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Weyl–Titchmarsh-type formula for periodic Schrödinger operator with Wigner–von Neumann potential

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The Schrödinger operator on the half-line with periodic background potential perturbed by a certain potential of Wigner–von Neumann type is considered. The asymptotics of generalized eigenvectors for $\lambda \in \mathbb{C}_+$ and on the absolutely continuous spectrum is established. The Weyl–Titchmarsh-type formula for this operator is proven.

1. Introduction

Consider the one-dimensional Schrödinger operator with a real potential which can be represented as a sum of three terms: a certain periodic function, a Wigner–von Neumann potential and a certain absolutely integrable function. More precisely, let q be a real periodic function with period a such that $q \in L_1(0,a)$ and let $q_1 \in L_1(\mathbb{R}_+)$. Then the Schrödinger operator $\mathcal{L}_\alpha$ is defined by the differential expression

$$\mathcal{L}_\alpha := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + \frac{c\sin(2\omega x + \delta)}{(x+1)^\gamma} + q_1(x) \tag{1.1}$$

on the set of functions satisfying the boundary condition

$$\psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0, \tag{1.2}$$

where $c, \omega, \delta, \in \mathbb{R}$, $\alpha \in [0,\pi)$ and $\gamma \in (\frac{1}{2}, 1]$. As shown by Kurasov and Naboko in [16], the absolutely continuous spectrum of such an operator coincides as a set with the spectrum of the corresponding periodic operator on $\mathbb{R}$,

$$\mathcal{L}_{\mathrm{per}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x). \tag{1.3}$$

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Note that the spectrum of $\mathcal{L}_{\mathrm{per}}$ has multiplicity 2, while the spectrum of $\mathcal{L}_\alpha$ is simple. Let $\psi_+(x,\lambda)$ and let $\psi_-(x,\lambda)$ be Bloch solutions for $\mathcal{L}_{\mathrm{per}}$ and $\varphi_\alpha(x,\lambda)$ be the solution of the Cauchy problem

$$-\varphi_\alpha''(x,\lambda)+\left(q(x)+\frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma}+q_1(x)\right)\varphi_\alpha(x,\lambda)=\lambda\varphi_\alpha(x,\lambda),$$
$$\varphi_\alpha(0,\lambda)=\sin\alpha,\qquad \varphi_\alpha'(0,\lambda)=\cos\alpha.$$

The main result of the present paper is the following theorem, which relates the spectral density ρ_α' of the operator $\mathcal{L}_\alpha$ and the asymptotics of the solution φ_α. We call it the Weyl–Titchmarsh formula.

THEOREM 1.1. *Let $2a\omega/\pi\notin\mathbb{Z}$ and $q_1\in L_1(\mathbb{R}_+)$, then for almost all $\lambda\in\sigma(\mathcal{L}_{\mathrm{per}})$ there exists $A_\alpha(\lambda)$ such that*

$$\varphi_\alpha(x,\lambda)=A_\alpha(\lambda)\psi_-(x,\lambda)+\overline{A_\alpha(\lambda)}\psi_+(x,\lambda)+o(1)\quad \text{as } x\to+\infty \tag{1.4}$$

and

$$\rho_\alpha'(\lambda)=\frac{1}{2\pi|W\{\psi_+(\lambda),\psi_-(\lambda)\}||A_\alpha(\lambda)|^2}.$$

Weyl–Titchmarsh formulae provide an efficient tool to study the behaviour of the spectral density. The absolutely continuous spectrum of the operator $\mathcal{L}_\alpha$ contains infinitely many critical (resonance) points (see (2.1)) where the type of the asymptotics of generalized eigenvectors changes and is not given by a linear combination of ψ_+ and ψ_- (as in (1.4)). The embedded eigenvalues of $\mathcal{L}_\alpha$ may occur precisely at these points. In the generic case no eigenvalue occurs, but it is natural to suspect that the spectral density of $\mathcal{L}_\alpha$ vanishes at these points.

Vanishing of the spectral density divides the absolutely continuous spectrum into independent parts and has a clear physical meaning. This phenomenon is called the pseudogap. In [19] we study zeros of the spectral density in more detail.

The study of Schrödinger operators with Wigner–von Neumann potentials began with the classical paper [21], where it was observed for the first time that the potential $c\sin(2\omega x+\delta)/(x+1)$ may produce an eigenvalue inside the absolutely continuous spectrum. Later on, such operators attracted the attention of many authors (see, for example, [1–4, 6, 12–15, 17, 18]). A phenomenon of this nature, an embedded eigenvalue ('bound state in the continuum'), was even observed experimentally in semiconductor heterostructures [7].

The Weyl–Titchmarsh formula for the spectral density in the case of zero periodic background potential follows directly from the results of [17]. This formula was proved once again in [3], where the method of Harris–Lutz transformations [11] was used. In the present paper we also use a modification of this method. Another approach was suggested in [5], but again in the case of zero periodic background potential.

Let us briefly outline our method. The Weyl–Titchmarsh-type formula relates the spectral density to the coefficient in the asymptotics (1.4) of the solution φ_α. We derive this formula in § 5 following the standard strategy from [20] using the uniform (in a proper sense) asymptotics of the solution φ_α given by theorem 5.1. It is important to note that establishing the asymptotics only on the absolutely continuous spectrum was done in [16] and is not enough. The main technical difficulty is to establish the asymptotics both on $\sigma_{\mathrm{ac}}(\mathcal{L}_{\mathrm{per}})$ and $\mathbb{C}_+$ and to show that the coefficient in the asymptotics is analytic and has boundary limits. In § 3, we transform the eigenfunction equation to a linear differential system in $\mathbb{C}^2$ of the special (Levinson) form, which is easy to analyse. The central part of these transformations uses the uniform Harris–Lutz method [11] and is similar to the transformation used by Behncke in [3] ('complex $I + Q$'). Our method uses a different regularization of the formula in order to make it work both on the real line and in $\mathbb{C}_+$. In § 4 we formulate and prove a variation (rather standard) of the classical Levinson asymptotic theorem [8, lemma 4.2]. We use it in § 5 to find 'uniform' asymptotics of φ_α and prove the Weyl–Titchmarsh-type formula.

