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Rayleigh estimates for differential operators on graphs

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Abstract. We study the spectral gap, i.e. the distance between the two lowest eigenvalues for Laplace operators on metric graphs. A universal lower estimate for the spectral gap is proven and it is shown that it is attained if the graph is formed by just one interval. Uniqueness of the minimizer allows to prove a geometric version of the Ambartsumian theorem derived originally for Schrödinger operators.

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1. Introduction

Quantum graphs – coupled differential equations on metric graphs – proved to be an important class of differential operators which can be used to model a wide variety of physical phenomena: from wave propagation in microwave cavities to electron transmission in nano systems. Such studies, motivated by possible applications, intensified in the 80s; see [1], [5], [14], and [17]. It appeared that quantum graphs are an ideal model for chaotic phenomena (see [20], [21], and [27]); their properties are connected with nodal domains for eigenfunctions: see [2] and [4]. More about how to define general quantum graphs and the description of their spectral and transmission properties can be found in [19] and [26], and recent surveys [3], [22], and [23].

Differential equations on metric graphs are also interesting from purely mathematical point of view and exhibit unusual spectral phenomena; see [6], [15], [24], and [25]. Standard intuition does not always work and conventional methods lead sometimes to unexpected results. This is shown in recent studies of the ground state for quantum graphs carried out by P. Exner and M. Jex [12] and may be explained

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by the fact that quantum graphs possess properties of both ordinary and partial differential equations. The present article is a nice illustration to the last statement. More precisely, we study the spectral gap, i.e. the difference between the two lowest eigenvalues for the Laplacian on a finite compact metric graph. The spectral gap is an important parameter describing the stability of the corresponding dynamical system (described by a non-stationary equation). For discrete graphs the spectral gap is proven to be an important characterization of graph connectivity. The spectral gap for differential operators on metric graphs has been studied by L. Friedlander, who gave, using symmetrization method, not only a universal estimate for the spectral gap, but universal estimates for all eigenvalues as well.

The main aim of this paper is to give an explicit geometric derivation of the universal lower estimate for the spectral gap, originally obtained in [16]. In our approach the graph Γ is compared to the "ball" in $\mathbb{R}$. i.e. the interval of the same total length as Γ. Surprisingly such a comparison does not give the upper estimate for the spectral gap, but the lower one. It is natural to ask whether the graph minimizing the spectral gap is unique, provided the total length is fixed. Answering this question we prove a geometric analog of the classical Ambartsumian theorem. More precisely, we prove that essentially only the graph formed by one interval gives the lowest possible spectral gap. The geometric character of our approach allows us to prove a new estimate for the spectral gap for balanced graphs, i.e. graphs having even valencies of all vertices.

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2. Rayleigh estimate

Let Γ be a compact connected metric graph formed by a finite number of compact edges $E_n = [x_{2n-1}, x_{2n}], n = 1, 2, \ldots, N$. Let us denote by $L(\Gamma)$ the corresponding free Laplace operator defined on W_2^2-functions u satisfying standard matching/boundary conditions at every vertex:

$$\begin{cases} u \text{ is continuous,} \\ \text{the sum of normal derivatives is equal to zero.} \end{cases} \tag{1}$$

The operator $L(\Gamma)$ is self-adjoint and is completely determined by the metric graph Γ. One may assume, without loss of generality, that no vertex of valency 2 is present. Standard matching conditions for such vertices imply that the function and its first derivative (N.B. not the normal derivative) are continuous along the vertex. Every

such vertex can be removed and the attaching edges can be substituted by one edge with the length equal to the sum of lengths in the two original edges. The free Laplace operators on such graphs are unitary equivalent and there is no reason to distinguish the corresponding graphs.

The spectrum of the free Laplacian is nonnegative and consists of an infinite sequence of eigenvalues λ_j of finite multiplicity tending to $+\infty$. The lowest eigenvalue (the ground state) is zero $\lambda_0 = 0$ and the corresponding eigenfunction is just a constant function on Γ. The multiplicity of λ_0 is one, since the graph is connected. Our main interest here is the spectral gap, i.e. the distance between the first two eigenvalues. Since $\lambda_0 = 0$ the spectral gap coincides with λ_1.

