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SCHRöDINGER OPERATORS ON GRAPHS: SYMMETRIZATION AND EULERIAN CYCLES.

G. KARRESKOG, P. KURASOV, AND I. TRYGG KUPERSMIDT

Abstract. Spectral properties of the Schrödinger operator on a finite compact metric graph with delta-type vertex conditions are discussed. Explicit estimates for the lowest eigenvalue (ground state) are obtained using two different methods: Eulerian cycle and symmetrization techniques. In the case of positive interactions even estimates for higher eigenvalues are derived.

1. Introduction

Schrödinger operators on metric graphs, or *quantum graphs* is an important direction in modern mathematical physics closely related to spectral theory of differential operators and discrete mathematics [3, 8, 9, 12, 15]. Such operators are used to model physical phenomena where dynamics is limited to a neighborhood of a metric graph Γ and thus can be approximately described by a certain differential equation considered directly on the graph. Such differential equations, one-dimensional on every edge, possess a lot of properties of partial differential equations due to non-trivial topology of the underlying graph. Current article is devoted to spectral properties of Schrödinger operators on compact metric graphs formed by a finite number of edges. Most of the existing literature on this subject is devoted to spectral properties of Laplacians with so-called standard vertex conditions (see definition below). The reason to study just Laplacians is not only that the corresponding eigenfunction equation on every edge can easily be solved in terms of exponentials, but also the fact that such an operator is uniquely determined by the underlying metric graph. Therefore such standard Laplacian is often considered as the free operator on the graphs Γ and it spectrum is refried to as the spectrum of the metric graph Γ.

In the case of standard Laplacian the lowest eigenvalue is $\lambda_0 = 0$ with the eigenfunction being a constant function Γ. Then an interesting question is the estimate of the spectral gap - the difference between the first two lowest eigenvalues (of course assuming that Γ is connected). In recent papers [6, 14] it was proven that among all metric graphs of the same total length, the spectral gap is minimal for the single interval graph. The authors used two different approaches: L. Friedlander applied symmetrization technique, while P. Kurasov and S. Naboko studied behavior of the spectral gap upon topological perturbations of Γ. These two approaches lead to

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an explicit estimate for the spectral gap in terms of the spectral gap for the single interval graph. While the first approach leads to an elegant and relatively short proof of the result, the second approach allows one to deform the original graph in several steps leading to graphs with smaller and smaller spectral gaps contributing to our intuition. One of the main goals of the current article is to understand how do these two approaches work in a more sophisticated case of Schrödinger operators on metric graphs with delta-type interactions at the vertices. The constant function in general is not an eigenfunction anymore and the role of the spectral gap is played by the energy of the ground state - one tries to estimate the value of the lowest eigenvalue in relation to the total length of the metric graph, sum of intensities of the delta interactions at the vertices and integrals of the potential.

Spectral properties of Schrödinger operators with nontrivial potential and other than standard vertex conditions are much less studied. For example one may show that the asymptotics of the spectrum is related to the Euler characteristic of the underlying metric graph [10, 11]. Already obtained results show, that spectral behavior may sometimes be unexpected, which makes research even more exciting. For example P. Exner and M. Jex [5] showed that making one of the edges longer may lead to an increase of the ground state, provided the vertex conditions are of delta-type. This is in contrast to the case of standard conditions where the eigenvalues of the Laplacian always decrease if one of the edges is made longer. Thus the second, more practical, goal of the article is to derive explicit estimates for the ground state of a Schrödinger operator with delta interactions at the vertices acting on a compact metric graph.

The plan of the paper is follows. In the next section we introduce necessary notations, discuss which parameters of the graph and of the interaction are important for the estimates and formulate the main result. Section 3 and 4 are devoted to the proofs of the main result using the two techniques mentioned above. The last section is devoted to explicit estimates for higher eigenvalues.

2. Notations and the main result

Consider any compact connected metric graph Γ obtained from a finite set of compact intervals - edges $E_n = [x_{2n-1}, x_{2n}],\ n = 1, 2, \ldots, N,$ - by joining together the ends points x_j at vertices $V_m,\ m = 1, 2, \ldots, M$. The vertices V_m can be considered as a partition of the set of all end points $\mathbf{V} = \{x_j\}_{j=1}^{2N}$. The corresponding Hilbert space is $\mathcal{H} = \oplus_{n=1}^{N} L_2(E_n)$.

On the edges E_n one considers the Schrödinger differential expression

$$-\frac{d^2}{dx^2} + q(x), \tag{1}$$

where q is a real integrable potential on Γ.

