Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

On vertex conditions for elastic systems

Jean-Claude Kiik, Pavel Kurasov *, Muhammad Usman

Department of Mathematics, Stockholm University, SE-10691, Stockholm, Sweden

ARTICLE INFO

Article history:
Received 12 January 2015
Received in revised form 7 May 2015
Accepted 8 May 2015
Available online xxxx
Communicated by P.R. Holland

Keywords:
Fourth-order differential operator
Matching conditions
Elasticity

ABSTRACT

In this paper vertex conditions for the differential operator of fourth derivative on the simplest metric graph – the Y-graph, – are discussed. In order to make the operator symmetric one needs to impose extra conditions on the limit values of functions and their derivatives at the central vertex. It is shown that such conditions corresponding to the free movement of beams depend on the angles between the beams in the equilibrium position.

1. Motivation

Quantum graphs – differential operators on metric graphs – form a well-established formalism to model wave propagation in thin channels and other physical systems where interesting phenomena occur in a neighborhood of a system of low-dimensional manifolds [1,12,21]. It appears that ordinary differential equations on graph edges coupled by certain vertex conditions give a rather good approximation of wave propagation in thin domains. Intensive research is devoted to understanding which particular vertex conditions give the best approximation. The set of all possible vertex conditions can be described using different mathematical languages [15,10,11,14,16,2]. One may expect that the geometry (first of all the angles between the edges) of the vertex should be reflected in these conditions. To study this question it is enough to consider the star graph and in this article we are going to restrict our consideration to the star graph Γ formed by 3 semi-infinite edges $[x_j, \infty)$, $j = 1, 2, 3$. There is just one vertex $V = \{x_1, x_2, x_3\}$.

In physics it is common to use quadratic forms instead of operators, since often it is easier to check that a quadratic (sesquilinear) form is semi-bounded and closed instead of proving that an operator is self-adjoint. The quadratic form corresponding to the Laplacian on Γ is given by the integral of $|u'(x)|^2$ plus possibly an additional term coming from the vertex. This additional term describes the interaction between the waves at the vertex and should be absent if one is looking for conditions corresponding to the free motion. We end up with the quadratic form

$$Q[u, v] = \int_\Gamma u'(x)\overline{v'(x)}dx = \sum_{j=1}^{3} \int_{x_j}^{\infty} u'(x)\overline{v'(x)}dx. \tag{1}$$

This form is positive and closed on the set of functions from the Sobolev space $W_2^1(\Gamma \setminus V) = \bigoplus_{j=1}^3 W_2^1(x_j, \infty)$. If no further conditions at the vertex are assumed, then the corresponding self-adjoint operator is the Laplace operator $L_{II} = -\frac{d^2}{dx^2}$ defined on the functions from $W_2^2(\Gamma \setminus V)$ satisfying Neumann conditions $u'(x_j) = 0$. The three edges are then independent of each other. Therefore the only way to introduce coupling between the edges is through restricting the domain of the quadratic form by certain conditions on the limiting values of the function at the vertex (to ensure that the operator is local). Since the functions from W_2^1 are continuous, but their derivatives are not necessarily continuous, these vertex conditions may involve only the values of the function at the end points $u(x_j)$. Here and in what follows we use the limiting values of functions and their derivatives from inside the edges defined as

$$u^{(n)}(x_j) := \lim_{\epsilon \searrow 0} u^{(n)}(x_j + \epsilon). \tag{2}$$

If all endpoints are equivalent, then it is natural to impose the continuity condition

$$u(x_1) = u(x_2) = u(x_3). \tag{3}$$

The self-adjoint operator corresponding to the quadratic form (1) defined on functions from $W_2^1(\Gamma \setminus V)$ satisfying (3) is precisely the Laplace operator L_{II} with the domain given by the standard conditions

* Corresponding author.
E-mail address: kurasov@math.su.se (P. Kurasov).

http://dx.doi.org/10.1016/j.physleta.2015.05.017
0375-9601/© 2015 Elsevier B.V. All rights reserved.

$$\begin{cases} u \text{ is continuous at the vertex } V; \\ \sum_{x_j \in V} u'(x_j) = 0. \end{cases} \tag{4}$$

Another possibility to restrict the quadratic form is to introduce Dirichlet conditions

$$u(x_j) = 0, \;\; x_j \in V \tag{5}$$

instead of (3). The corresponding operator is self-adjoint but there is no coupling between the edges.