The classical Rayleigh theorem states that the maximum for the gap between the lowest two eigenvalues of the Neumann Laplacian in a domain of fixed area is attained if the domain is a circle. One might expect that for differential operators on graphs the spectral gap attains its maximum for the graph being just one interval, provided the total length is fixed. On the contrary it appears that the Laplacian on a single interval can be used to estimate the spectral gap not from above, but from below. In other words, the spectral gap among all graphs of the same total length is minimal for the single interval. This fact shows once again that differential operators on graphs possess properties of both ordinary and partial differential operators.

The following theorem provides a universal lower estimate for the spectral gap in contrast to the Rayleigh theorem giving an upper estimate.

Theorem 1. *Let Γ be a connected finite metric graph with total length $\mathcal{L}(\Gamma)$ and let $L(\Gamma)$ be the corresponding free Laplace operator defined on the domain of functions satisfying standard matching conditions at the vertices. Consider the graph $\Delta_{\mathcal{L}(\Gamma)}$ formed by one interval of length $\mathcal{L}(\Gamma)$ and the corresponding Neumann Laplacian $L(\Delta_{\mathcal{L}(\Gamma)})$. The spectral gap for the Laplacian on Γ can be estimated as follows:*[1]

$$\lambda_1(\Gamma) \geq \lambda_1(\Delta_{\mathcal{L}(\Gamma)}) = \Big(\frac{\pi}{\mathcal{L}(\Gamma)}\Big)^2. \tag{2}$$

Proof. The first nontrivial eigenvalue of $L(\Gamma)$ can be calculated by minimizing the Rayleigh quotient

$$\lambda_1(\Gamma) = \min_{u \perp 1} \frac{\displaystyle\int_\Gamma |u'(x)|^2 dx}{\displaystyle\int_\Gamma |u(x)|^2 dx}, \tag{3}$$

where the minimum is taken over all functions u belonging to the Sobolev space W_2^1 on every edge and continuous on the whole Γ. The first eigenfunction ψ_1 is the

[1]As we already mentioned this estimate is a particular case of the estimate obtained in [16].

minimizer of (3) and therefore it satisfies

$$\lambda_1(\Gamma) = \frac{\displaystyle\int_\Gamma |\psi_1'(x)|^2 dx}{\displaystyle\int_\Gamma |\psi_1(x)|^2 dx}. \tag{4}$$

Consider the graph Γ^* – a certain "double cover" of Γ – obtained from Γ by doubling each edge. The new edges connect the same vertices as before, so that the set of vertices is preserved. The corresponding valencies are just doubled as well.

Any function u from $L_2(\Gamma)$ can be lifted up to a function $u^* \in L_2(\Gamma^*)$ in a symmetric way by assigning it the same values on any new pair of edges as on the original edge in Γ. More precisely, consider any edge $E_n \in \Gamma$ and let us denote by E_n' and E_n'' the corresponding edge pair in Γ^*. It is natural to use the same parametrization of the intervals E_n, E_n', and E_n''.Then we have

$$u^*|_{E_n'} = u^*|_{E_n''} = u|_{E_n}.$$

The function ψ_1^* obtained from ψ_1 in this way obviously satisfies

$$\lambda_1(\Gamma) = \frac{\displaystyle\int_{\Gamma^*} |\psi_1^{*\prime}(x)|^2 dx}{\displaystyle\int_{\Gamma^*} |\psi_1^*(x)|^2 dx},$$

where the numerator and denominator gain factor 2 compared to (4).

Every vertex in Γ^* has even valency and therefore there exists a closed (Eulerian) path $\mathcal{P}$ on Γ^* coming along every edge in Γ^* precisely one time; see [10] and [18]. The path goes through certain vertices several times, but we identify it with the loop $S_{2\mathcal{L}(\Gamma)}$ of length $2\mathcal{L}(\Gamma)$. The loop is a metric graph and we consider the corresponding Laplace operator $L(S_{2\mathcal{L}(\Gamma)})$. The ground state for $L(S_{2\mathcal{L}(\Gamma)})$ is again $\lambda_0 = 0$. Its first nontrivial eigenvalue can be calculated by minimizing the corresponding Rayleigh quotient. The set of trial functions consists of $W_2^1(S_{2\mathcal{L}(\Gamma)})$ functions having mean value zero. The set of trial functions can be increased by considering all continuous piece-wise W_2^1 functions. The corresponding minimizer will be the same as before, since every minimizer will have equal limits of the first derivative on different sides of possible points of discontinuity.