2. Preliminaries

The spectrum of $\mathcal{L}_{\mathrm{per}}$ consists of infinitely many intervals (see [9, theorem 2.3.1])

$$\sigma(\mathcal{L}_{\mathrm{per}}) := \bigcup_{j=0}^{\infty}([\lambda_{2j}, \mu_{2j}] \cup [\mu_{2j+1}, \lambda_{2j+1}]),$$

where

$$\lambda_0 < \mu_0 \leqslant \mu_1 < \lambda_1 \leqslant \lambda_2 < \mu_2 \leqslant \mu_3 < \lambda_3 \leqslant \lambda_4 < \cdots,$$

where λ_j and μ_j are the eigenvalues of the Schrödinger differential equation on the interval $[0, a]$ with periodic and antiperiodic boundary conditions. Spectral properties of $\mathcal{L}_{\mathrm{per}}$ are related to the entire function $D(\lambda)$ (discriminant) and the function $k(\lambda)$ (quasi-momentum), where

$$k(\lambda) := -\mathrm{i} \ln\left(\frac{\operatorname{tr} D(\lambda) + \sqrt{\operatorname{tr}^2 D(\lambda) - 4}}{2}\right).$$

We can choose the branch of $k(\lambda)$ so that (as follows from the properties of $D(\lambda)$, see [9, theorem 2.3.1])

$$\begin{aligned} &k(\lambda_0) = 0, \\ &k(\mu_0) = k(\mu_1) = \pi, \\ &k(\lambda_1) = k(\lambda_2) = 2\pi, \\ &\qquad\vdots \end{aligned}$$

where

$$\begin{aligned} &k(\lambda) \in \mathbb{R} \quad \text{if } \lambda \in \sigma(\mathcal{L}_{\mathrm{per}}), \\ &k(\lambda) \in \mathbb{C}_+ \quad \text{if } \lambda \in \mathbb{C}_+. \end{aligned}$$

The eigenfunction equation for $\mathcal{L}_{\mathrm{per}}$,

$$-\psi''(x) + q(x)\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R},$$

has two solutions (Bloch solutions) $\psi_+(x,\lambda)$ and $\psi_-(x,\lambda)$ satisfying quasi-periodic conditions

$$\psi_+(x+a,\lambda) \equiv \mathrm{e}^{\mathrm{i}k(\lambda)}\psi_+(x,\lambda),$$
$$\psi_-(x+a,\lambda) \equiv \mathrm{e}^{-\mathrm{i}k(\lambda)}\psi_-(x,\lambda).$$

They are determined uniquely up to multiplication by coefficients depending on λ. It is possible to choose these coefficients so that Bloch solutions have the following properties:

(1) $\psi_+(x,\lambda)$, $\psi_-(x,\lambda)$ for every $x \geqslant 0$ and their Wronskian $W\{\psi_+(\lambda),\psi_-(\lambda)\}$ are analytic functions of λ in $\mathbb{C}_+$ and continuous up to $\sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j,\mu_j, j \geqslant 0\}$;

(2) for $\lambda \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j,\mu_j, j \geqslant 0\}$ one has $\psi_+(x,\lambda) \equiv \overline{\psi_-(x,\lambda)}$;

(3) the Wronskian does not have zeros and for $\lambda \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j,\mu_j, j \geqslant 0\}$ one has $W\{\psi_+(\lambda),\psi_-(\lambda)\} \in \mathrm{i}\mathbb{R}_+$.

Bloch solutions can also be written in the form

$$\psi_+(x,\lambda) = \mathrm{e}^{\mathrm{i}k(\lambda)x/a}p_+(x,\lambda),$$
$$\psi_-(x,\lambda) = \mathrm{e}^{-\mathrm{i}k(\lambda)x/a}p_-(x,\lambda),$$

where the functions $p_+(x,\lambda)$ and $p_-(x,\lambda)$ have period a in the variable x and the same properties as $\psi_+(x,\lambda)$ and $\psi_-(x,\lambda)$ with respect to the variable λ.

As we mentioned earlier, the operator $\mathcal{L}_\alpha$ was studied in [16], where the asymptotics of the generalized eigenvectors was obtained. Kurasov and Naboko showed that in every band of $\sigma(\mathcal{L}_{\mathrm{per}})$ ($[\lambda_j,\mu_j]$ if j is even and $[\mu_j,\lambda_j]$ if j is odd) there exist two critical points $\nu_{j,+}$ and $\nu_{j,-}$ determined by the equalities

$$k(\nu_{j,+}) = \pi\left(j+1-\left\{\frac{a\omega}{\pi}\right\}\right) \quad \text{and} \quad k(\nu_{j,-}) = \pi\left(j+\left\{\frac{a\omega}{\pi}\right\}\right), \tag{2.1}$$

where by $\{\cdot\}$ we denote the fractional part. Critical points do not coincide with each other and with the ends of bands, if

$$\frac{2a\omega}{\pi} \notin \mathbb{Z}. \tag{2.2}$$

3. Reduction of the eigenfunction equation to a linear system of Levinson form

In this section we transform the eigenfunction equation for $\mathcal{L}_\alpha$ to a linear 2×2 system with the coefficient matrix being a sum of the diagonal and summable matrices.

Consider the eigenfunction equation for $\mathcal{L}_\alpha$,

$$-\psi''(x) + \left(q(x) + \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma} + q_1(x)\right)\psi(x) = \lambda\psi(x). \tag{3.1}$$

For every

$$\lambda \in \mathbb{C}_+ \cup (\sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j,\mu_j, j \geqslant 0\})$$

let us make the following substitution:

$$\begin{pmatrix}\psi(x)\\ \psi'(x)\end{pmatrix} = \begin{pmatrix}\psi_-(x,\lambda) & \psi_+(x,\lambda)\\ \psi'_-(x,\lambda) & \psi'_+(x,\lambda)\end{pmatrix} u(x). \tag{3.2}$$

Writing (3.1) as

$$\begin{pmatrix}\psi(x)\\ \psi'(x)\end{pmatrix}' = \begin{pmatrix}0 & 1\\ q(x) + \dfrac{c\sin(2\omega x+\delta)}{(x+1)^\gamma} + q_1(x) - \lambda & 0\end{pmatrix}\begin{pmatrix}\psi(x)\\ \psi'(x)\end{pmatrix}$$

and substituting (3.2) into it, we get

$$u'(x) = \frac{c\sin(2\omega x+\delta)/(x+1)^\gamma + q_1(x)}{W\{\psi_+(\lambda),\psi_-(\lambda)\}} \times \begin{pmatrix}-\psi_+(x,\lambda)\psi_-(x,\lambda) & -\psi_+^2(x,\lambda)\\ \psi_-^2(x,\lambda) & \psi_+(x,\lambda)\psi_-(x,\lambda)\end{pmatrix} u(x). \tag{3.3}$$