Connections between the edges are introduced via vertex matching conditions. We are going to use vertex conditions of delta-type

$$\begin{cases} u \text{ is continuous at } V_m, \\ \sum_{x_j \in V_m} \partial u(x_j) = \alpha_m u(V_m), \end{cases} \tag{2}$$

at every vertex V_m and $\alpha_m \in \mathbb{R}$. Note that the value $u(V_m)$ is well-defined, since u is supposed to be continuous on Γ. The real parameter α_m is called the strength of the delta interaction. Conditions corresponding to $\alpha_m = 0$ are called *standard* or free.

The operator L_q defined by the differential expression (1) on the domain of functions from the Sobolev space $W_2^2(\Gamma \setminus \mathbf{V}) = \oplus_{n=1}^N W_2^2(E_n)$ satisfying delta-type vertex conditions 2) at all vertices $V_m,\ m = 1, 2, \ldots, M$ is self-adjoint and has discrete spectrum [1, 3, 12, 15]. The operator is semibounded from below and the eigenvalues accumulate to $+\infty$ satisfying Weyl asymptotics $\lambda_n \sim \frac{\pi^2}{\mathcal{L}^2} n^2$, where $\mathcal{L}$ is the total length of the graph (the sum of the lengths of all edges $\mathcal{L} = \sum_{n=1}^N x_{2n} - x_{2n-1}$). Hence interesting estimates on the eigenvalues may be obtained only if the total length is fixed. For example in the case of standard vertex conditions ($\alpha_m = 0$) and zero potential all eigenvalues scale uniformly if one stretches all the edges.

It appears that effective eigenvalue estimates can be obtained in terms of the positive and negative parts of the interaction introduced below. Under the interaction we understand the potential q and the strengths α_m of the interactions at the vertices. One may consider these interactions as delta-potentials concentrated at the vertices. The **total interaction** I, **total positive interaction** I_+ and **total negative interaction** I_- of the Schrödinger operator with vertex δ-conditions is defined as

$$\begin{aligned} I &= \int_\Gamma q(x)dx + \sum_{m=1}^M \alpha_m, \\ I_+ &= \int_\Gamma q_+(x)dx + \sum_{\alpha_m > 0} \alpha_m, \\ I_- &= \int_\Gamma q_-(x)dx - \sum_{\alpha_m < 0} \alpha_m, \end{aligned} \tag{3}$$

where $q_+(x) = (q(x) + |q(x)|)/2 \geq 0$ and $q_-(x) = (-q(x) + |q(x)|)/2 \geq 0$ are the positive and negative parts of $q(x) = q_+(x) - q_-(x)$.

We are going to use Rayleigh estimate, which involves the quadratic form Q_{L_q} of the operator L_q. The domain of the quadratic form is given by the set of functions from the Sobolev space $W_2^1(\Gamma \setminus \mathbf{V})$ which are in addition continuous at the vertices (note that the second vertex condition in (2) is not preserved). It is very important for our calculations that the domain of the quadratic form is independent of the values of the strengths of delta interactions α_m. The quadratic form is given by the formula

$$Q_{L_q}(u, u) = \int_\Gamma |u'(x)|^2 dx + \int_\Gamma q(x)|u(x)|^2 dx + \sum_{m=1}^M \alpha_m |u(V_m)|^2. \tag{4}$$

As we already mentioned our goal is to obtain explicit universal estimates on the lowest eigenvalue, also called the ground state. The Rayleigh quotient gives the following explicit upper estimate

$$\lambda_0(L_q) = \min \frac{Q_{L_q}(u,u)}{\int_\Gamma |u(x)|^2 dx} \leq \frac{I}{\mathcal{L}}, \tag{5}$$

if one uses $u(x) \equiv 1$ as a trial function. It is clear that this estimate is sharp, since it is realized when $q(x) = I/\mathcal{L}$ and conditions at the vertices are standard.

Our main result is the following lower estimate for the ground state:

Theorem 1. *Let Γ be a finite compact metric graph of total length $\mathcal{L}$ and let L_q be a Schrödinger operator on Γ defined by (1) and delta vertex conditions (2) with the total negative strength I_-, and the total positive strengths I_+ (given by (3)). Then the ground state $\lambda_0(L_q)$ is bounded by $\lambda_0(\mathbf{L})$, where $\mathbf{L}$ is the Laplace operator $\mathbf{L} = -\frac{d^2}{dx^2}$ on the single interval $[0,\mathcal{L}]$ defined on functions satisfying the following Robin boundary conditions*

$$u'(0) = I_+ u(0), \qquad u'(\mathcal{L}) = I_- u(\mathcal{L}). \tag{6}$$

In other words $\lambda_0(L_q)$ is always greater or equal to the lowest solution to the secular equation

$$\left(k + \frac{I_- I_+}{k}\right) \tan k\mathcal{L} = I_+ - I_-, \;\; k = \sqrt{\lambda}. \tag{7}$$

In what follows we present two different proofs of the theorem following approaches due to S. Naboko and one of the authors in the first proof (Eulerian cycle technique) and due to L. Friedlander in the second (symmetrization technique). The proofs are equivalent, but shed light on different spectral properties of quantum graphs.

