \end{cases}$$

i.e. a wave from the boundary point γ_1 reaches the boundary point γ_2 at exactly $t = (b_1 - \gamma_1) + (b_2 - \gamma_2)$. Considering all boundary edges $[\gamma_j, b_j]$, $j = 1, 2, ..., m-1$ let us select all those edges joined together with $[\gamma_1, b_1]$. Assume that these edges are $[\gamma_2, b_2], ..., [\gamma_{m_0}, b_{m_0}]$. These edges form a star subgraph together with another one edge, which will be denoted by $[c_1, c_2]$. Now the knowledge of the potential on the edges $[\gamma_j, b_j]$, $j = 1, 2, ..., m_0$ allows one to calculate the reduced Titchmarsh-Weyl matrix associated with the tree Γ' obtained from Γ by removing the edges $[\gamma_j, b_j]$, $j = 1, 2, ..., m_0$ in exactly the same way as it was done for the star graph during the proof of Lemma 3.2. Hence we are faced with solving the original reconstruction problem but for a smaller graph. Since the tree Γ is finite, the problem will be reduced to the classical problem of reconstructing the potential q on $[\gamma_m, b_m]$ from the Titchmarsh-Weyl coefficient associated with the right end point. This problem has unique solution. □

This theorem shows that the knowledge of the whole Titchmarsh-Weyl matrix function allows one to reconstruct the tree and the potential. This theorem in general does not hold for arbitrary graph. The reason is that the response operator does not contain any information concerning the eigenfunctions with the support separated from the boundary points.

4.2. **Reconstruction of the tree from the diagonal of the Titchmarsh-Weyl matrix.** Note that the non-diagonal elements of the Titchmarsh-Weyl matrix were used only in order to establish the way the edges are connected to each other. This structure can be described using the connectivity matrix. Let a graph Γ have N edges, then the connectivity matrix C is the $N \times N$ matrix with entries equal to 0 or 1 determined as follows

$$C_{ij} = \begin{cases} 0, & \text{if the edges } E_i \text{ and } E_j \text{ have no common points,} \\ 1, & \text{if the edges } E_i \text{ and } E_j \text{ have one common point.} \end{cases}$$

The following theorem holds.

Theorem 4.2. *Let Γ be a tree with m boundary points and $H = L + q$ be the corresponding Schrödinger operator, where $q \in L_1(\Gamma)$ is a real valued function. Assume in addition that the connectivity matrix C is known. Then the lengths of edges and the potential q are uniquely determined by one of the following sets of data:*

- *the diagonal of the reduced $(m-1)\times(m-1)$ Titchmarsh-Weyl matrix $\mathbf{M}_{m-1}(\lambda)$ (defined by (4.1));*
- *by all except one back-scattering coefficients;*
- *the diagonal of the reduced $(m-1)\times(m-1)$ response operator $\mathbf{R}^T_{m-1}$ (defined by (4.2)) for sufficiently large T.*

Considering star graph one may conclude that the theorem in general cannot be improved in the following sense: reducing the dimension of the Titchmarsh-Weyl function by two dimensions may not allow to recover the potential completely.

Our results can be generalized to include graphs with cycles (not trees). In this case our method allows to recover any loose branch (any subtree on the graph which can be separated from the rest of the graph by cutting just one vertex) of such graphs together with the potential on it.

Acknowledgements. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for hospitality, where this work was accomplished. We are grateful to Victor Mikhaylov for fruitful discussions.

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Received April 2007; revised June 2007.

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