The function ψ_1^* defined originally on the graph Γ^* can be considered as a function on the loop $S_{2\mathcal{L}(\Gamma)}$. It is a continuous piece-wise W_2^1 function with zero mean value and therefore gives an upper estimate for the Laplacian eigenvalue on the loop

$$\lambda_1(S_{2\mathcal{L}}) \le \frac{\displaystyle\int_{S_{2\mathcal{L}}} |\psi_1^{*\prime}(x)|^2 dx}{\displaystyle\int_{S_{2\mathcal{L}}} |\psi_1^*(x)|^2 dx} = \lambda_1(\Gamma).$$

We obtain the result by noticing that

$$\lambda_1(S_{2\mathcal{L}}) = \lambda_1(\Delta_{\mathcal{L}}). \qquad \square$$

In other words, we have proven that the minimum of the spectral gap for metric graphs of fixed total length is attained when the graph is formed by just one interval.

The obtained estimate can be improved if the original graph Γ possesses special properties. For example, if we assume that all vertices are balanced, i.e. that they have even valency, then there is no need to consider the "double covering" and the Euler theorem can be applied to the graph Γ directly. We would like to note that balanced vertices were considered recently by P. Exner in connection with momentum operators on graphs [11]. It appeared that the momentum operator can be introduced on a graph Γ if and only if it is balanced. Another recent example where balanced vertices play an important role concerns asymptotics of resonances on graphs having several infinite leads attached to a compact part. It appears that the asymptotics is of Weyl type if and only if every external vertex (i.e. a vertex to which external leads are attached) is not balanced; see [13], [8], and [9]. Our result shows that if we know that the graph is balanced then the lower estimate for the spectral gap can be improved by a factor 4.

Theorem 2. *Let all assumptions of Theorem 1 be satisfied. Assume in addition that all vertices in Γ have even valencies. Then the spectral gap for the Laplacian on Γ can be estimated as follows:*

$$\lambda_1(\Gamma) \geq \lambda_1(\Delta_{\mathcal{L}(\Gamma)/2}) = \Big(\frac{2\pi}{\mathcal{L}(\Gamma)}\Big)^2. \tag{5}$$

Proof. The proof is almost identical to the one of Theorem 1. Let ψ_1 be the eigenfunction corresponding to the eigenvalue $\lambda_1(\Gamma)$. Since the vertices in Γ are balanced (the valencies are even) there exists a closed (Eulerian) path coming along each edge in Γ precisely once. The length of any such path is $\mathcal{L}(\Gamma)$ and the function ψ_1 can be identified with a unique function on $S_{\mathcal{L}(\Gamma)}$. The function ψ_1 in $L_2(S_{\mathcal{L}(\Gamma)})$ is a continuous piecewise W_2^1 function and can be used to estimate the first eigenvalue for the corresponding Laplacian

$$\lambda_1(S_{\mathcal{L}}) \leq \frac{\displaystyle\int_{S_{\mathcal{L}}} |\psi_1{}'(x)|^2 dx}{\displaystyle\int_{S_{\mathcal{L}}} |\psi_1(x)|^2 dx} = \lambda_1(\Gamma). \tag{6}$$

Taking into account that $\lambda_1(\Delta_{\mathcal{L}/2}) = \lambda_1(S_{\mathcal{L}})$ we get the estimate (5). $\square$

The second estimate shows that for balanced graphs the spectral gap is minimal if the graph is a loop, provided the total length is preserved.

Summing up our results we conclude that the minimum of the spectral gap is attained for graphs having minimal branching under allowed conditions. If no condition on the valency of vertices is imposed, then such graph is an interval. If one requires that the vertices are balanced, then the minimizer is a loop.

3. Geometric version of Ambartsumian theorem

The classical Ambartsumian theorem states that the Schrödinger operator on a finite interval has the same spectrum as the Neumann Laplacian if and only if the potential is identically equal to zero. This theorem has been generalized for the case of graphs in [7], where the spectrum of the Schrödinger and Laplace operators on the same metric graph were compared. Our goal here is to compare the spectra of Laplacians on two different graphs having the same total length. It appears that the spectral gap for $L(\Gamma)$ coincides with the spectral gap for Neumann Laplacian on the single interval of length $\mathcal{L}(\Gamma)$ if an only if Γ itself is an interval. Remember that we agreed to remove all vertices of valency 2 and therefore a series of chain-coupled intervals is identified with one interval of length equal to the sum of lengths in the series. It is interesting to note that the theorem does not require that all eigenvalues coincide (like in the classical Ambartsumian theorem), but just the first two (the ground state and λ_1).