3. Eulerian cycle technique

We start this section by proving that eigenvalues of a Schrödinger operator on a graph depend monotonically upon topological perturbations of the metric graph. Corresponding results are elementary and have already been proven in [2, 13] for the Laplacian with standard vertex conditions.

Theorem 2. *Let Γ be a finite compact metric graph, and let Γ' be another metric graph obtained from Γ by splitting one of the vertices, say vertex V_m into two vertices $V_{m'}$ and $V_{m''}$. The corresponding strengths $\alpha_{m'}$ and $\alpha_{m''}$ at the new vertices are chosen arbitrarily subject to the condition*

$$\alpha_m = \alpha_{m'} + \alpha_{m''}, \tag{8}$$

where α_m is the strength of the delta interaction at the original vertex V_m. Let us denote by L_q and L'_q the Schrödinger operators defined by the same differential expression on the graphs Γ and Γ', the same strengths of the delta interactions at all preserved vertices and with delta interactions at the vertices α_m, $\alpha_{m'}$, and $\alpha_{m''}$ as described above. Then the eigenvalues for L' are majorized by the corresponding eigenvalues for L:

$$\lambda_n(L) \geq \lambda_n(L'). \tag{9}$$

Proof. Observe that any function defined on the metric graph Γ can also be considered as a function on Γ' and *vice versa*. Moreover if a function u on Γ is continuous at the vertex V_m, then the corresponding function on Γ' is continuous at $V_{m'}$ and $V_{m''}$. On the other hand a function continuous on Γ' does not necessarily lead to a function continuous on Γ, since the values of the original function at the vertices $V_{m'}$ and $V_{m''}$ may be different. The above analysis shows that the following inclusion holds for the domains of the quadratic forms $D_{Q(L_q)} \subset D_{Q(L'_q)}$. The n-th eigenvalue can be calculated using *min-max* principle for the Rayleigh quotient

$$\lambda_n(L_q) = \max_{\substack{A_n \subset D_Q \\ \dim(A_n)=n}} \left(\min_{\substack{u \perp A_n \\ u \in D_Q}} \frac{Q_{L_q}(u,u)}{\int_\Gamma |u(x)|^2 dx} \right). \tag{10}$$

Taking into account that the Rayleigh quotients for the operator L_q and L'_q are identical while the set of allowed trial functions is bigger for L'_q we conclude that (9) holds. □

Note that we did not use the particular form of matching conditions at the preserved vertices. It follows that the theorem holds even if other than delta matching conditions are introduced at these vertices. We are ready to give the first proof of Theorem 1.

Proof of Theorem 1 using Eulerian cycle. Let $\lambda_0(L)$ denote the lowest eigenvalue of the Schrödinger operator L_q with δ-conditions, acting on a metric graph Γ of the total length $\mathcal{L}$, with the total positive and negative interaction equal to I_+ and I_+ respectively.

Following [14] consider the graph $\Gamma^{(2)}$, constructed from Γ by for every edge E_n between two vertices of Γ adding another edge between the two vertices of the same length as E_n, and doubling the strength of the delta conditions in all the vertices. We also define the value of the potential in each point on a new edge, by mirroring the values of the potential on the original edge. This doubles the total positive and negative interaction. $\Gamma^{(2)}$ may be called the double cover of Γ. The corresponding operator will be denoted by $L_q^{(2)}$. For any eigenfunction u_n on Γ, another function $u_n^{(2)}$ can be defined on $\Gamma^{(2)}$ by letting it assume the same value as u_n on the old and new edges. Clearly all $u_n^{(2)}$ fulfill the eigenvalue equation for the same eigenvalues, and the vertex conditions, giving that each eigenvalue of Γ, is also an eigenvalue of $\Gamma^{(2)}$. The reverse is however not always true. This implies that

$$\lambda_0(L_q) \geq \lambda_0(L_q^{(2)}).$$

All vertices of $\Gamma^{(2)}$ are of even degree. By a well-known theorem by Euler [4, 7], there must thus be an Eulerian cycle on the graph, meaning that there exists a path starting and ending in the same vertex, that visits every edge exactly once. This cycle can be obtained by cutting the graph $\Gamma^{(2)}$ at certain vertices. The resulting graph to be denoted by $\widehat{\Gamma}$ is a loop of length $2\mathcal{L}$ with several delta interactions on it. We assume that the cutting is performed in such a way that positive and negative interaction are preserved. The corresponding Schrödinger operator will be denoted by $\widehat{L}$. By Theorem 2 we have the following estimates:

$$\lambda_0(L_q) \geq \lambda_0(L_q^{(2)}) \geq \lambda_0(\widehat{L}_q).$$

We have proven that the ground state is always greater or equal to the ground state of the Schrödinger operator on the loop of double length and doubled interaction. Our immediate aim is to prove that concentrating all negative and all positive interactions to two points on the loop only diminishes the ground state energy.