We have seen that the derived vertex conditions do not contain any information about the geometry of the junction, especially the angles between the edges are not reflected. In what follows we turn to problems of elasticity and will show how the geometry of the junction may be reflected in vertex conditions.

2. Elasticity

Elasticity problems are usually described by fourth order differential operators [19]. Consider therefore the differential operator $L_{IV} := \frac{d^4}{dx^4}$ first on the interval $[x_0, \infty)$ and later on the graph Γ. The corresponding wave equation

$$\frac{\partial^4}{\partial x^4} u = -\frac{\partial^2}{\partial t^2} u$$

can be used to describe the dynamics of small deflection of beams. Thus the operator L_{IV} in $L^2(x_0, \infty)$ corresponds to a single long beam. The operator can be made self-adjoint by choosing appropriate boundary conditions at the origin. For example, in the case when the end point of the beam is clamped the corresponding self-adjoint boundary condition is

$$u(x_0) = u'(x_0) = 0.$$

And in the case when the end point is free we can impose the following condition to make the operator self-adjoint

$$u''(x_0) = u'''(x_0) = 0. \tag{6}$$

One can describe all possible boundary conditions at the end point x_0 by the following families of matching conditions (see [23]):

0) $u(x_0) = u'(x_0) = 0,$

i) $u(x_0) = 0, \qquad u''(x_0) = \gamma_1 u'(x_0),$

ii) $u'(x_0) = \gamma_3 u(x_0), \qquad -u'''(x_0) + \overline{\gamma_3} u''(x_0) = \gamma_2 u(x_0),$

iii) $\begin{cases} u''(x_0) = \gamma_3 u(x_0) + \gamma_1 u'(x_0) \\ u'''(x_0) = -\gamma_2 u(x_0) - \overline{\gamma_3} u'(x_0), \end{cases}$

where γ_1, γ_2 are arbitrary real numbers and γ_3 is an arbitrary complex number. This formalism can be carried out to the case where several beams are connected together at one junction. To describe all possible vertex conditions leading to self-adjoint operators one may use either von Neumann theory [22], boundary triples [8] or scattering matrix approach [12–14]. Our goal here is not to carry out such description, but to derive conditions corresponding to the free dynamics of beams.

Consider first the case where just two beams are connected. Assume that the equilibrium position corresponds to the case when the two beams form a line. Such beams may be parameterized as $(-\infty, x_0]$, and $[x_0, \infty)$. The quadratic form corresponding to the free motion is given by

$$\begin{aligned} Q[u, v] &= \int_{-\infty}^{\infty} u''(x) \overline{v''(x)} dx \\ &= \int_{-\infty}^{x_0} u''(x) \overline{v''(x)} dx + \int_{x_0}^{+\infty} u''(x) \overline{v''(x)} dx. \end{aligned} \tag{7}$$

As before it is natural to assume that u is continuous at x_0, otherwise the beams do not touch each other:

$$u(x_0 - 0) = u(x_0 + 0). \tag{8}$$

The beams are free, no external force is applied if the graph of $u(x)$ does not form any angle at x_0, *i.e.* the first derivative is continuous:

$$u'(x_0 - 0) = u'(x_0 + 0). \tag{9}$$

The quadratic form on the domain of functions from $W_2^2(\mathbb{R} \setminus x_0)$ satisfying conditions (8) and (9) is semi-bounded and closed. Let us calculate the self-adjoint operator corresponding to this quadratic form. First of all one needs to determine the set of all u, for which the quadratic form $Q[u, v]$ is a bounded linear functional with respect to v. Taking v from $C_0^\infty(\mathbb{R} \setminus x_0)$ we see that the second derivative of the function u'' should be from $L_{2,\mathrm{loc}}(\mathbb{R} \setminus x_0)$. In other words $u \in W_2^4(\mathbb{R} \setminus x_0)$ and one may carry out integration by parts:

