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Generalized perturbations and operator relations.[1]

1 Introduction

Selfadjoint extensions of symmetric operators are used to obtain operators with complicated spectral structure. In particular generalized extensions of symmetric operators having finite deficiency indices have been investigated. Let A_0^0 be such symmetric operator acting in a certain Hilbert space $\mathcal{H}$ and let $\mathcal{H}^+$ be an extended Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$. Then a selfadjoint operator $\mathcal{A}$ acting in $\mathcal{H}^+$ is called generalized extension of A_0^0 if the operator A_0^0 is a symmetric restriction of $\mathcal{A}$. Usually physically relevant is the restriction of the resolvent of the total operator $\mathcal{A}$ acting in $\mathcal{H}^+$ to the original Hilbert space $\mathcal{H}$. These operators are called the generalized resolvents. The generalized resolvent equal to the resolvent of a certain selfadjoint operator in $\mathcal{H}$ is called orthogonal. In [9] it was suggested to use the following approach to obtain generalized extensions of symmetric operators. Let A_0^0 be a symmetric operator acting in the Hilbert space $\mathcal{H}^0 = \mathcal{H}$. Consider arbitrary extension Hilbert space $\mathcal{H}^1$ and arbitrary symmetric operator A_1^0 acting in this space. Define the symmetric operator $\mathcal{A}^0 = A_0^0 \oplus A_1^0$ acting in the Hilbert space $\mathbf{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. Then any selfadjoint extension of the symmetric operator $\mathcal{A}^0$ is a generalized perturbation of the operator A_0^0. We are going to define two classes of selfadjoint perturbations. We call a selfadjoint extension $\mathcal{A}$ of $\mathcal{A}^0$ **separated** if it is equal to the orthogonal sum of selfadjoint extensions of A_0^0 and A_1^0, i.e. when the total operator or operator relation $\mathcal{A}$ is equal to the orthogonal sum of operators or operator relations defined in $\mathcal{H}^0$ and $\mathcal{H}^1$. All the other selfadjoint perturbations will be called **connected**.

Two important questions concerning this model arise:

- Is it possible to obtain all generalized extensions of the operator A_0^0 considering arbitrary extension spaces and operators?
- Can selfadjoint operator relations appear if the original operator A_0^0 is densely defined?

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We are going to study these two questions in the case, where the operator A_0^0 has deficiency indices $(1, 1)$. We prove that all generalized resolvents of the operator A_0^0 can be obtained even if one considers only the operators A_1^0 with the deficiency indices $(1, 1)$. It will be shown that if the original operator A_0^0 is densely defined then every generalized resolvent can be obtained considering only the selfadjoint extensions of the operator $\mathcal{A}^0$ that are selfadjoint operators, not operator relations. In other words if the selfadjoint extension $\mathcal{A}$ is an operator relation, then it is necessarily separated. But each selfadjoint extension of a densely defined symmetric operator is an operator. Thus the restricted resolvent of every such operator relation to the original space $\mathcal{H}^0$ is orthogonal, i.e. coincides with the resolvent of a certain selfadjoint operator in $\mathcal{H}^0$. Therefore the case of selfadjoint operator relations is not interesting, since physically relevant is the restriction of the resolvent to the original Hilbert space $\mathcal{H}^0$.

Current paper is organized as follows. First we describe in detail how to define the selfadjoint perturbation in the case where the operator A_1^0 is not densely defined. We prove that the operator relation can occur only in the case where the total selfadjoint perturbation is equal to the orthogonal sum of an operator in $\mathcal{H}^0$ and operator relation (or operator) in $\mathcal{H}^1$. In the last section we show that every generalized resolvent can be obtained using the model defined in the case where the deficiency indices of the restricted operator A_0^0 are equal to $(1, 1)$.