Let us introduce another vector-valued function v,

$$v(x) := \begin{pmatrix}\mathrm{e}^{-\mathrm{i}k(\lambda)(x/a)} & 0\\ 0 & \mathrm{e}^{\mathrm{i}k(\lambda)(x/a)}\end{pmatrix} u(x), \tag{3.4}$$

and the matrix

$$R^{(1)}(x,\lambda) := \frac{q_1(x)}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\begin{pmatrix}-p_+(x,\lambda)p_-(x,\lambda) & -p_+^2(x,\lambda)\\ p_-^2(x,\lambda) & p_+(x,\lambda)p_-(x,\lambda)\end{pmatrix}. \tag{3.5}$$

Then system (3.3) is equivalent to

$$v'(x) = \left(\begin{pmatrix}-\mathrm{i}k(\lambda)/a & 0\\ 0 & \mathrm{i}k(\lambda)/a\end{pmatrix} + \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}} \times \begin{pmatrix}-p_+(x,\lambda)p_-(x,\lambda) & -p_+^2(x,\lambda)\\ p_-^2(x,\lambda) & p_+(x,\lambda)p_-(x,\lambda)\end{pmatrix} + R^{(1)}(x,\lambda)\right)v(x). \tag{3.6}$$

Let us search for a differentiable matrix-valued function $Q(x)$ such that

$$Q(x), Q'(x) = O(1/x^\gamma)$$

as $x \to +\infty$ and such that the substitution

$$v(x) = \mathrm{e}^{Q(x)}\tilde{v}(x) \tag{3.7}$$

leads to a system for the vector-valued function $\tilde{v}$ of the form

$$\tilde{v}'(x) = \left(\begin{pmatrix}-\mathrm{i}k/a & 0\\ 0 & \mathrm{i}k/a\end{pmatrix} + \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+,\psi_-\}} \times \begin{pmatrix}-p_+(x)p_-(x) & 0\\ 0 & p_+(x)p_-(x)\end{pmatrix} + R^{(2)}(x)\right)\tilde{v}(x),$$

where the remainder $R^{(2)}(x)$ also belongs to $L_1(0,\infty)$. Using that

$$\mathrm{e}^{\pm Q(x)} = I \pm Q(x) + O\left(\frac{1}{x^{2\gamma}}\right),$$
$$(\mathrm{e}^{\pm Q(x)})' = \pm Q'(x) + O\left(\frac{1}{x^{2\gamma}}\right)$$

as $x \to +\infty$ we obtain

$$\begin{aligned}\tilde{v}'(x) = \Bigg(\begin{pmatrix} -\mathrm{i}k/a & 0 \\ 0 & \mathrm{i}k/a \end{pmatrix} &+ \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+,\psi_-\}} \\ &\times \begin{pmatrix} -p_+(x)p_-(x) & -p_+^2(x) \\ p_-^2(x) & p_+(x)p_-(x) \end{pmatrix} - Q'(x) \\ &- \left[Q(x), \begin{pmatrix} -\mathrm{i}k/a & 0 \\ 0 & \mathrm{i}k/a \end{pmatrix} \right] + R^{(1)}(x) + O\left(\frac{1}{x^{2\gamma}}\right)\Bigg)\tilde{v}(x), \qquad (3.8)\end{aligned}$$

where

$$\left[Q(x), \begin{pmatrix} -\mathrm{i}k/a & 0 \\ 0 & \mathrm{i}k/a \end{pmatrix} \right]$$

is the commutator of the two matrices. Our aim is to cancel the anti-diagonal entries of

$$\begin{pmatrix} -p_+(x)p_-(x) & -p_+^2(x) \\ p_-^2(x) & p_+(x)p_-(x) \end{pmatrix}$$

in (3.8) by properly choosing Q. To this end, Q has to satisfy the following equation:

$$Q'(x) + \left[Q(x), \begin{pmatrix} -\mathrm{i}k/a & 0 \\ 0 & \mathrm{i}k/a \end{pmatrix} \right] = \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+,\psi_-\}} \begin{pmatrix} 0 & -p_+^2(x) \\ p_-^2(x) & 0 \end{pmatrix}. \qquad (3.9)$$

The latter is equivalent (after multiplication by

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}kx/a} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}kx/a} \end{pmatrix}$$

from the right and by its inverse from the left) to

$$\begin{aligned}&\left(\begin{pmatrix} \mathrm{e}^{\mathrm{i}kx/a} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kx/a} \end{pmatrix} Q(x) \begin{pmatrix} \mathrm{e}^{-\mathrm{i}kx/a} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}kx/a} \end{pmatrix} \right)' \\ &\qquad = \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+,\psi_-\}} \begin{pmatrix} 0 & -p_+^2(x)\mathrm{e}^{2\mathrm{i}kx/a} \\ p_-^2(x)\mathrm{e}^{-2\mathrm{i}kx/a} & 0 \end{pmatrix}. \qquad (3.10)\end{aligned}$$

For every

$$\mu \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, \nu_{j,+}, \nu_{j,-}, n \geqslant 0\}$$

and for the values of λ from some neighbourhood of the point μ (which we will specify later) let us take the following solution of (3.10):

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}kx/a} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kx/a} \end{pmatrix} Q(x,\lambda,\mu) \begin{pmatrix} \mathrm{e}^{-\mathrm{i}kx/a} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}kx/a} \end{pmatrix}$$
$$= \frac{c}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}$$
$$\times \begin{pmatrix} 0 & \displaystyle\int_x^\infty \frac{\sin(2\omega t+\delta)p_+^2(t,\lambda)\mathrm{e}^{2\mathrm{i}k(\lambda)t/a}\,\mathrm{d}t}{(t+1)^\gamma} \\ \displaystyle\int_0^x \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\lambda)t/a}\,\mathrm{d}t}{(t+1)^\gamma} - \int_0^\infty \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\mu)t/a}\,\mathrm{d}t}{(t+1)^\gamma} & 0 \end{pmatrix}$$

(this is our choice of constants of integration that depend on μ). This leads to

$$Q(x,\lambda,\mu) := \frac{c}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}$$
$$\times \begin{pmatrix} 0 & \mathrm{e}^{-2\mathrm{i}k(\lambda)x/a}\displaystyle\int_x^\infty \frac{\sin(2\omega t+\delta)p_+^2(t,\lambda)\mathrm{e}^{2\mathrm{i}k(\lambda)t/a}\,\mathrm{d}t}{(t+1)^\gamma} \\ \mathrm{e}^{2\mathrm{i}k(\lambda)x/a}\left(\displaystyle\int_0^x \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\lambda)t/a}\,\mathrm{d}t}{(t+1)^\gamma} - \int_0^\infty \frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\mu)t/a}\,\mathrm{d}t}{(t+1)^\gamma}\right) & 0 \end{pmatrix}. \quad (3.11)$$