$$\begin{aligned} Q[u, v] = &\int_{-\infty}^{x_0} u^{(iv)}(x) \overline{v(x)} dx + \int_{x_0}^{+\infty} u^{(iv)}(x) \overline{v(x)} dx \\ &- u''(x_0 + 0) \overline{v'}(x_0 + 0) + u'''(x_0 + 0) \overline{v}(x_0 + 0) \\ &+ u''(x_0 - 0) \overline{v'}(x_0 - 0) - u'''(x_0 - 0) \overline{v}(x_0 - 0) \\ = &\int_{-\infty}^{x_0} u^{(iv)}(x) \overline{v(x)} dx + \int_{x_0}^{+\infty} u^{(iv)}(x) \overline{v(x)} dx \\ &- (u''(x_0 + 0) - u''(x_0 - 0)) \overline{v'}(x_0) \\ &+ (u'''(x_0 + 0) - u'''(x_0 - 0)) \overline{v}(x_0), \end{aligned} \tag{10}$$

where on the last step we used that the function v satisfies (8) and (9). The integral terms are bounded linear functionals with respect to v, while the functionals

$$v \mapsto v(x_0) \text{ and } v \mapsto v'(x_0)$$

are not. It follows that the coefficients in front of these functionals must be equal to zero

$$\begin{cases} u''(x_0 - 0) = u''(x_0 + 0), \\ u'''(x_0 - 0) = u'''(x_0 + 0). \end{cases} \tag{11}$$

The operator $\frac{d^4}{dx^4}$ defined on the set of functions from $W_2^4(\mathbb{R} \setminus x_0)$ satisfying matching conditions (8), (9) and (11) is the self-adjoint operator corresponding to the quadratic form. As in the case of Laplacian the vertex x_0 can be removed and the two edges may be substituted by one edge $(-\infty, \infty)$.

Our goal is to understand how the matching conditions (8), (9), (11) can be generalized to the case when several beams are joined at one vertex, having different angles in-between in the equilibrium position. The function u describing small deflections of the system from the equilibrium can be considered as a function on the graph Γ made up of three half-lines connected at the vertex V and making angles α, β and γ, none of which is equal to 0 or π (see Fig. 1). The special cases when one of the angles is equal to 0 or π is considered in Section 4. The Hilbert space is $L_2(\Gamma) := \bigoplus_{i=1}^3 L_2[x_j, \infty)$ and we are interested in vertex

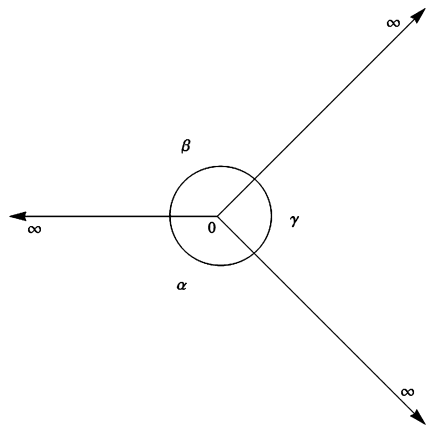

Fig. 1. A Y-graph with certain angles between the branches.

conditions for the operator of fourth differentiation $L_{IV} = \frac{d^4}{dx^4}$ corresponding to the free motion of beams. We shall use the limiting values of the function and its derivatives defined by (2).

The literature devoted to this problem is rather limited especially using rigorous mathematical language. We have found the articles [6,7], where the following two types of vertex conditions are considered without giving much motivation (Eqs. (2)–(5) from [6] and (4)–(8) from [7] adopted to our notations)

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_1) + u'(x_2) + u'(x_3) = 0, \\ u''(x_1) = u''(x_2) = u''(x_3), \\ u'''(x_1) + u'''(x_2) + u'''(x_3) = 0, \end{cases} \tag{12}$$

and

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_1) = u'(x_2) = u'(x_3), \\ u''(x_1) + u''(x_2) + u''(x_3) = 0, \\ u'''(x_1) + u'''(x_2) + u'''(x_3) = 0. \end{cases} \tag{13}$$