2 Domain of the perturbed operator.

Let us consider a pair of Hilbert spaces $\mathcal{H}^0, \mathcal{H}^1$ and a pair of selfadjoint operators A_0, A_1 defined respectively in $\mathcal{H}^0, \mathcal{H}^1$. The scalar product in the spaces $\mathcal{H}^\beta, \quad \beta = 0, 1$ will be denoted by $\langle \cdot, \cdot \rangle_\beta$. The orthogonal sum $\mathcal{A} = A_0 \oplus A_1$ is a selfadjoint operator acting in the orthogonal sum of the Hilbert spaces $\mathbf{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$

$$\mathcal{A} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} A_0\psi_0 \\ A_1\psi_1 \end{pmatrix}.$$

In what follows we are going to use the natural embedding of the spaces $\mathcal{H}^\beta$ into the space $\mathbf{H}$. Thus every element $\psi_0 \in \mathcal{H}^0$ will be identified with the element $(\psi_0, 0) \in \mathbf{H}$. Similarly we identify $\psi_1 \in \mathcal{H}^1$ and $(0, \psi_1) \in \mathbf{H}$. Similar identification will be used for the operators. For example the operator A_0 in the Hilbert space $\mathcal{H}^0$ and the operator $A_0(\psi_0, \psi_1) = (A_0\psi_0, 0)$ in $\mathbf{H}$ will be identified.

Let us denote by $\mathcal{H}_2(A_\beta)$ the domain of the operator A_β equiped with the norm

$$\| \varphi \|^2_{\mathcal{H}_2(A_\beta)} = \| \varphi \|^2 + \| A_\beta \varphi \|^2 .$$

The conjugated space of all bounded functionals will be denoted by $\mathcal{H}_{-2}(A_\beta)$. The norm in this space is defined by

$$\| \varphi \|^2_{\mathcal{H}_{-2}(A_\beta)} = \langle \varphi, \frac{1}{A_\beta^2 + 1} \varphi \rangle .$$

Consider two elements $\varphi_\beta \in \mathcal{H}_{-2}(A_\beta), \beta = 0, 1$ with the unit norms

$$\| \varphi_\beta \|_{\mathcal{H}_{-2}(A_\beta)} = 1, \ \beta = 0, 1.$$

The restrictions A^0_β of the operators $A_\beta, \beta = 0, 1$ to the domains

$$\mathrm{Dom}_{\varphi_\beta}(A_\beta) = \{ \psi \in D(A_\beta) : \langle \psi, \varphi_\beta \rangle = 0 \}; \ \beta = 0, 1$$

are symmetric operators. We are going to study the case where one of the restricted operators is densely defined. Suppose that the vector φ_0 does not belong to the Hilbert space $\mathcal{H}^0$. Then the operator A^0 is densely defined.

We denote by $\mathrm{Dom}(\mathcal{A}^0)$ the domain of the restricted operator

$$\mathrm{Dom}(\mathcal{A}^0) = \{ \Psi = (\psi_0, \psi_1) \in \mathrm{Dom}(\mathcal{A}) : \langle \psi_0, \varphi_0 \rangle_0 = 0, \langle \psi_1, \varphi_1 \rangle_1 = 0 \}.$$

The operator $\mathcal{A}^0$ coincides with the operator $\mathcal{A}$ restricted to the domain $\mathrm{Dom}(\mathcal{A}^0)$.

If the vector φ_1 does not belong to the Hilbert space $\mathcal{H}^1$ then the total operator $\mathcal{A}^0$ is densely defined and every its selfadjoint extension is an operator, not an operator relation. Therefore we restrict our consideration to the case where the operator A^0_1 is not densely defined, i.e. $\varphi_1 \in \mathcal{H}^1$.

Let M be the two dimensional subspace of $\mathbf{H}$ spanned by the vectors $\frac{1}{A_\beta + i} \varphi_\beta, \ \beta = 0, 1$:

$$M = \mathcal{L} \{ \frac{1}{A_0 + i} \varphi_0, \frac{1}{A_1 + i} \varphi_1 \}.$$

Let Γ be a Hermitean operator acting in M. We denote by Dom_Γ the set of elements Ψ from $\mathrm{Dom}(A_0^{0*}) \oplus \mathrm{Dom}(A_1) \subset \mathbf{H}$ possessing the following representation

$$\Psi = \tilde{\Psi} + \frac{\mathcal{A}}{\mathcal{A} - i} \Xi_+(\Psi) + \frac{1}{\mathcal{A} - i} \Xi_-(\Psi) \tag{1}$$

with $\tilde{\Psi} \in \mathrm{Dom}(\mathcal{A}^0),\ \Xi_\pm(\Psi) \in M,\ \Xi_- = \Gamma\Xi_+$.