In particular, for $\lambda=\mu$,

$$Q(x,\mu,\mu) = \frac{c}{W\{\psi_+(\mu),\psi_-(\mu)\}} \times \begin{pmatrix} 0 & \mathrm{e}^{-2\mathrm{i}k(\mu)x/a}\displaystyle\int_x^\infty \frac{\sin(2\omega t+\delta)\psi_+^2(t,\mu)\,\mathrm{d}t}{(t+1)^\gamma} \\ -\mathrm{e}^{2\mathrm{i}k(\mu)x/a}\displaystyle\int_x^\infty \frac{\sin(2\omega t+\delta)\psi_-^2(t,\mu)\,\mathrm{d}t}{(t+1)^\gamma} & 0 \end{pmatrix}. \quad (3.12)$$

Formula (3.12) does not make sense if $\mu \in \mathbb{C}_+$, due to the divergence of the integral in the lower entry. But it has analytic continuation in the spectral parameter from the point μ. As was mentioned in § 1, a similar transformation was used in [3] and was called the 'complex $I+Q$' transformation. Nevertheless, here we need analyticity with respect to λ, and to this end we introduce an additional parameter μ. The rest of this section is devoted to proving uniform (with respect to λ) estimates for the entries of Q, which we need to establish uniform asymptotics of the solution.

Let us denote

$$\varepsilon(\mu) := \frac{1}{2}\min_{n\in\mathbb{Z}}\left\{\left|\frac{2k(\mu)}{a}+2\omega+\frac{2\pi n}{a}\right|, \left|\frac{2k(\mu)}{a}-2\omega+\frac{2\pi n}{a}\right|\right\}.$$

Consider some $\beta > 0$ and the set

$$U(\beta,\mu) := \{\lambda \in \overline{\mathbb{C}_+} \colon 2\varepsilon(\lambda) \geqslant \varepsilon(\mu),\ 0 \leqslant \operatorname{Im} 2k(\lambda)/a \leqslant 1, \\ |\operatorname{Re} k(\lambda) - k(\mu)| \leqslant \beta \operatorname{Im} k(\lambda)\}.$$

The set $U(\beta,\mu)$ is compact and contains the point μ. For every $\beta_1 < \beta$ it contains some neighbourhood of the vertex of the sector

$$|\operatorname{Re}\lambda - \mu| \leqslant \frac{\beta_1}{k'(\mu)} \operatorname{Im}\lambda.$$

Note that $k'(\mu)$ is positive for $\mu \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, j \geqslant 0\}$.

Lemma 3.1. *Let $\beta > 0$ and*

$$\mu \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, \nu_{j,+}, \nu_{j,-}, j \geqslant 0\}.$$

Then there exists $c_1(\beta,\mu,\gamma)$ such that, for every $x \geqslant 0$ and $\lambda \in U(\beta,\mu)$,

$$\|Q(x,\lambda,\mu)\|, \|Q'(x,\lambda,\mu)\| < \frac{c_1(\beta,\mu,\gamma)}{(x+1)^{\gamma}}$$

holds.

Proof. Note first that

$$k(\lambda) \in \mathbb{R} \qquad \text{for } \lambda \in \sigma(\mathcal{L}_{\mathrm{per}})$$

and

$$k(\lambda) \in \mathbb{C}_+ \quad \text{for } \lambda \in \mathbb{C}_+.$$

Let us denote the entries of $Q(x,\lambda,\mu)$ as follows:

$$Q(x,\lambda,\mu) = \begin{pmatrix} 0 & Q_{12}(x,\lambda) \\ Q_{21}(x,\lambda,\mu) & 0 \end{pmatrix}.$$

Let us first estimate the entry Q_{12}. Let f be a periodic function with period a such that $f \in L_1(0,a)$. Its Fourier coefficients will be denoted by

$$f_n := \frac{1}{a}\int_0^a f(x)\mathrm{e}^{-2\pi i n x/a}\,\mathrm{d}x.$$

LEMMA 3.2. *If*

$$\{f_n\}_{n=-\infty}^{+\infty} \in l^1(\mathbb{Z})$$

and $\xi \in \overline{\mathbb{C}_+}$ *is such that*

$$\frac{a\xi}{2\pi} \notin \mathbb{Z},$$

then

$$\left| \mathrm{e}^{-\mathrm{i}\xi x} \int_x^\infty \frac{\mathrm{e}^{\mathrm{i}\xi t} f(t)\,\mathrm{d}t}{(t+1)^\gamma} \right| \leqslant 2\biggl(\sum_{n=-\infty}^{+\infty} \frac{|f_n|}{|\xi - 2\pi n/a|} \biggr) \frac{1}{(x+1)^\gamma} \tag{3.13}$$

(i.e. the expression on the left-hand side exists and is estimated by that on the right-hand side).

Proof. Consider $x_1 > x$. Since the Fourier series converges absolutely, we have

$$\mathrm{e}^{-\mathrm{i}\xi x} \int_x^{x_1} \frac{\mathrm{e}^{\mathrm{i}\xi t}}{(t+1)^\gamma} \biggl(\sum_{n=-\infty}^{+\infty} f_n \mathrm{e}^{2\pi n t/a} \biggr) \mathrm{d}t = \sum_{n=-\infty}^{+\infty} f_n \mathrm{e}^{-\mathrm{i}\xi x} \int_x^{x_1} \frac{\mathrm{e}^{\mathrm{i}(\xi + 2\pi n/a)t}}{(t+1)^\gamma}\,\mathrm{d}t. \tag{3.14}$$

Integrating by parts and estimating the absolute value, we get

$$\begin{aligned} &\left| \mathrm{e}^{-\mathrm{i}\xi x} \int_x^{x_1} \frac{\mathrm{e}^{\mathrm{i}(\xi+2\pi n/a)t}\,\mathrm{d}t}{(t+1)^\gamma} \right| \\ &\qquad \leqslant \frac{1}{\|\xi + 2\pi n/a\|} \times \left(\frac{1}{(x+1)^\gamma} + \frac{1}{(x_1+1)^\gamma} + \gamma \int_x^{x_1} \frac{\mathrm{d}t}{(t+1)^{\gamma+1}} \right) \\ &\qquad = \frac{2}{|\xi + 2\pi n/a|(x+1)^\gamma}. \end{aligned}$$