Probably the main motivation to use these vertex conditions is just the analogy with the standard conditions (4) widely used for the second order differential operators. Another type of vertex conditions is used in [20] (Eqs. (1.3.16), (1.3.17), (1.3.19) and (1.3.21) again adopted to our notations and changing the number of joined together beams from 4 to 3)

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ -u'(x_1) = u'(x_2) = u'(x_3), \\ -u''(x_1) + u''(x_2) + u''(x_3) = 0, \\ u'''(x_1) + u'''(x_2) + u'''(x_3) = 0. \end{cases} \tag{14}$$

Our goal is to derive vertex conditions corresponding to different possible angles between the beams in the equilibrium. It appears that the angles between the beams make their way into the matching conditions in contrast to the case of second order differential operator. All presented above vertex conditions appear as special cases of the vertex conditions to be derived, provided the angles between the beams are properly chosen. For example conditions (12) correspond to the case where all three angles are equal and conditions (13) and (14) – to the case, where at least two beams are parallel. See Section 4 for details.

Here is the main result of our paper:

Theorem 1. *The free system of three beams having angles α, β, and γ in the equilibrium position in a plane (see Fig. 1) is described by the fourth order differential operator L_{IV}, on the domain of functions from the Sobolev space $W_2^4(\Gamma\backslash V) := \bigoplus_{i=1}^3 W_2^4(x_i, \infty)$ satisfying the following vertex conditions*

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ \sin\alpha \cdot u'(x_1) + \sin\beta \cdot u'(x_2) + \sin\gamma \cdot u'(x_3) = 0, \\ \dfrac{u''(x_1)}{\sin\alpha} = \dfrac{u''(x_2)}{\sin\beta} = \dfrac{u''(x_3)}{\sin\gamma}, \\ \sum\limits_{j=1}^{3} u'''(x_j) = 0, \end{cases} \tag{15}$$

provided none of the angles α, β and γ is equal to 0 or π.

Note that it is not our intention to describe all possible self-adjoint vertex conditions for the differential operator L_{IV}. We are discussing the most natural vertex conditions. We expect that precisely these conditions appear if one considers approximation of the system by narrow channels. We plan to return back to this question in one of our future publications.

3. Derivation of the matching conditions

We consider the system of three beams joined together at the vertex V. We assume that in the equilibrium the beams are placed in a plane forming graph Γ presented in Fig. 1. Small displacements of the beams in the direction perpendicular to the plane are described by the function $u(x)$, $x \in \Gamma$. If no external forces are applied to the system of beams, then at every point $x_0 \in \Gamma$ there is a tangential plane to the graph of u. As we already seen if x_0 is an internal point on one of the edges, then the requirement that there exists a tangential plane (line) at x_0 leads to conditions (8), (9), and (11).

Our main goal now is to study the vertex of Γ and matching conditions there. We require that there is a tangential plane to the graph of u even at the vertex V. Such plane may exist only if the function u is continuous there

$$u(x_1) = u(x_2) = u(x_3). \tag{16}$$

Another additional condition on the first derivative of functions is obtained by requiring that the tangential lines to the graph of the function u at the point V lie in the same plane. Observe that the graph of u consists of three curves, since the function is defined on the Y-graph. To find the tangent vectors along each edge we assume the Y-graph is lying in xy-plane in $\mathbb{R}^3$ as shown in Fig. 2. It is natural to choose one of the axis along one of the beams in equilibrium. Then, the tangent vectors $\vec{a}^1$, $\vec{a}^2$ and $\vec{a}^3$ to the graph of the function u are given by

$$\begin{aligned} \vec{a}^1 &= (-\cos(\pi - \beta), \sin(\pi - \beta), u'(x_1)) = (\cos\beta, \sin\beta, u'(x_1)), \\ \vec{a}^2 &= (-\cos(\pi - \alpha), -\sin(\pi - \alpha), u'(x_2)) \\ &= (\cos\alpha, -\sin\alpha, u'(x_2)), \\ \vec{a}^3 &= (1, 0, u'(x_3)). \end{aligned}$$