The operator Γ is acting in the two dimensional subspace $\Gamma : M \to M$. The vectors $e_0 = \frac{1}{A_0 - i}\varphi_0, e_1 = \frac{1}{A_1 - i}\varphi_1$ form an orthonormal basis in the two dimensional subspace M. The operator Γ in this basis is the operator of multiplication by the 2×2 Hermitean matrix

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix}, \quad \gamma_{\beta_0\beta_1} = \langle \frac{1}{A_{\beta_0} - i}\varphi_{\beta_0}, \Gamma_{\beta_0\beta_1}\frac{1}{A_{\beta_1} - i}\varphi_{\beta_1}\rangle_{\beta_0}. \tag{2}$$

Lemma 2.1 *If*

$$\gamma_{01} \neq 0 \tag{3}$$

and/or

$$\gamma_{11} + \langle \frac{1}{A_1 + i}\varphi_1, \frac{A_1}{A_1 + i}\varphi_1\rangle \neq 0, \tag{4}$$

then every element $\Psi \in \mathrm{Dom}_\Gamma$ *possesses the unique representation*

$$\Psi = \tilde{\Psi} + \frac{1}{\mathcal{A} - i}\Gamma\Xi_+ + \frac{\mathcal{A}}{\mathcal{A} - i}\Xi_+, \tag{5}$$

where $\tilde{\Psi} \in \mathrm{Dom}(\mathcal{A}^0), \Xi_+ \in M_+$.

Proof. One needs to prove only the uniqueness of representation (1) because the domain Dom_Γ was defined as the set of all vectors possessing the representation (5). Let Ψ be an element of Dom_Γ. Suppose that there exist vectors $\tilde{\Psi}, \tilde{\Psi}' \in \mathrm{Dom}(\mathcal{A}^0)$ and $\Xi_+, \Xi'_+ \in M$ such that

$$\Psi = \tilde{\Psi} + \frac{1}{\mathcal{A} - i}(\Gamma + \mathcal{A})\Xi_+ = \tilde{\Psi}' + \frac{1}{\mathcal{A} - i}(\Gamma + \mathcal{A})\Xi'_+.$$

This implies that

$$\tilde{\Psi} - \tilde{\Psi}' = -\frac{1}{\mathcal{A} - i}(\Gamma + \mathcal{A})(\Xi_+ - \Xi'_+). \tag{6}$$

The latter equality implies the equality

$$\tilde{\Psi} - \tilde{\Psi}' + \frac{1}{\mathcal{A} - i}(\Gamma + i)(\Xi_+ - \Xi'_+) = -(\Xi_+ - \Xi'_+).$$

The left-hand side of the latter equality belongs to the domain $\mathrm{Dom}(\mathcal{A})$. The right hand side is a linear combination of the vectors $\frac{1}{A_0 - i}\varphi_0$ and $\frac{1}{A_1 - i}\varphi_1$. The element $\frac{1}{A_0-i}\varphi_0$ does not belong to the domain $\mathrm{Dom}(\mathcal{A})$. Thus the equality holds only if there exist some constant c such that

$$\Xi_+ - \Xi'_+ = c\frac{1}{A_1 - i}\varphi_1.$$

The difference $\tilde{\Psi} - \tilde{\Psi}'$ belongs to the domain $\mathrm{Dom}(\mathcal{A}^0)$ and one can apply the operator $\mathcal{A} - i$ to the equality (6)

$$(\mathcal{A} - i)(\tilde{\Psi} - \tilde{\Psi}') = -c(\Gamma + \mathcal{A})\frac{1}{A_1 - i}\varphi_1.$$

Projection into the space M leads to the equation

$$0 = -cP_M(\Gamma + A_1)\frac{1}{A_1 - i}\varphi_1.$$

The latter equation written in the basis $\{\frac{1}{A_0+i}\varphi_0, \frac{1}{A_1+i}\varphi_1\}$ is equivalent to the following 2×2 linear system

$$\begin{array}{rcl} 0 & = & -c\gamma_{01} \\ 0 & = & -c(\gamma_{11} + \langle \frac{1}{A_1+i}\varphi_1, \frac{A_1}{A_1+i}\varphi_1\rangle_1) \end{array}.$$

Thus the constant c is trivial if at least one of the equations (3) and/or (4) is not satisfied. This ends the proof of the lemma. $\square$