For any graph with the ground state eigenfunction u_0 there must be two points $x_{\max}, x_{\min} \in \Gamma$, such that $|u_0(x_{\max})| \geq |u_0(x)|$ and $|u_0(x)| \geq |u_0(x_{\min})|$ for all x. Together with the original loop graph $\widehat{\Gamma}$ and Schrödinger operator on it, consider the same loop with Laplace operator $\tilde{L}$ having two delta interactions of strength $-2I_-$ and $2I_+$ introduced at the points $x_{\max}$ and $x_{\min}$ respectively. Then the eigenfunction u_0 on $\widehat{\Gamma}$ is a perfect trial function for the new operator $\tilde{L}$ and the corresponding Rayleigh quotient gives an upper estimate for the ground state:

$$\begin{aligned}
\lambda_0(\widehat{L}_q) &= \frac{\displaystyle\int_{\widehat{\Gamma}} |u_0'(x)|^2 dx + \int_{\widehat{\Gamma}} q(x)|u_0(x)|^2 dx + \sum_v \alpha_v |u(v)|^2}{\displaystyle\int_{\widehat{\Gamma}} |u_0|^2 dx} \\
&\geq \frac{\displaystyle\int_{\widehat{\Gamma}} |u_0'(x)|^2 dx - 2I_-|u(x_{\max})|^2 + 2I_+|u(x_{\min})|^2}{\displaystyle\int_{\widehat{\Gamma}} |u_0|^2 dx} \geq \lambda_0(\tilde{L}).
\end{aligned}$$

Our last step is to prove that the ground state is minimal if the points $x_{\max}$ and $x_{\min}$ are situated symmetrically (so that the distance in-between is precisely $\mathcal{L}$), of course provided the total length $2\mathcal{L}$ and total positive and negative parts of the interaction $2I_\pm$ are preserved. Consider the general non-symmetric case and let us denote by ℓ_1 and ℓ_2 the lengths of the two edges between the points $x_{\min}$ and $x_{\max}$ on the loop, $\ell_1 + \ell_2 = 2\mathcal{L}$.

Let us parametrize the two edges as $[0, \ell_1]$ and $[0, \ell_2]$ so that the left ends points and the right end points are joined at two vertices carrying positive and negative delta interactions of strengths $2I_+$ and $-2I_-$ respectively. It will be convenient to denote the components of functions on these intervals using lower indices as $u_1(x)$ and $u_2(x)$. Every eigenfunction not only satisfies the equation $-u_j''(x) = \lambda u_j(x)$, $j = 1, 2$, on each interval, but also delta-conditions at the end points

$$\left\{ \begin{array}{l} u_1(0) = u_2(0), \\ u_1'(0) + u_2'(0) = 2I_+ u_1(0), \end{array} \right. \qquad \left\{ \begin{array}{l} u_1(\ell_1) = u_2(\ell_2), \\ u_1'(\ell_1) + u_2'(\ell_2) = 2I_- u_1(\ell_1). \end{array} \right.$$

Writing general solution to the differential equation as

$$u_j = a_j \cos kx + b_j \frac{\sin kx}{k}$$

we arrive at the following system of equations on a_j, b_j

$$\begin{cases} a_1 = a_2, \\ b_1 + b_2 = 2I_+ a_1, \\ a_1 \cos k\ell_1 + b_1 \dfrac{\sin k\ell_1}{k} = a_2 \cos k\ell_2 + b_2 \dfrac{\sin k\ell_2}{k}, \\ -a_1 k \sin k\ell_1 + b_1 \cos k\ell_1 - a_2 k \sin k\ell_2 + b_2 \cos k\ell_2 = 2I_- \left(a_1 \cos k\ell_1 + b_1 \dfrac{\sin k\ell_1}{k} \right). \end{cases}$$

The system has a nontrivial solution if and only if the determinant is zero

$$-(1 - \cos 2k\mathcal{L}) - \frac{I_- - I_+}{k} \sin 2k\mathcal{L} = 2\frac{I_- I_+}{k^2} \sin k\ell_1 \sin k\ell_2. \tag{11}$$