These three tangent vectors lie in the same plane if and only if the following determinant is equal to zero

$$\det\begin{pmatrix} \cos\beta & \sin\beta & u'(x_1) \\ \cos\alpha & -\sin\alpha & u'(x_2) \\ 1 & 0 & u'(x_3) \end{pmatrix} = 0$$

$$\Leftrightarrow \sin\alpha \cdot u'(x_1) + \sin\beta \cdot u'(x_2) - (\cos\beta \cdot \sin\alpha + \sin\beta \cdot \cos\alpha) \cdot u'(x_3) = 0.$$

Using the trigonometric reduction $\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta = \sin(\alpha + \beta)$ we get the following matching condition

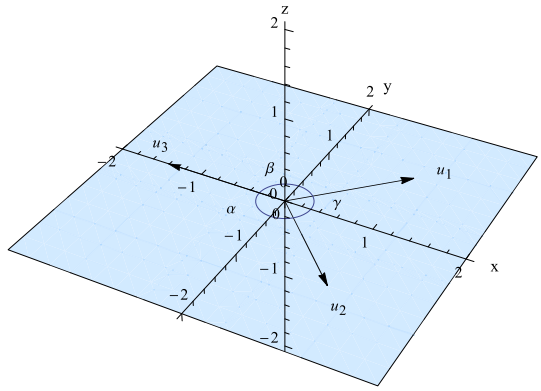

Fig. 2. Y-graph in $\mathbb{R}^3$ with angles α, β and γ.

$$\sin\alpha \cdot u'(x_1) + \sin\beta \cdot u'(x_2) - \sin(\alpha+\beta) \cdot u'(x_3) = 0.$$

Further, noting that $\sin(\alpha+\beta) = \sin(2\pi-\gamma) = -\sin\gamma$ this condition can be written as

$$\sin\alpha \cdot u'(x_1) + \sin\beta \cdot u'(x_2) + \sin\gamma \cdot u'(x_3) = 0.$$

Thus, so far we have the following three matching conditions

$$\begin{cases} u(x_1) = u(x_2) = u(x_3) \\ \sin\alpha \cdot u'(x_1) + \sin\beta \cdot u'(x_2) + \sin\gamma \cdot u'(x_3) = 0. \end{cases} \tag{17}$$

Proof of Theorem 1. With the differential expression $\frac{d^4}{dx^4}$ one associates two differential operators: the maximal and the minimal ones. It is standard to define the *minimal* operator, $L^{\min}$, on smooth functions with the support separated from the vertex, that is:

$$\mathrm{Dom}(L^{\min}) = C_0^\infty(\Gamma\backslash V) := \bigoplus_{i=1}^{3} C_0^\infty(x_i, \infty).$$

Using integration by parts, it is easy to see that the minimal operator is symmetric.

The *maximal* operator, $L^{\max}$, is defined on the domain of all functions from the Hilbert space $L^2(\Gamma)$ such that their images under the differential operator are still in the Hilbert space, *i.e.*

$$\mathrm{Dom}(L^{\max}) = W_2^4(\Gamma\backslash V) = \bigoplus_{i=1}^{3} W_2^4(x_i, \infty).$$

It consists of all square integrable functions having square integrable fourth generalized derivatives. The maximal operator is not self-adjoint, or even symmetric.

Any self-adjoint operator corresponding to the graph Γ should have a domain lying inside the domain of the maximal operator, but should definitely contain the domain of the minimal one. The differential operator $\frac{d^4}{dx^4}$ can be made self-adjoint by restricting the maximal operator with the help of certain matching conditions at the vertex. Hence the following inclusion for the domains of the minimal and maximal operators hold

$$\mathrm{Dom}(L^{\min}) \subset \mathrm{Dom}(L) \subset \mathrm{Dom}(L^{\max}),$$

where L is any self-adjoint operator associated with $\frac{d^4}{dx^4}$.

Let us consider the quadratic form

$$Q[u,v] = \int_\Gamma u''(x)\overline{v''}(x)\,dx = \sum_{i=1}^{3}\int_{x_i}^{\infty} u''(x)\overline{v''}(x)\,dx \tag{18}$$

defined on the domain of functions from $W_2^2(\Gamma \setminus V)$ satisfying vertex conditions (17). Clearly, $Q[u,u] \geq 0$ and closed. Therefore, there exists a self-adjoint operator L such that $Q[u,v] = \langle Lu, v\rangle$. If $u \in \mathrm{Dom}(L)$ then by Riesz-representation theorem there exists a function f in $L^2(\Gamma)$ such that $\langle Lu, v\rangle = \langle f, v\rangle$ and $\langle f, v\rangle$ is a bounded linear functional with respect to v.