We have proven the uniqueness of the representation (5) under some conditions. The element $\tilde{\Psi}$ can be obtained by the formula

$$\tilde{\Psi} = \Psi - \frac{1}{\mathcal{A} - i}(\mathcal{A} + \Gamma)\frac{1}{P_M\mathcal{A}P_M + \Gamma}P_M(\mathcal{A} - i)\Psi.$$

One has to prove now that $\tilde{\Psi} \in (\mathcal{A} - i)^{-1}[\mathbf{H} \ominus M_-]$. The following calculation prove the desired property

$$P_M(\mathcal{A} - i)\tilde{\Psi} = P_M(\mathcal{A} - i)\Psi - P_M(\mathcal{A} + \Gamma)\frac{1}{P_M\mathcal{A}P_M + \Gamma}P_M(\mathcal{A} - i)\Psi = 0.$$

We are going to discuss the conditions (3) and (4) in more detail. The condition (3) implies that the components ψ_0 and ψ_1 of the elements from the domain Dom_Γ are independent. This implies that every selfadjoint operator $\mathcal{A}_\Gamma$ with the domain Dom_Γ is equal to a certain extension of the symmetric operator $\mathcal{A}^0$ which is equal to the orthogonal sum of two selfadjoint operators acting in the Hilbert spaces $\mathcal{H}^0$ and $\mathcal{H}^1$ (so-called separated extension). The resolvent of such extension restricted to the Hilbert space $\mathcal{H}^0$ is an orthogonal resolvent. Such resolvents have been studied in [2, 3, 4].

The conditions (3) and (4) together imply that the resolvent restricted to the space $\mathcal{H}^1$ is a resolvent of a selfadjoint relation, not an operator. But in this case the selfadjoint extension of $\mathcal{A}^0$ is separated. Physically relevant are only connected extensions and extensions with trivial extension space $\mathcal{H}^1$, since only the resolvent restricted to the subspace $\mathcal{H}^0$ is used in applications.

3 Perturbed operator.

The Γ-modified operator $\mathcal{A}_\Gamma$ is defined on Dom_Γ using the representation (5) by the formula

$$\begin{aligned} \mathcal{A}_\Gamma \Psi &= \mathcal{A}_\Gamma \left(\tilde{\Psi} + \frac{1}{\mathcal{A}-i} \Gamma \Xi_+(\Psi) + \frac{\mathcal{A}}{\mathcal{A}-i} \Xi_+(\psi) \right) \\ &= \mathcal{A}\tilde{\Psi} + (\mathcal{A}-i)^{-1}(-1+\mathcal{A}\Gamma)\Xi_+(\Psi). \end{aligned} \tag{7}$$

Theorem 3.1 *Let the operator Γ in M be Hermitean. If $\Gamma_{01} \neq 0$ then the Γ-modified operator $\mathcal{A}_\Gamma$ is selfadjoint on the domain* Dom_Γ.

Proof. If $\Gamma_{01} \neq 0$, then every element from the domain Dom_Γ possesses the unique representation (5). We show first that the operator $\mathcal{A}_\Gamma$ is symmetric.