This equation can be rewritten as

$$\sin^2 k\mathcal{L} + (I_- - I_+)\frac{\sin k\mathcal{L}}{k} \cos k\mathcal{L} + (I_- I_+)\frac{\sin^2 k\mathcal{L}}{k^2} = I_- I_+ \frac{\sin^2 k\Delta\ell}{k^2}, \tag{12}$$

where we used the notation $\Delta\ell = (\ell_1 - \ell_2)/2$. The equation determines all nonzero eigenvalues of the operator, since we used cos and sin functions as basic solutions to the differential equation. (The case $\lambda = 0$ is special, since the solutions to the eigenfunction equation are linear functions.) Similarly the nonzero spectrum in the symmetric case $\ell_1 = \ell_2 = \mathcal{L}$ is given by the smallest zero of the analytic function

$$f(\lambda) := \sin^2 k\mathcal{L} + (I_- - I_+)\frac{\sin k\mathcal{L}}{k} \cos k\mathcal{L} + \frac{I_- I_+}{2} \frac{\sin^2 k\mathcal{L}}{k^2},$$

which is precisely the left hand side of equation (12). The function $f(\lambda)$ is real on the real axis and tends to $-\infty$ as $\lambda \to -\infty$. Hence the lowest eigenvalue occurs when the graph of the function crosses the real axis. The right hand side of equation (12) is a nonnegative function, since $I_\pm \geq 0$ and therefore the smallest solution to equation (12) lies to the right of the smallest zero of the function f. To prove our claim it remains to check that $\lambda = 0$ is an eigenvalue for the non symmetric problem $\ell_1 \neq \ell_2$ only if it is also an eigenvalue for the symmetric case $\ell_1 = \ell_2$, which is elementary.

The statement of the theorem follows from the fact that the ground state of the Laplacian on the loop with two delta interactions situated symmetrically coincides with the ground state of the Laplacian on the interval of half length with Robin boundary conditions (6). □

The estimate can be improved if we know that the original graph is balanced, *i.e.* all vertices have even degree. Using the first proof we do not need to double the edges before Eulerian cycle is chosen. Hence the estimate from below is given by the ground state of the Laplace operator on the loop of length $\mathcal{L}$ (instead of $2\mathcal{L}$) and we have the following

Corollary 1. *Let all assumptions of Theorem 1 be satisfied. Assume in addition that the metric graph Γ is balanced. Then the ground state for the Schrödinger operator on Γ is estimated from below by the ground state of the Laplace operator on the interval $[0, \mathcal{L}/2]$ with Robin boundary conditions at the end points:*

$$u'(0) = I_+/2\, u(0), \qquad u'(\mathcal{L}/2) = I_-/2\, u(\mathcal{L}/2). \tag{13}$$

4. Symmetrization technique

In this section we follow closely the article [6], where symmetrization technique was applied to obtain estimates for the spectral gap for the standard Laplacian.

Proof of Theorem 1 using symmetrization technique. We first use the same idea as in the first proof and substitute the original Schrödinger operator L on Γ with the Laplace operator $\mathbb{L}$ with just two delta interactions of strengths $-I_-$ and I_+. Estimate for the ground state of $\mathbb{L}$ is done following ideas from the article [6].

Let u_0 be the ground state eigenfunction for the Schrödinger operator L. Let us denote by $x_{\min}$ and $x_{\max}$ the points of minimum and maximum for $|u(x)|^2$. Consider the Laplace operator $\mathbb{L}$ on Γ defined on the domain of functions satisfying delta boundary conditions at the points $x_{\min}$ and $x_{\max}$

$$\left\{ \begin{array}{lcl} \sum_{x_{\min}} \partial u(x_{\min}) & = & I_+ u(x_{\min}); \\ \sum_{x_{\max}} \partial u(x_{\min}) & = & -I_- u(x_{\max}); \end{array} \right. \tag{14}$$

where the sums are taken over all end points of the edges joined together at $x_{\min}, x_{\max}$. If any of these points is an inner point on an edge, then the sum contains just two terms, otherwise the number of terms is equal to degree of the vertex.

The function u_0 is a continuous function on Γ and therefore can be used to estimate the ground state of the operator $\mathbb{L}$

$$\begin{array}{rcl} \lambda_0(\mathbb{L}) & \leq & \dfrac{\displaystyle\int_\Gamma |u_0'(x)|^2 dx - I_- |u(x_{\max})|^2 + I_+ |u(x_{\min})|^2}{\displaystyle\int_\Gamma |u_0|^2 dx} \\ & \leq & \dfrac{\displaystyle\int_\Gamma |u_0'(x)|^2 dx + \int_\Gamma q(x) |u_0(x)|^2 dx + \sum_v \alpha_v |u(v)|^2}{\displaystyle\int_{\widehat{\Gamma}} |u_0|^2 dx} \\ & = & \lambda_0(L). \end{array} \tag{15}$$

It follows that to estimate the ground state it is not only enough to consider Laplacians with two delta interactions, but it is also enough to consider trial functions having their minima and maxima at the points supporting the interactions. Following ideas of symmetrization technique we are going to compare the Rayleigh quotient for the operator $\mathbb{L}$ and trial function u_0 with the Rayleigh quotient for the Laplace operator $\mathbf{L}$ in $L_2[0, \mathcal{L}]$ (introduced above in the formulation of Theorem 1) and a certain "symmetrized" trial function u^* constructed from u_0.