It is clear that every $u \in \mathrm{Dom}(L)$ is from $W_2^4(\Gamma\backslash V)$. (This can be derived by considering $v \in \bigoplus_{j=1}^{3} C_0^\infty(x_j, \infty)$.) Integrating twice by parts on the right-hand side of (18) we get

$$Q[u,v] = \sum_{i=1}^{3}\left(u'''(x_i)\overline{v(x_i)} - u''(x_i)\overline{v'(x_i)}\right) + \int_\Gamma u^{(iv)}(x)\overline{v}(x)\,dx. \tag{19}$$

Now, using the conditions (17) for the function v and assuming $\alpha, \beta, \gamma \neq 0, \pi$ we arrive at

$$Q[u,v] = \left(\sum_{i=1}^{3} u'''(x_i)\right)\overline{v(x_1)} + \left(u''(x_3)\frac{\sin\alpha}{\sin\gamma} - u''(x_1)\right)\overline{v'(x_1)} + \left(u''(x_3)\frac{\sin\beta}{\sin\gamma} - u''(x_2)\right)\overline{v'(x_2)} + \int_\Gamma u^{(iv)}(x)\overline{v}(x)\,dx.$$

$Q[u,v]$ is a bounded linear functional with respect to v if and only if the coefficients in front of $\overline{v(x_1)}$, $\overline{v'(x_1)}$ and $\overline{v'(x_2)}$ are equal to 0. This gives us the following three more conditions

$$\begin{cases} \sin\gamma \cdot u''(x_1) - \sin\alpha \cdot u''(x_3) = 0, \\ \sin\gamma \cdot u''(x_2) - \sin\beta \cdot u''(x_3) = 0, \\ \sum_{j=1}^{3} u'''(x_j) = 0. \end{cases} \tag{20}$$

The above set of conditions along with (17) make the operator L_{IV} self-adjoint defined on the Sobolev class $W_2^4(\Gamma\backslash V)$. This completes the proof of Theorem 1. □

4. Special cases and examples

In the proof of Theorem 1 it was assumed that none of the angles α, β and γ is equal to 0 or π. Let us consider now the case when one of the angles is π or 0.

Case 1. One of the angles is π, the other two angles are different from 0 and π.

Assume without loss of generality that $\gamma = \pi$ and $\alpha, \beta \neq 0, \pi$. The condition that the tangent vectors to the functions u at V lie in the same plane is equivalent to

$$u'(x_1) = -u'(x_2). \tag{21}$$

Observe that we get no restriction on $u'(x_3)$. Using this condition along with the continuity condition $u(x_1) = u(x_2) = u(x_3)$ into the quadratic form (19) we get

$$Q[u,v] = \left(\sum_{i=1}^{3} u'''(x_j)\right)\overline{v}(x_1) - (u''(x_1) - u''(x_2))\overline{v'(x_1)} - u''(x_3)\overline{v'(x_3)} + \int_\Gamma u^{(iv)}(x)\overline{v}(x)\,dx. \tag{22}$$

As before, it is a bounded linear functional with respect to v if and only if the coefficients in front of $\overline{v(x_1)}$, $\overline{v'(x_1)}$ and $\overline{v'(x_3)}$ are equal to 0. This gives us the following set of conditions

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_1) = -u'(x_2), \\ u''(x_1) = u''(x_2), \qquad u''(x_3) = 0, \\ \sum_{j=1}^{3} u'''(x_j) = 0. \end{cases} \tag{23}$$

Case 2. One of the angles is 0, the other two angles are different from 0 and π.

Assume without loss of generality that $\alpha = 0$ and $\beta, \gamma \neq 0, \pi$. The tangential plane at V exists only if the following conditions are satisfied:

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_2) = u'(x_3). \end{cases}$$

We get a bounded linear functional if and only if the following conditions are satisfied:

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_2) = u'(x_3), \\ u''(x_2) = -u''(x_3), \qquad u''(x_1) = 0, \\ \sum_{j=1}^{3} u'''(x_j) = 0. \end{cases} \tag{24}$$

Case 3. One of the angles is zero, the other two are equal to π.