Substituting into (3.14) yields

$$\left| \mathrm{e}^{-\mathrm{i}\xi x} \int_x^{x_1} \frac{\mathrm{e}^{\mathrm{i}\xi t} f(t)\,\mathrm{d}t}{(t+1)^\gamma} \right| \leqslant 2\biggl(\sum_{n=-\infty}^{+\infty} \frac{|f_n|}{|\xi - 2\pi n/a|} \biggr) \frac{1}{(x+1)^\gamma}.$$

By Cauchy's criterion the integral

$$\int_x^\infty \frac{\mathrm{e}^{\mathrm{i}\xi t} f(t)\,\mathrm{d}t}{(t+1)^\gamma}$$

exists and the desired estimate (3.13) follows. □

Formula (3.11) implies that

$$\begin{aligned} &Q_{12}(x,\lambda) \\ &\quad = \frac{c\mathrm{e}^{2\mathrm{i}\omega x + \mathrm{i}\delta}}{2\mathrm{i}W\{\psi_+(\lambda), \psi_-(\lambda)\}} \mathrm{e}^{-\mathrm{i}(2k(\lambda)/a + 2\omega)x} \int_x^\infty \frac{p_+^2(t,\lambda)\mathrm{e}^{\mathrm{i}(2k(\lambda)/a+2\omega)t}\,\mathrm{d}t}{(t+1)^\gamma} \\ &\qquad - \frac{c\mathrm{e}^{-2\mathrm{i}\omega x - \mathrm{i}\delta}}{2\mathrm{i}W\{\psi_+(\lambda), \psi_-(\lambda)\}} \mathrm{e}^{-\mathrm{i}(2k(\lambda)/a - 2\omega)x} \int_x^\infty \frac{p_+^2(t,\lambda)\mathrm{e}^{\mathrm{i}(2k(\lambda)/a-2\omega)t}\,\mathrm{d}t}{(t+1)^\gamma}. \end{aligned} \tag{3.15}$$

Denote the Fourier coefficients of the function $p_+^2(\cdot,\lambda)$ by $b_n(\lambda)$,

$$b_n(\lambda) := \frac{1}{a}\int_0^a p_+^2(x,\lambda)\mathrm{e}^{-2\pi i n x/a}\,\mathrm{d}x.$$

Then lemma 3.2 applied to (3.15) gives

$$\begin{aligned}
&|Q_{12}(x,\lambda)| \\
&\quad\leqslant \frac{|c|}{|W\{\psi_+(\lambda),\psi_-(\lambda)\}|}\frac{1}{(x+1)^\gamma} \\
&\qquad\times\sum_{n=-\infty}^{+\infty}|b_n(\lambda)|\left(\frac{1}{|2k(\lambda)/a+2\omega+2\pi n/a|}+\frac{1}{|2k(\lambda)/a-2\omega+2\pi n/a|}\right) \\
&\quad\leqslant \frac{2|c|}{\varepsilon(\mu)|W\{\psi_+(\lambda),\psi_-(\lambda)\}|}\left(\sum_{n=-\infty}^{+\infty}|b_n(\lambda)|\right)\frac{1}{(x+1)^\gamma}. \qquad (3.16)
\end{aligned}$$

Let us now estimate the entry Q_{21}. Formula (3.11) implies that

$$\begin{aligned}
&Q_{21}(x,\lambda,\mu) \\
&= \frac{c\mathrm{e}^{2\mathrm{i}k(\lambda)x/a}}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\left(\int_0^x\frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\lambda)t/a}\,\mathrm{d}t}{(t+1)^\gamma}\right. \\
&\qquad\qquad\left.-\int_0^\infty\frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\mu)t/a}\,\mathrm{d}t}{(t+1)^\gamma}\right) \\
&= \frac{c\mathrm{e}^{2\mathrm{i}k(\lambda)x/a}}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\int_0^x\frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)(\mathrm{e}^{-2\mathrm{i}k(\lambda)t/a}-\mathrm{e}^{-2\mathrm{i}k(\mu)t/a})\,\mathrm{d}t}{(t+1)^\gamma} \\
&\qquad -\frac{c\mathrm{e}^{2\mathrm{i}k(\lambda)x/a}}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\int_x^\infty\frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\mu)t/a}\,\mathrm{d}t}{(t+1)^\gamma}.
\end{aligned}$$

Define

$$\begin{aligned}
Q_{21}^I(x,\lambda,\mu) := {}&\frac{c\mathrm{e}^{2\mathrm{i}k(\lambda)x/a}}{W\{\psi_+(\lambda),\psi_-(\lambda)\}} \\
&\times\int_0^x\frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)(\mathrm{e}^{-2\mathrm{i}k(\lambda)t/a}-\mathrm{e}^{-2\mathrm{i}k(\mu)t/a})\,\mathrm{d}t}{(t+1)^\gamma}
\end{aligned}$$

and

$$Q_{21}^{\mathrm{II}}(x,\lambda,\mu) := -\frac{c\mathrm{e}^{2\mathrm{i}k(\lambda)x/a}}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\int_x^\infty\frac{\sin(2\omega t+\delta)p_-^2(t,\lambda)\mathrm{e}^{-2\mathrm{i}k(\mu)t/a}\,\mathrm{d}t}{(t+1)^\gamma},$$

so that

$$Q_{21}(x,\lambda,\mu) = Q_{21}^I(x,\lambda,\mu)+Q_{21}^{\mathrm{II}}(x,\lambda,\mu).$$

The second term can be estimated in the same manner as $Q_{12}(x,\lambda)$ using lemma 3.2. Denote by

$$\hat{b}_n(\lambda) := \frac{1}{a}\int_0^a p_-^2(x,\lambda)\mathrm{e}^{-2\pi i n x/a}\,\mathrm{d}x$$

the Fourier coefficients of $p_-^2(\cdot,\lambda)$. Then,

$$
\begin{aligned}
|Q_{21}^{\mathrm{II}}&(x,\lambda,\mu)| \\
&\leqslant \frac{|c|}{|W\{\psi_+(\lambda),\psi_-(\lambda)\}|}\frac{1}{(x+1)^\gamma} \\
&\quad\times \sum_{n=-\infty}^{+\infty}|\hat{b}_n(\lambda)|\biggl(\frac{1}{|2k(\lambda)/a-2\omega-2\pi n/a|}+\frac{1}{|2k(\lambda)/a+2\omega-2\pi n/a|}\biggr) \\
&\leqslant \frac{2|c|}{\varepsilon(\mu)|W\{\psi_+(\lambda),\psi_-(\lambda)\}|}\biggl(\sum_{n=-\infty}^{+\infty}|\hat{b}_n(\lambda)|\biggr)\frac{1}{(x+1)^\gamma} \qquad (3.17)
\end{aligned}
$$

(using that $k(\mu)\in\mathbb{R}$ and $k(\lambda)\in\overline{\mathbb{C}_+}$).