The function u^* on the interval $[0, \mathcal{L}]$ is the unique nondecreasing continuous function such that

$$u^*(0) = u_0(x_{\min}), \quad u^*(\mathcal{L}) = u_0(x_{\max})$$

and

$$m(t) := \text{measure}\, \{x \in S : u_0(x) < t\} = \text{measure}\, \{s \in [0, \mathcal{L}] : u^*(s) < t\}\,.$$

The function u^* constructed in this way satisfies

$$\int_\Gamma |u_0(x)|^2 dx = \int_0^{\mathcal{L}} |u^*(x)|^2 dx. \tag{16}$$

The number of preimages of t under $u_0(x)$ is finite, since u satisfies the eigenfunction equation on each interval. Let us denote the number of preimages by $n(t)$. Obviously

$$n(t) \geq 1, \tag{17}$$

since the function u_0 is continuous.

The co-area formula implies

$$\int_S |u_0'(x)|^2 dx = \int_{u_0(x_{\min})}^{u_0(x_{\max})} \sum_{x:u_0(x)=t} |u_0'(t)| dt.$$

Cauchy-Schwartz inequality then gives

$$\begin{aligned} \sum_{x:u_0(x)=t} |u_0'(x)| &\geq n(t)^2 \left(\sum_{x:u_0(x)=t} \frac{1}{|u_0'(x)|} \right)^{-1} \\ &\geq \left(\sum_{x:u_0(x)=t} \frac{1}{|u_0'(x)|} \right)^{-1} \\ &= \frac{1}{m'(t)} . \end{aligned} \tag{18}$$

where we also used (17). Therefore we have

$$\int_S |u_0'(x)|^2 dx \geq \int_{u_0(x_{\min})}^{u_0(x_{\max})} \frac{dt}{m'(t)}. \tag{19}$$

The same argument can be applied to the function u^* with the only difference that all inequalities turn into equalities and there is no need to use (17). Finally we get:

$$\int_S |u_0'(x)|^2 dx \geq \int_0^{\mathcal{L}} |(u^*)'(s)|^2 ds. \tag{20}$$

It follows that the Rayleigh quotients satisfy the inequality:

$$\begin{aligned} \frac{Q_{\mathbb{L}}(u_0, u_0)}{\| u_0 \|^2} &\geq \frac{\displaystyle\int_\Gamma |u_0'(x)|^2 dx + I_+ |u_0(x_{\min})|^2 - I_- |u_0(x_{\max})|^2}{\int_\Gamma |u_0(x)|^2 dx} \\ &\geq \frac{\displaystyle\int_0^{\mathcal{L}} |u^{*\prime}(x)|^2 dx + I_+ |u^*(0)|^2 - I_- |u^*(\mathcal{L})|^2}{\int_0^{\mathcal{L}} |u^*(x)|^2 dx} = \frac{Q_{\mathbf{L}}(u^*, u^*)}{\| u^* \|^2} \end{aligned} \tag{21}$$

and therefore we have

$$\lambda_0(L) \geq \lambda_0(\mathbf{L}). \tag{22}$$

□

Corollary 1 can also be proven using symmetrization technique but requires slightly more work than in the first approach. We first note that the number $n(t)$ of preimages of t under $u_0(t)$ is at least 2

$$n(t) \geq 2, \tag{23}$$

since there exists an Eulerian cycle on Γ and u_0 can be considered as a continuous function on such cycle. Then formula (18) can be modified as follows

$$\sum_{x: u_0(x)=t} |u_0'(x)| \geq \frac{n(t)^2}{m'(t)} \geq \frac{4}{m'(t)}. \tag{24}$$

Consider now another one function u^{**} defined on the interval $s \in [0, \mathcal{L}/2]$

$$u^{**}(s) = u^*(2|s|), \quad s \in [0, \mathcal{L}/2].$$

It is clear that

$$\begin{aligned} \int_0^{\mathcal{L}/2} |u^{**}(s)|^2 ds = \frac{1}{2} \int_0^{\mathcal{L}} |u^*(s)|^2 ds = \frac{1}{2} \int_\Gamma |u_0(x)|^2 dx, \\ \int_0^{\mathcal{L}/2} |(u^{**})'(s)|^2 ds = 2 \int_0^{\mathcal{L}} |(u^*)'(s)|^2 ds \leq \frac{1}{2} \int_\Gamma |u_0'(x)|^2 dx, \end{aligned}$$

and the ground state λ_0 for Laplacian on $[0, \mathcal{L}/2]$ with Robin conditions (13) at the end points provides a lower estimate

$$\begin{aligned} \lambda_0 & \leq \frac{\int_0^{\mathcal{L}/2} |u^{**\prime}(x)|^2 dx + I_+/2\ |u^{**}(0)|^2 - I_-/2\ |u^{**}(\mathcal{L}/2)|^2}{\int_0^{\mathcal{L}/2} |u^{**}(x)|^2 dx} \\ & \leq \frac{1/2 \left(\int_\Gamma |u_0'(x)|^2 dx + I_+\ |u_0(x_{\min})|^2 - I_- |u_0(x_{\max})|^2 \right)}{1/2 \int_\Gamma |u_0(x)|^2 dx} \\ & \leq \lambda_0(L). \end{aligned} \tag{25}$$