Assume without loss of generality that $\alpha = 0$ and $\beta = \gamma = \pi$. The tangential plane exists only if the function is continuous and the first derivatives are the same if taken in the same direction

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ -u'(x_1) = u'(x_2) = u'(x_3). \end{cases}$$

Writing the quadratic form and determining the corresponding self-adjoint operator leads to vertex conditions (14).

Case 4. Two angles are zero, the third angle is equal to 2π.

Assume that $\alpha = \beta = 0$ and $\gamma = 2\pi$. Existence of the tangential plane gives us the geometric conditions:

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_1) = u'(x_2) = u'(x_3). \end{cases}$$

The vertex conditions determining the corresponding self-adjoint operator are then given by (13).

We have covered now all possible values for the angles between the three beams in a plane. Let us consider few enlightening examples.

Example 1. For the symmetric case when $\gamma = 2\delta$, with $\delta \neq 0, \pi$ and $\alpha = \beta = \pi - \delta$ our matching conditions read as

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_1) + u'(x_2) + 2\cos\delta \cdot u'(x_3) = 0, \\ 2\cos\delta \cdot u''(x_1) = 2\cos\delta \cdot u''(x_2) = u''(x_3), \\ \sum_{j=1}^{3} u'''(x_j) = 0. \end{cases}$$

Example 2. In the case when all three angles between the branches are equal, that is $\alpha = \beta = \gamma = \frac{2\pi}{3}$ we have the following set of six matching conditions

$$\begin{cases} u(x_1) = u(x_2) = u(x_3), \\ u'(x_1) + u'(x_2) + u'(x_3) = 0, \\ u''(x_1) = u''(x_2) = u''(x_3), \\ \sum_{j=1}^{3} u'''(x_j) = 0, \end{cases} \tag{25}$$

which coincide with conditions (12) considered earlier.

Note that, the differential operator L_{IV} with the matching conditions (25) is the square of the Laplace operator L_{II} subject to standard matching conditions (4). The only other cases for which $L_{IV} = L_{II}^2$ are when L_{IV} is accompanied with the matching conditions $u(x_i) = u''(x_i) = 0$ or $u'(x_i) = u'''(x_i) = 0$ for $i = 1, 2, 3$, or any combination of these conditions. The corresponding L_{II} operator satisfies the matching conditions $u(x_i) = 0$ and $u'(x_i) = 0$ respectively.

The operator L_{IV} with vertex conditions (25) is the only operator from the described family, which is square of a Laplace operator. It is easy to see that the vertex conditions corresponding to square of a Laplacian connect together separately the values of the functions and first derivatives on one hand and values of the second and third derivatives on the other hand. Moreover the conditions should be the same. Comparing the first and third lines in (15) we conclude that $\sin\alpha = \sin\beta = \sin\gamma$. It follows that $\alpha = \beta = \gamma = 2\pi/3$, since $\alpha + \beta + \gamma = 2\pi$ and $\alpha \neq \pi$, $\beta \neq \pi$, $\gamma \neq \pi$.

5. Conclusions and perspectives

We derived vertex conditions that appear as natural counterpart of free conditions in the case where the beams form nontrivial angles in the equilibrium position. We expect that precisely these conditions appear when approximating of beams by narrow channels is carried out similar to the quantum graph case (second order differential operator) [5,9,17,18]. It is even more interesting to construct similar approximations for the whole class of matching conditions following ideas developed in [4] for quantum graphs. Note that all vertex conditions considered earlier appear as special cases of the derived conditions, provided the angles are chosen properly. It is straightforward to generalize obtained conditions to the case where the number of beams is greater than 3. It is important to study spectral properties of differential operators on metric graphs with described vertex conditions which depend on the angles between the edges. One may require that the graphs considered are realizable in an Euclidean space, then changing of edge lengths lead to a change of angles and therefore even of vertex conditions. We expect to observe new phenomena, extending ideas developed in [3] for the second order systems. It is even more interesting to understand what is going on when the beams in the equilibrium do not lie in the same plane.