Let $U, V \in \mathrm{Dom}_\Gamma(\mathcal{A})$

$$\begin{aligned}
& \langle U, \mathcal{A}_\Gamma V\rangle - \langle \mathcal{A}_\Gamma U, V\rangle \\
= \; & \langle \tilde{U} + \frac{1}{\mathcal{A}-i}(\Gamma+\mathcal{A})\Xi_+(U), \mathcal{A}^0\tilde{V} + \frac{1}{\mathcal{A}-i}(-1+\mathcal{A}\Gamma)\Xi_+(V)\rangle - \\
& -\langle \mathcal{A}^0\tilde{U} + \frac{1}{\mathcal{A}-i}(-1+\mathcal{A}\Gamma)\Xi_+(U), \tilde{V} + \frac{1}{\mathcal{A}-i}(\Gamma+\mathcal{A})\Xi_+(V)\rangle \\
= \; & \langle \tilde{U}, \mathcal{A}^0\tilde{V}\rangle + \langle \frac{1}{\mathcal{A}-i}(\Gamma+\mathcal{A})\Xi_+(U), \mathcal{A}^0\tilde{V}\rangle \\
& +\langle \tilde{U}, \frac{1}{\mathcal{A}-i}(-1+\mathcal{A}\Gamma)\Xi_+(V)\rangle \\
& +\langle \frac{1}{\mathcal{A}-i}(\Gamma+\mathcal{A})\Xi_+(U), \frac{1}{\mathcal{A}-i}(-1+\mathcal{A}\Gamma)\Xi_+(V)\rangle \\
& -\langle \mathcal{A}^0\tilde{U}, \tilde{V}\rangle - \langle \frac{1}{\mathcal{A}-i}(-1+\mathcal{A}\Gamma)\Xi_+(U), \tilde{V}\rangle \\
& -\langle \mathcal{A}^0\tilde{U}, \frac{1}{\mathcal{A}-i}(\Gamma+\mathcal{A})\Xi_+(V)\rangle \\
& -\langle \frac{1}{\mathcal{A}-i}(-1+\mathcal{A}\Gamma)\Xi_+(U), \frac{1}{\mathcal{A}-i}(\Gamma+\mathcal{A})\Xi_+(V)\rangle \\
= \; & \langle \Xi_+(U), (\mathcal{A}-i)\tilde{V}\rangle - \langle (\mathcal{A}-i)\tilde{U}, \Xi_+(V)\rangle \\
& +\langle (-1+\Gamma\mathcal{A})\frac{1}{\mathcal{A}^2+1}(\Gamma+\mathcal{A}) \\
& -(\Gamma+\mathcal{A})\frac{1}{\mathcal{A}^2+1}(-1+\mathcal{A}\Gamma)\Xi_+(U), \Xi_+(V)\rangle \\
= \; & 0.
\end{aligned}$$

The first two scalar products in the latter formula are equal to zero because $\Xi_+(U), \Xi_-(V) \in M$ and $\tilde{U}, \tilde{V} \in \mathrm{Dom}(\mathcal{A}^0)$. The third scalar product is equal to zero due to the following operator equality

$$(-1+\Gamma\mathcal{A})\frac{1}{\mathcal{A}^2+1}(\Gamma+\mathcal{A}) = ((\mathcal{A}+\Gamma)\mathcal{A} - \mathcal{A}^2 - 1)\frac{1}{\mathcal{A}^2+1}(\Gamma+\mathcal{A}) =$$

$$= (\mathcal{A}+\Gamma)\frac{\mathcal{A}}{\mathcal{A}^2+1}(\Gamma+\mathcal{A}) - (\Gamma+\mathcal{A}) = (\mathcal{A}+\Gamma)\frac{1}{\mathcal{A}^2+1}(-1+\mathcal{A}\Gamma).$$

Thus we have proved that the operator $\mathcal{A}_\Gamma$ is symmetric.

To prove that the operator $\mathcal{A}_\Gamma$ is selfadjoint we are going to calculate its resolvent, i.e. the solution of the equation

$$(\mathcal{A}_\Gamma - \lambda)^{-1}F = U$$

for arbitrary $F \in \mathbf{H}$ and arbitrary λ, $\Im\lambda \neq 0$. The latter equality implies that

$$F = (\mathcal{A}_\Gamma - \lambda) \left(\tilde{U} + \frac{1}{\mathcal{A} - i} (\mathcal{A} + \Gamma) \Xi_+(U) \right) =$$

$$= (\mathcal{A} - \lambda)\tilde{U} + \frac{1}{\mathcal{A} - i} \left(-1 + \mathcal{A}\Gamma - \lambda(\mathcal{A} + \Gamma)\right) \Xi_+(U).$$

Applying the operator $\frac{\mathcal{A}-i}{\mathcal{A}-\lambda}$ and projecting into the subspace M one gets the following equation

$$P_M \frac{\mathcal{A} - i}{\mathcal{A} - \lambda} F = (\Gamma - Q(\lambda))\Xi_+(U),$$

where

$$Q(\lambda) = P_M \frac{1 + \lambda\mathcal{A}}{\mathcal{A} - \lambda} P_M.$$

The vector $\Xi_+(U)$ can be calculated, since the operator $\Gamma - Q(\lambda)$ has nontrivial imaginary part and is therefore invertible

$$\Xi_+(U) = \frac{1}{\Gamma - Q(\lambda)} P_M \frac{\mathcal{A} - i}{\mathcal{A} - \lambda} F. \tag{8}$$