Note that this proof involves both symmetrization and Eulerian cycle and shows how these two methods may be combined.

5. Bounds on higher eigenvalues $\lambda_n, \ n \geq 1$.

We first prove that the eigenvalues depend monotonically upon the potential q and strengths of the delta interactions α_m. Monotonicty of the eigenvalues follow from the monotonicity of the corresponding quadratic forms.

Proposition 1. *Each of the eigenvalues of a Schrödinger operator on a metric graph Γ with δ-conditions at the vertices depend positively (non-negatively) on the strengths of each of the matching conditions, and the value of the potential on the* *edges. In other words let L_q and $\widehat{L}_{\widehat{q}}$ be two Schrödinger operators sharing the same underlying metric graph and such that*

$$\alpha_m \leq \widehat{\alpha}_m \quad m = 1, 2, \ldots, M, \tag{26}$$

$$q(x) \leq \widehat{q}(x) \quad \forall x \in \Gamma, \tag{27}$$

then

$$\lambda_n(L_q) \leq \lambda_n(\widehat{L}_{\widehat{q}}) \tag{28}$$

for any $n = 0, 1, 2, \ldots$

Proof. Let us remind that the domains of the quadratic forms for L and $\widehat{L}_{\widehat{q}}$ coincide and the following obvious inequality holds

$$\begin{aligned} Q_{L_q}(u,u) &= \int_\Gamma |u'(x)|^2 + \int_\Gamma q(x)|u(x)|^2 dx + \sum_{m=1}^{M} \alpha_m |u(V_m)|^2 \\ &\leq \int_\Gamma |u'(x)|^2 + \int_\Gamma \widehat{q}(x)|u(x)|^2 dx + \sum_{m=1}^{M} \widehat{\alpha}_m |u(V_m)|^2 \quad = \quad Q_{\widehat{L}_{\widehat{q}}}(u,u), \end{aligned} \tag{29}$$

for any admissible function u from the domain of the quadratic form. Taking into account that the n-th eigenvalue can be estimated via the Rayleigh quotient (10) as before. Then (28) follows from the inequality for the quadratic forms, since the sets of admissible functions are equal. □

Let us continue our analysis assuming that the interactions are nonnegative, *i.e.* $I_- = 0$. This means that the potential is nonnegative and all strengths of vertex interactions are nonnegative.) The proof extends upon a result of L. Friedlander in [6] which gives a lower bounds for Laplace operators on metric graphs with standard vertex conditions. In our notations this case corresponds to $I_- = I_+ = 0$.

Proposition 2 (following Theorem 1 from [6])**.** *The eigenvalues λ_n, $n = 0, 1, \ldots,$ of the Laplace operator L on a metric graph Γ, of total length $\mathcal{L}$ and with standard vertex conditions is bounded from below by*

$$\lambda_n(L) \geq \left(\frac{(n+1)\pi}{2\mathcal{L}} \right)^2, \; n = 1, 2, 3... \tag{30}$$

with equality only when Γ is the star graph with $n+1$ edges of the same length if $n \geq 2$, and equality for $n = 1$ only if Γ is the single interval of length $\mathcal{L}$.

Proof. A proof of this proposition using symmetrization technique can be found in [6]. We present here a sketch of the proof using Eulerian cycles, but for odd eigenvalues only, the same method applied to even eigenvalues does not give the best possible estimate. We follow closely the proof of Theorem 1 presented in Section 3. Let Γ be any finite compact metric graph, $\Gamma^{(2)}$ - its "double" cover, and $\widehat{\Gamma}$ - an Eulerian cycle. As before we have the following estimates

$$\lambda_n(L) \geq \lambda_n(L^{(2)}) \geq \lambda_n(\widehat{L}) = \frac{\pi^2}{\mathcal{L}^2} \left[\frac{n+1}{2} \right]^2,$$

where the last equality holds, since $\widehat{L}$ is nothing else than the Laplacian on the loop of length $2\mathcal{L}$. The integer part $\left[\frac{n+1}{2}\right]$ can be simplified if n is odd leading precisely to estimate (30). For even n we get $\lambda_n(L) \geq \left(\frac{n\pi}{2\mathcal{L}}\right)^2$ instead of (30) which is not optimal. □

Precisely the same bound can be proven to hold for any quantum graph with $I_- = 0$.