Acknowledgements

The authors would like to thank A. Watson for inspiring discussions as well as providing Ref. [20]. The work of P.K. was partially supported by the Swedish Research Council (Grant D0497301) and ZiF – Zentrum für interdisziplinäre Forschung, Bielefeld (Cooperation Group *Discrete and continuous models in the theory of networks*).

References

[1] G. Berkolaiko, P. Kuchment, Introduction to Quantum Graphs, Mathematical Surveys and Monographs, vol. 186, Amer. Math. Soc., Providence, RI, 2013, xiv+270 pp.

[2] T. Cheon, P. Exner, O. Turek, Tripartite connection condition for a quantum graph vertex, Phys. Lett. A 375 (2) (2010) 113–118.

[3] P. Exner, M. Jex, On the ground state of quantum graphs with attractive δ-coupling, Phys. Lett. A 376 (5) (2012) 713–717.

[4] P. Exner, O. Post, A general approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds, Commun. Math. Phys. 322 (1) (2013) 207–227.

[5] P. Exner, O. Post, Convergence of spectra of graph-like thin manifolds, J. Geom. Phys. 54 (1) (2005) 77–115.

[6] B. Dekoninck, S. Nicaise, Control of networks of Euler–Bernoulli beams, ESAIM Control Optim. Calc. Var. 4 (1999) 57–81 (electronic).

[7] B. Dekoninck, S. Nicaise, The eigenvalue problem for networks of beams, Linear Algebra Appl. 314 (1–3) (2000) 165–189.

[8] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991) 1–95.

[9] D. Grieser, Spectra of graph neighborhoods and scattering, Proc. Lond. Math. Soc. (3) 97 (3) (2008) 718–752.

[10] M. Harmer, Hermitian symplectic geometry and the factorization of the scattering matrix on graphs, J. Phys. A 33 (49) (2000) 9015–9032.

[11] M. Harmer, Hermitian symplectic geometry and extension theory, J. Phys. A 33 (50) (2000) 9193–9203.

[12] P. Kurasov, Quantum Graphs: Spectral Theory and Inverse Problems, in press.

[13] P. Kurasov, M. Enerbäck, Aharonov–Bohm ring touching a quantum wire: how to model it and to solve the inverse problem, Rep. Math. Phys. 68 (3) (2011) 271–287 (English summary).

[14] P. Kurasov, M. Nowaczyk, Geometric properties of quantum graphs and vertex scattering matrices, Opusc. Math. 30 (3) (2010) 295–309.

[15] V. Kostrykin, R. Schrader, Kirchhoff's rule for quantum wires, J. Phys. A 32 (4) (1999) 595–630.

[16] P. Kuchment, Quantum graphs. I. Some basic structures, in: Special Section on Quantum Graphs, Waves Random Media 14 (1) (2004) S107–S128.

[17] P. Kuchment, H. Zeng, Asymptotics of spectra of Neumann Laplacians in thin domains, in: Advances in Differential Equations and Mathematical Physics, Birmingham, AL, 2002, in: Contemp. Math., vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 199–213.

[18] P. Kuchment, H. Zeng, Convergence of spectra of mesoscopic systems collapsing onto a graph, J. Math. Anal. Appl. 258 (2) (2001) 671–700.

[19] L.D. Landau, E.M. Lifshitz, Course of Theoretical Physics, vol. 7, Theory of Elasticity, third edition, Pergamon Press, Oxford, 1986 (translated from Russian by J.B. Sykes and W.H. Reid).

[20] I. Mulins, Natural frequencies of tree shaped graphs, Master Thesis (11MAD300 – MMath), Loughborough Univ., 2012.

[21] O. Post, Spectral Analysis on Graph-Like Spaces, Lecture Notes in Mathematics, vol. 2039, 2012.

[22] M. Reed, B. Simon, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness, Academic Press, New York–London, 1975.

[23] D.R. Yafaev, Spectral and Scattering Theory of Fourth Order Differential Operators, Am. Math. Soc. Transl., Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 265–299.