Theorem 3. *Let $L(\Gamma)$ be the free Laplace operator on a connected finite compact metric graph Γ of total length $\mathcal{L}(\Gamma)$. Assume that the first (nonzero) eigenvalue of $L(\Gamma)$ coincides with the first (nonzero) eigenvalue of the Laplacian on the interval of length $\mathcal{L}(\Gamma)$*

$$\lambda_1(\Gamma) = \lambda_1(\Delta_{\mathcal{L}(\Gamma)}); \tag{7}$$

then the graph Γ coincides with the interval $\Delta_{\mathcal{L}(\Gamma)}$.

Proof. Consider the functions ψ_1 and ψ_1^* introduced in the proof of Theorem 1. These functions are defined on Γ and $S_{2\mathcal{L}}$ respectively. Since $\lambda_1(\Gamma) = \lambda_1(\Delta_{\mathcal{L}(\Gamma)})$ the function ψ_1^* itself is an eigenfunction for the Laplacian on the loop. Choosing proper parameterization of the loop this function just coincides with $\cos \frac{\pi}{\mathcal{L}} x$. The function ψ_1 can be reconstructed from ψ_1^* by gluing its values on intervals from the same pair. But the values of ψ_1^* cover the interval $[-1, 1]$ twice, implying that there exists just one way to glue points on the loop together to get Γ back. It follows that Γ is essentially just one interval. It might happen that Γ is formally given by a series of intervals, but then there exists just one way to glue these intervals together keeping ψ continuous and having $\psi' = 0$ at the end points. Since we agreed to remove vertices of valency 2 the unique graph Γ is the interval of length $\mathcal{L}(\Gamma)$. □

The last theorem implies that if the spectral gap for $L(\Gamma)$ coincides with the spectral gap for the single interval of the same total length, then all other eigenvalues coincide as well.

It is interesting to note that this theorem cannot be generalized by using higher eigenvalues instead of the spectral gap, i.e. the first eigenvalue. The second eigenvalue λ_2 for Laplacians on the interval $\Delta_{\mathcal{L}}$ and on the loop $S_{\mathcal{L}}$ coincide (all nonzero eigenvalues on the loop are double degenerate). The same holds for all even eigenvalues. Consider the graph Γ_1 shown in Figure 1.

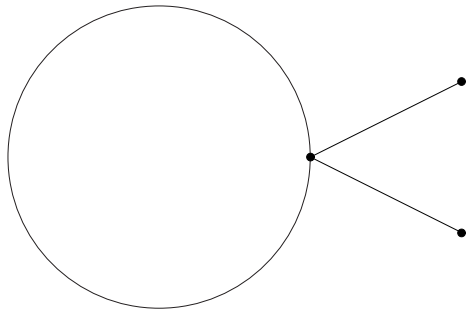

Figure 1. Graph Γ_1: a loop with two intervals attached.

We assume that the length of the loop is $2/3\mathcal{L}$ and the lengths of the outgrowths are $1/6\mathcal{L}$. The eigenvalues of the Laplacian on Γ_1 are

$$\lambda_0 = 0, \quad \lambda_1 = \left(\frac{2\pi}{\mathcal{L}}\right)^2, \quad \lambda_2 = \lambda_3 = \left(\frac{3\pi}{\mathcal{L}}\right)^2, \quad \dots .$$

We see that the third eigenvalue coincides with the third eigenvalue for the interval of the same length.

The theorem cannot be generalized directly to include balanced graphs. The graph Γ_2 shown in Figure 2 has the same spectral gap as the loop graph of the same total length, also the first eigenvalue is not degenerate.

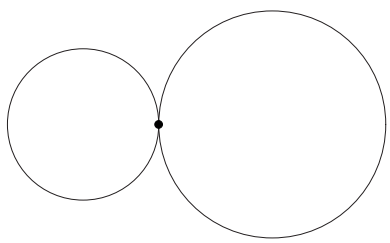

Figure 2. Graph Γ_2: two loops attached.

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