To estimate Q_{21}^{I} we shall need the following lemma.

LEMMA 3.3. *Let $\varepsilon,\beta>0$, then there exists $c_2(\varepsilon,\beta,\gamma)$ such that, for every ξ_1 and ξ_2,*

$$
\begin{aligned}
0\leqslant \operatorname{Im}\xi_1\leqslant 1,&\quad |\xi_1|\geqslant\varepsilon,\\
\xi_2\in\mathbb{R},&\quad |\xi_2|\geqslant\varepsilon,\\
|\operatorname{Re}\xi_1-\xi_2|\leqslant\beta\operatorname{Im}\xi_1&
\end{aligned}
$$

and for every $x\geqslant 0$

$$\left|\mathrm{e}^{\mathrm{i}\xi_1 x}\int_0^x\frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t}-\mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^\gamma}\right|<\frac{c_2(\varepsilon,\beta,\gamma)}{(x+1)^\gamma}$$

holds.

Proof. Integrating by parts, we get

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}\xi_1 x}\int_0^x\frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t}-\mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^\gamma}
&=\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\xi_1 x}(\xi_1-\xi_2)}{\mathrm{i}\xi_1\xi_2}+\frac{\mathrm{e}^{\mathrm{i}(\xi_1-\xi_2)x}}{\mathrm{i}\xi_2(x+1)^\gamma}\\
&\quad+\frac{\mathrm{i}}{\xi_1(x+1)^\gamma}+\frac{\gamma\mathrm{e}^{\mathrm{i}\xi_1 x}(\xi_1-\xi_2)}{\mathrm{i}\xi_1\xi_2}\int_0^x\frac{\mathrm{e}^{-\mathrm{i}\xi_2 t}\,\mathrm{d}t}{(t+1)^{\gamma+1}}\\
&\quad-\frac{\gamma\mathrm{e}^{\mathrm{i}\xi_1 x}}{\mathrm{i}\xi_1}\int_0^x\frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t}-\mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^{\gamma+1}}.
\end{aligned}
$$

Consider the new constant

$$c_3(\gamma):=\max_{x\geqslant 0}x^\gamma\mathrm{e}^{-x}.$$

For every $x\geqslant 0$ and ξ_1 considered,

$$(\operatorname{Im}\xi_1)^\gamma\mathrm{e}^{-\operatorname{Im}\xi_1 x}\leqslant\frac{c_3(\gamma)\mathrm{e}^{\operatorname{Im}\xi_1}}{(x+1)^\gamma}\leqslant\frac{\mathrm{e}c_3(\gamma)}{(x+1)^\gamma}.$$

Using that

$$|\operatorname{Re}\xi_1-\xi_2|\leqslant\beta\operatorname{Im}\xi_1,$$

which is equivalent to

$$|\xi_1 - \xi_2| \leqslant \sqrt{\beta^2+1}\,\mathrm{Im}\,\xi_1,$$

the integral can be estimated as

$$\left|\mathrm{e}^{\mathrm{i}\xi_1 x}\int_0^x \frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t} - \mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^\gamma}\right| \leqslant \frac{2\mathrm{e}c_3(\gamma)\sqrt{\beta^2+1}}{\varepsilon^2(x+1)^\gamma} + \frac{2}{\varepsilon(x+1)^\gamma} + \frac{\gamma}{\varepsilon}\left|\mathrm{e}^{\mathrm{i}\xi_1 x}\int_0^x \frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t} - \mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^{\gamma+1}}\right|.$$

The last term can be split into three parts as follows:

$$\left|\mathrm{e}^{\mathrm{i}\xi_1 x}\int_0^x \frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t} - \mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^{\gamma+1}}\right| \leqslant \mathrm{e}^{-\mathrm{Im}\,\xi_1 x}\left[\int_0^{1/\mathrm{Im}\,\xi_1} + \int_{1/\mathrm{Im}\,\xi_1}^{x/2} + \int_{x/2}^{x}\right]\frac{|\mathrm{e}^{-\mathrm{i}\xi_1 t} - \mathrm{e}^{-\mathrm{i}\xi_2 t}|\,\mathrm{d}t}{(t+1)^{\gamma+1}}.$$

Let us estimate these three integrals separately.

(1)

$$\mathrm{e}^{-\mathrm{Im}\,\xi_1 x}\int_0^{1/\mathrm{Im}\,\xi_1}\frac{|\mathrm{e}^{-\mathrm{i}\xi_1 t} - \mathrm{e}^{-\mathrm{i}\xi_2 t}|\,\mathrm{d}t}{(t+1)^{\gamma+1}} = \mathrm{e}^{-\mathrm{Im}\,\xi_1 x}\int_0^{1/\mathrm{Im}\,\xi_1}\frac{|\mathrm{e}^{-\mathrm{i}(\xi_1-\xi_2)t} - 1|\,\mathrm{d}t}{(t+1)^{\gamma+1}}.$$

Introduce the constant

$$c_4(\beta) := \max_{|x|\leqslant\sqrt{\beta^2+1}}\frac{|\mathrm{e}^x - 1|}{|x|}.$$

Since for the first interval

$$|-\mathrm{i}(\xi_1-\xi_2)t| \leqslant \frac{|\xi_1 - \xi_2|}{\mathrm{Im}\,\xi_1} \leqslant \sqrt{\beta^2+1},$$

we have

$$\begin{aligned}
\mathrm{e}^{-\mathrm{Im}\,\xi_1 x}&\int_0^{1/\mathrm{Im}\,\xi_1}\frac{|\mathrm{e}^{-\mathrm{i}\xi_1 t} - \mathrm{e}^{-\mathrm{i}\xi_2 t}|\,\mathrm{d}t}{(t+1)^{\gamma+1}} \\
&\leqslant \mathrm{e}^{-\mathrm{Im}\,\xi_1 x}\int_0^{1/\mathrm{Im}\,\xi_1}\frac{c_4(\beta)\sqrt{\beta^2+1}\,\mathrm{Im}\,\xi_1 t\,\mathrm{d}t}{(t+1)^{\gamma+1}} \\
&\leqslant \frac{\mathrm{e}c_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}(\mathrm{Im}\,\xi_1)^{1-\gamma}}{(x+1)^\gamma}\int_0^{1/\mathrm{Im}\,\xi_1}\frac{\mathrm{d}t}{(t+1)^\gamma} \\
&= \frac{\mathrm{e}c_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}((1+\mathrm{Im}\,\xi_1)^{1-\gamma} - (\mathrm{Im}\,\xi_1)^{1-\gamma})}{(x+1)^\gamma(1-\gamma)} \\
&\leqslant \frac{2^{1-\gamma}\mathrm{e}c_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}}{(1-\gamma)(x+1)^\gamma}.
\end{aligned}$$

