On the δ' -interaction in one dimension. *

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Abstract

The boundary conditions for the Schroedinger operator with the δ'-interaction are discussed. A new class of selfadjoint extensions is investigated.

This note is devoted to the Schroedinger operator with δ'-potential

$$H = -\frac{d^2}{dx^2} + c\delta'(x) \tag{1}$$

where c is a coupling constant and $\delta'(x)$ is the derivative of the δ-function. This operator can be understood as a selfadjoint extension of the operator

$$H_{0,0} = -\frac{d^2}{dx^2},$$

$$Dom(H_{0,0}) = \{f \in W_2^2(\mathbf{R}) \mid f(0) = f'(0) = 0\}. \tag{2}$$

The family of selfadjoint extensions is a four parameter one and there are differente possibilities to define the δ'-interaction. The standard definition ([1], [2], [3]) is a one-parameter family of selfadjoint extensions:

$$H_\beta = -\frac{d^2}{dx^2},$$

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$$Dom(H_\beta) = \{f \in W_2^2(\mathbf{R} \setminus \{0\}) \mid f'(-0) = f'(+0), f(+0) - f(-0) = \beta f'(0)\}, \tag{3}$$

where parameter β is connected with the coupling constant c. As it was shown in [3] this selfadjoint extension corresponds to the heuristic operator

$$-\frac{d^2}{dx^2} + \beta \mid \delta'(x) >< \delta'(x) \mid . \tag{4}$$

In the recent paper [4] the author pointed out that this definition is not very good to describe the δ'-interaction. Unfortunately, definition of the δ'-interaction proposed in [4] can not be used, because the corresponding operator is not selfadjoint. In this note we try to understand these calculations from the theory of selfadjoint operators point of view. On this way a new class of boundary conditions is selected from the four-parameter family of selfadjoint extensions of the operator $H_{0,0}$. We note that the heuristic operator (1) was analized as a strong resolvent limit of the operator with two δ-function interactions in [3]. Unfortunately this approach does not lead to a new class of selfadjoint extensions.

In our approach we use the following generalization of the δ-function for the case of the discontinuous trial function f:

$$\int_{-\infty}^{+\infty} \delta(x) f(x) dx \equiv \frac{f(+0) + f(-0)}{2}, \tag{5}$$

where $f(+0), f(-0)$ denote the limit values of the function from the right side and from the left side of the point zero. Integrating by parts one can obtain the following equality:

$$\delta'(x) f(x) = \frac{f(+0) + f(-0)}{2} \delta'(x) - \frac{f'(+0) + f'(-0)}{2} \delta(x) \tag{6}$$

Then as in [4] we integrate the Schroedinger equation

$$-\frac{d^2 f(x)}{dx^2} + c\delta'(x) f(x) = E f(x) \tag{7}$$

from -0 to $+0$ and twice from $-L < 0$ to x and from -0 to $+0$ and obtain the following conditions on the function f:

$$f'(+0) - f'(-0) = -\frac{c}{2}(f'(+0) + f'(-0)) \tag{8}$$

$$f(+0) - f(-0) = \frac{c}{2}(f(+0) + f(-0)) \tag{9}$$

Operator of the second derivative in $W_2^2(\mathbf{R} \setminus \{0\})$, restricted on the linear set picket out by these boundary conditions is a selfadjoint extension of the operator $H_{0,0}$. We shall define this selfadjoint operator as a Schroedinger operator with the δ'-interaction. Continuous spectrum eigenfunctions of the operator have the form, $k \in [0, \infty)$:

$$f_-(k,x) = \begin{array}{ll} e^{\imath kx} + R_- e^{-\imath kx}, & x < 0 \\ T_- e^{\imath kx}, & x > 0. \end{array} \tag{10}$$

$$f_+(k,x) = \begin{array}{ll} T_+ e^{-\imath kx}, & x < 0 \\ e^{-\imath kx} + R_+ e^{\imath kx}, & x > 0. \end{array} \tag{11}$$

where the reflection and transmission coefficients can be easily calculated from the boundary conditions (8,9):

$$T_- = T_+ = \frac{4 - c^2}{4 + c^2} \tag{12}$$

$$R_- = -R_+ = \frac{-4c}{4 + c^2} \tag{13}$$

The corresponding unitary scattering matrix is independent on the energy. The discrete spectrum of the operator is empty. When $c = \pm 2$ then the transition coefficient is equal to zero and the problems on the half axises are decoupled. Functions from the domains of the operators satisfy Dirichelet boundary condition on one side of the point zero and Neuwmann boundary condition on the other side. The case $c = 0$ leads to the nonperturbed operator of the second derivative in $L_2(\mathbf{R})$.

This model can be easily generalized to the case of finite or infinite number of point interactions. The boundary conditions similar to (8, 9) were used in [5], [6] during the construction of the selfadjoint extension with extra internal channel of interaction.

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