Projection into the orthogonal complement of M gives the equality

$$\tilde{U} = \frac{1}{\mathcal{A} - i} (1 - P_M) \frac{\mathcal{A} - i}{\mathcal{A} - \lambda} F + \frac{1}{\mathcal{A} - i} (1 - P_M) \frac{1 + \lambda\mathcal{A}}{\mathcal{A} - \lambda} \Xi_+(U). \tag{9}$$

Combining formulas (8) and (9) one gets the solution

$$U = \frac{1}{\mathcal{A} - \lambda} F + \frac{\mathcal{A} + i}{\mathcal{A} - \lambda} \frac{1}{\Gamma - Q(\lambda)} P_M \frac{\mathcal{A} - i}{\mathcal{A} - \lambda} F. \tag{10}$$

The element U belongs to the domain Dom_Γ. The domain of the resolvent coincides with the Hilbert space $\mathbf{H}$. It is necessary to prove that the kernel of the calculated resolvent operator is trivial. Suppose that

$$\frac{1}{\mathcal{A} - \lambda} F + \frac{\mathcal{A} + i}{\mathcal{A} - \lambda} \frac{1}{\Gamma - Q(\lambda)} P_M \frac{\mathcal{A} - i}{\mathcal{A} - \lambda} F = 0.$$

The first term in the latter equality is an element from the domain $\mathrm{Dom}(\mathcal{A})$. The second term is equal to a linear combination of the vectors $\frac{1}{A_0 - \lambda}\varphi_0$ and $\frac{1}{A_1 - \lambda}\varphi_1$. The vector $\frac{1}{A_0 - \lambda}\varphi_0$ does not belong to the domain $\mathrm{Dom}(\mathcal{A})$. Thus there exists a certain constant c such that

$$F = c\varphi_0.$$

This implies that

$$\frac{1}{A_0-\lambda}\varphi_0+\frac{\mathcal{A}+i}{\mathcal{A}-\lambda}\frac{1}{\Gamma-Q(\lambda)}\langle\frac{1}{A_0+i}\varphi_0,\frac{A_0-i}{A_0-\lambda}\varphi_0\rangle_0\frac{1}{A_0+i}\varphi_0=0.$$

The latter equality holds only if the operator $(\Gamma-Q(\lambda))^{-1}$ is diagonal. This implies that the matrix Γ has to be diagonal, i.e. $\Gamma_{01}=0$. We got thus a contradiction which proves the statement, ending the proof of the theorem. □

The resolvent of the constructed selfadjoint operator $\mathcal{A}_\Gamma$ is given by the formula (10). That formula coincides with the formula for the resolvent of the operator $\mathcal{A}_\Gamma$ constructed using a densely defined restricted operator.

4 Operators with internal structure and generalized resolvents.

The following theorem can be proven (see [5] for details):

Theorem 4.1 *Let A_0 be a selfadjoint operator acting in the Hilbert space $\mathcal{H}^0$ and let A_0^0 be its restriction to the domain $\mathrm{Dom}_{\varphi_0}=\{\psi\in\mathrm{Dom}(A):\langle\psi,\varphi_0\rangle_0=0\}$ where $\varphi_0\in\mathcal{H}_{-2}(A_0)\setminus\mathcal{H}^0$. Let $\mathbf{R}(\lambda)$ be a generalized resolvent corresponding to a certain generalized extension of the operator A_0^0. Then there exists a generalized perturbation with internal structure of the operator A_0 such that its resolvent restricted to the Hilbert space $\mathcal{H}^0$ coincides with the generalized resolvent $\mathbf{R}(\lambda)$.*

We note that it is enough to consider only connected selfadjoint extensions of the operator $\mathcal{A}^0$ in $\mathbf{H}$ and selfadjoint extensions of the operator A_0^0 in $\mathcal{H}^0$ to prove the latter theorem. Therefore to obtain all generalized perturbations of the densely defined operator A_0^0 one can consider only the selfadjoint extensions that are operators, not operator relations. The result announced here for arbitrary symmetric operator A_0^0 with deficiency indices $(1,1)$ has been already proven for the point perturbations of the Laplace operator in [10].

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