(2) For $x \geqslant 2/\operatorname{Im}\xi_1$ we have

$$\begin{aligned}
\mathrm{e}^{-\operatorname{Im}\xi_1 x}\int_{1/\operatorname{Im}\xi_1}^{x/2}\frac{|\mathrm{e}^{-\mathrm{i}\xi_1 t}-\mathrm{e}^{-\mathrm{i}\xi_2 t}|\,\mathrm{d}t}{(t+1)^{\gamma+1}} &\\
&\leqslant \mathrm{e}^{-\operatorname{Im}\xi_1 x/2}\int_{1/\operatorname{Im}\xi_1}^{x/2}\frac{(\mathrm{e}^{\operatorname{Im}\xi_1(t-x/2)}+\mathrm{e}^{-\operatorname{Im}\xi_1 x/2})\,\mathrm{d}t}{(t+1)^{\gamma+1}}\\
&\leqslant 2\mathrm{e}^{-\operatorname{Im}\xi_1 x/2}\int_{1/\operatorname{Im}\xi_1}^{\infty}\frac{\mathrm{d}t}{t^{\gamma+1}}\\
&\leqslant \frac{2^{\gamma+1}\mathrm{e}c_3(\gamma)}{\gamma(x+2)^{\gamma}}.
\end{aligned}$$

For $x < 2/\operatorname{Im}\xi_1$ the integral is negative.

(3)
$$\mathrm{e}^{-\operatorname{Im}\xi_1 x}\int_{x/2}^{x}\frac{|\mathrm{e}^{-\mathrm{i}\xi_1 t}-\mathrm{e}^{-\mathrm{i}\xi_2 t}|\,\mathrm{d}t}{(t+1)^{\gamma+1}} \leqslant \int_{x/2}^{x}\frac{2\,\mathrm{d}t}{(t+1)^{\gamma+1}} < \frac{2^{\gamma+1}}{\gamma(x+2)^{\gamma}}.$$

Combining these estimates, we get

$$\left|\mathrm{e}^{\mathrm{i}\xi_1 x}\int_0^x\frac{(\mathrm{e}^{-\mathrm{i}\xi_1 t}-\mathrm{e}^{-\mathrm{i}\xi_2 t})\,\mathrm{d}t}{(t+1)^{\gamma+1}}\right| < \frac{c_2(\varepsilon,\beta,\gamma)}{(x+1)^{\gamma}}$$

with

$$\begin{aligned}
c_2(\varepsilon,\beta,\gamma) := {}&\frac{2\mathrm{e}c_3(\gamma)\sqrt{\beta^2+1}}{\varepsilon^2}+\frac{2}{\varepsilon}\\
&+\frac{\gamma}{\varepsilon}\left(\frac{2^{1-\gamma}\mathrm{e}c_3(\gamma)c_4(\beta)\sqrt{\beta^2+1}}{1-\gamma}+\frac{2^{\gamma+1}\mathrm{e}c_3(\gamma)}{\gamma}+\frac{2^{\gamma+1}}{\gamma}\right).
\end{aligned}$$

This completes the proof of the lemma. □

Let us continue to estimate Q_{21}^{I}

$$\begin{aligned}
Q_{21}^{\mathrm{I}}(x,\lambda,\mu) = {}&\sum_{n=-\infty}^{+\infty}\frac{c\hat{b}_n(\lambda)\mathrm{e}^{\mathrm{i}\delta+2\mathrm{i}\omega+2\pi\mathrm{i}nx/a}}{2\mathrm{i}W\{\psi_+(\lambda),\psi_-(\lambda)\}}\mathrm{e}^{\mathrm{i}(2k(\lambda)/a-2\omega-2\pi n/a)x}\\
&\times\int_0^x\frac{(\mathrm{e}^{-\mathrm{i}(2k(\lambda)/a-2\omega-2\pi n/a)t}-\mathrm{e}^{-\mathrm{i}(2k(\mu)/a-2\omega-2\pi n/a)t})\,\mathrm{d}t}{(t+1)^{\gamma}}\\
&-\sum_{n=-\infty}^{+\infty}\frac{c\hat{b}_n(\lambda)\mathrm{e}^{-\mathrm{i}\delta-2\mathrm{i}\omega+2\pi\mathrm{i}nx/a}}{2\mathrm{i}W\{\psi_+(\lambda),\psi_-(\lambda)\}}\mathrm{e}^{\mathrm{i}(2k(\lambda)/a+2\omega-2\pi n/a)x}\\
&\times\int_0^x\frac{(\mathrm{e}^{-\mathrm{i}(2k(\lambda)/a+2\omega-2\pi n/a)t}-\mathrm{e}^{-\mathrm{i}(2k(\mu)/a+2\omega-2\pi n/a)t})\,\mathrm{d}t}{(t+1)^{\gamma}}.
\end{aligned}$$

Applying lemma 3.3, we get

$$|Q_{21}^{\mathrm{I}}(x,\lambda,\mu)| \leqslant \frac{|c|c_2(\varepsilon(\mu),\beta,\gamma)}{|W\{\psi_+(\lambda),\psi_-(\lambda)\}|}\left(\sum_{n=-\infty}^{+\infty}|\hat{b}_n(\lambda)|\right)\frac{1}{(x+1)^{\gamma}}. \tag{3.18}$$

Therefore, combining the estimates (3.16), (3.18) and (3.17) the matrix Q can be estimated as follows:

$$\begin{aligned}
&\|Q(x,\lambda,\mu)\| \\
&\quad\leqslant \frac{1}{(x+1)^{\gamma}}\frac{|c|}{|W\{\psi_+(\lambda),\psi_-(\lambda)\}|} \\
&\quad\quad\times\sqrt{\frac{4}{\varepsilon^2(\mu)}\left(\sum_{n=-\infty}^{+\infty}|b_n(\lambda)|\right)^2+\left(\frac{2}{\varepsilon(\mu)}+c_2(\varepsilon(\mu),\beta,\gamma)\right)^2\left(\sum_{n=-\infty}^{+\infty}|\hat{b}_n(\lambda)|\right)^2}.
\end{aligned}$$