Theorem 3. *Let L_q be a Schrödinger operator on a finite compact metric graph Γ of total length $\mathcal{L}$ with delta vertex conditions. If the interaction is nonnegative ($I_- = 0$), then the n-th eigenvalue $\lambda_n(L_q)$, $n \geq 1$, is bounded from below by the n-th eigenvalue of the regular star graph with $n+1$ edges and standard conditions at all vertices, i.e. inequality*

$$\lambda_n(L_q) \geq \left(\frac{(n+1)\pi}{2\mathcal{L}}\right)^2, \; n = 1, 2, 3... \tag{31}$$

holds. The equality is realized only for the regular star graph with $n+1$ edges, $q(x) \equiv 0$, and a δ-condition of strength I at the middle vertex and standard conditions at all other vertices.

Proof. Let us denote by L the Laplace operator on Γ defined by standard vertex conditions. Then by Theorem 1

$$\lambda_n\left(L_q\right) \geq \lambda_n(L),$$

since the interaction is nonnegative ($q(x) \geq 0$, $\alpha_m \geq 0$). Then estimate (31) follows easily from Friedlander's result (Proposition 2).

The important question is whether the bound is sharp, *i.e.* whether there exists a graph with length $\mathcal{L}$ and a Schrödinger operator of nonnegative strength I for which the inequality becomes an equality.

The eigenvalues of the regular star graph with $n+1$ edges, with $q(x) \equiv 0$, and with $\alpha_m = 0$ for all V_m but the middle vertex V_1 where $\alpha_1 = I$, can be calculated using straightforward calculations. The first excited eigenvalue has multiplicity n and therefore the eigenvalue λ_n

$$\lambda_n = \left(\frac{(n+1)}{2}\right)^2 \left(\frac{\pi}{\mathcal{L}}\right)^2.$$

The corresponding eigenfunction is equal to zero at the central vertex and therefore does not feel the delta interaction there. As the result $I = \alpha_1$ does not enter the estimate, which coincides with the estimate obtained by Friedlander for the case of standard Laplacian. It was also proven in [6] that the graph minimizing the n-th eigenvalue of the standard Laplacian is unique. We get automatically uniqueness for the Schrödinger operator as well, since any graph minimizing the n-th eigenvalue of the Schrödinger operator with nonnegative interactions minimizes also the n-th eigenvalue of the standard Laplacian. □

It is not possible to find an upper bound for each eigenvalue for a general quantum graph only restricted by a given length and a given interaction. The first and the n:th eigenvalue of the regular star graph with $n+1$ edges, and the interaction concentrated to the middle vertex, are the same and are given by $\left(\frac{(n+1)}{2}\right)^2 \left(\frac{\pi}{\mathcal{L}}\right)^2$. If we let the number of edges go to infinity we see that also λ_1 goes to infinity, so no upper bound can exist. If we however take into account the specific set of edges $\{E_n\}_{n=1}^N$ of the graph, and the potential on them, then the upper bound is given by the corresponding eigenvalue for the Schrödinger operator on the flower graph Γ_{F} obtained from the original graph Γ by identifying all its vertices

$$\lambda_n(L_q(\Gamma)) \leq \lambda_n(L_q(\Gamma_{\mathrm{F}})). \tag{32}$$

This follows directly from Theorem 2 if one assumes that the strength of the delta interaction in the unique vertex of the flower graph is taken equal to the sum of stregths in the original graph $\sum_{m=1}^M \alpha_n$ keeping the total interaction unchanged.

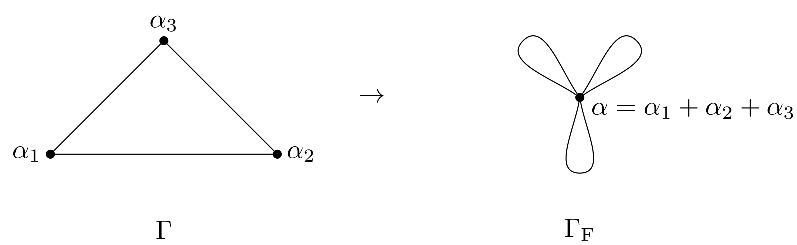

FIGURE 1. An example of a graph and its corresponding flower graph.

6. Conclusions

We have compared the two methods to obtain explicit estimates for eigenvalues for quantum graphs. The symmetrization technique allows one to obtain effective estimates not only for the ground state, but for all higher eigenvalues. Eulerian cycle approach provides an interesting insight to the problem, but applied to higher eigenvalues it gives not the best estimates for all even eigenvalues. It might be interesting to see how this method can be generalized to get estimates for λ_{2n}, $n = 1, 2, \ldots$

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