Let us now estimate the Fourier coefficients. For $n \neq 0$ we have

$$b_n(\lambda)=\frac{1}{a}\int_0^a p_+^2(x,\lambda)\mathrm{e}^{-2\pi\mathrm{i}nx/a}\,\mathrm{d}x=-\frac{a}{4\pi^2n^2}\int_0^a(p_+^2(x,\lambda))''\mathrm{e}^{-2\pi\mathrm{i}nx/a}\,\mathrm{d}x.$$

Thus,

$$|b_n(\lambda)|\leqslant\frac{a}{4\pi^2n^2}\int_0^a|(p_+^2(x,\lambda))''|\,\mathrm{d}x. \tag{3.19}$$

In terms of the corresponding Bloch solution, the second derivative of p_+^2 is

$$\begin{aligned}
(p_+^2(x,\lambda))''=2\mathrm{e}^{-2\mathrm{i}k(\lambda)x/a}\biggl(&\psi_+''(x,\lambda)\psi_+(x,\lambda)-\frac{2k^2(\lambda)}{a^2}\psi_+^2(x,\lambda)\\
&+(\psi_+'(x,\lambda))^2-\frac{4\mathrm{i}k(\lambda)}{a}\psi_+'(x,\lambda)\psi_+(x,\lambda)\biggr).
\end{aligned}$$

Let us estimate the norm in $L_1(0,a)$ of the function $\psi_+''(\cdot,\lambda)$. From the equation

$$\psi_+''(x,\lambda)=(q(x)-\lambda)\psi_+(x,\lambda)$$

we see that

$$\|\psi_+''(\cdot,\lambda)\|_{L_1(0,a)}\leqslant(\|q\|_{L_1(0,a)}+|\lambda|a)\max_{x\in[0,a]}|\psi_+(x,\lambda)|.$$

Since the functions

$$\mathrm{e}^{-\mathrm{i}k(\lambda)x/a},\qquad\psi_+(x,\lambda),\qquad\psi_+'(x,\lambda)$$

are continuous in both variables on the set

$$[0,a]\times U(\beta,\mu),$$

and hence attain their maximums, we have that the integral

$$\int_0^a|(p_+^2(x,\lambda))''|\,\mathrm{d}x$$

is bounded on $U(\beta,\mu)$. For $n=0$,

$$b_0(\lambda)=\frac{1}{a}\int_0^a p_+^2(x,\lambda)\,\mathrm{d}x=\frac{1}{a}\int_0^a\mathrm{e}^{-2\mathrm{i}k(\lambda)x/a}\psi_+^2(x,\lambda)\,\mathrm{d}x$$

and is also bounded. The same argument is valid for $\hat{b}_n$. Finally, we see that there exists $c_5(\beta,\mu)$ such that, for every $\lambda \in U(\beta,\mu)$,

$$|b_n(\lambda)|, |\hat{b}_n(\lambda)| \leqslant \frac{c_5(\beta,\mu)}{n^2+1}.$$

The Wronskian $W\{\psi_+(\lambda),\psi_-(\lambda)\}$ does not have zeros in $U(\beta,\mu)$. Also, in the formula for the derivative of Q,

$$\begin{aligned} Q'(x,\lambda,\mu) = {} & \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}} \begin{pmatrix} 0 & -p_+^2(x,\lambda) \\ p_-^2(x,\lambda) & 0 \end{pmatrix} \\ & - \left[Q(x,\lambda,\mu), \begin{pmatrix} -\mathrm{i}k(\lambda)/a & 0 \\ 0 & \mathrm{i}k(\lambda)/a \end{pmatrix} \right], \end{aligned}$$

the functions $\pm k(\lambda)/a$, $\pm c\sin(2\omega x+\delta)p_\pm^2(x,\lambda)$ are bounded for $(x;\lambda) \in [0;+\infty) \times U(\beta,\mu)$. Hence, there exists $c_1(\beta,\mu,\gamma)$ such that for every $\lambda \in U(\beta,\mu)$ and $x \geqslant 0$ the following estimates hold:

$$\|Q(x,\lambda,\mu)\|, \|Q'(x,\lambda,\mu)\| \leqslant \frac{c_1(\beta,\mu,\gamma)}{(x+1)^\gamma}.$$

This completes the proof of the theorem. □

Let us study the properties of the remainder

$$\begin{aligned} & R^{(2)}(x,\lambda,\mu) \\ & := \mathrm{e}^{-Q(x,\lambda,\mu)} \\ & \quad \times \left(\begin{pmatrix} -\mathrm{i}k(\lambda)/a & 0 \\ 0 & \mathrm{i}k(\lambda)/a \end{pmatrix} + \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}} \right. \\ & \qquad \left. \times \begin{pmatrix} -p_+(x,\lambda)p_-(x,\lambda) & -p_+^2(x,\lambda) \\ p_-^2(x,\lambda) & p_+(x,\lambda)p_-(x,\lambda) \end{pmatrix} + R^{(1)}(x,\lambda) \right) \mathrm{e}^{Q(x,\lambda,\mu)} \\ & \qquad - \begin{pmatrix} -\mathrm{i}k(\lambda)/a & 0 \\ 0 & \mathrm{i}k(\lambda)/a \end{pmatrix} - \mathrm{e}^{-Q(x,\lambda,\mu)}(\mathrm{e}^{Q(x,\lambda,\mu)})' \\ & \qquad - \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}} \begin{pmatrix} -p_+(x,\lambda)p_-(x,\lambda) & 0 \\ 0 & p_+(x,\lambda)p_-(x,\lambda) \end{pmatrix}. \end{aligned} \tag{3.20}$$

LEMMA 3.4. *The remainder $R^{(2)}$ given by (3.20) possesses the following properties:*

(1) $R^{(2)}(\cdot,\lambda,\mu) \in L_1(0,\infty)$ *and the integral*

$$\int_0^\infty \|R^{(2)}(x,\lambda,\mu)\|\,\mathrm{d}x$$

converges uniformly with respect to $\lambda \in U(\beta,\mu)$;

(2) *one has*

$$\left.\begin{aligned}(R^{(2)}(x,\mu,\mu))_{21} &= \overline{(R^{(2)}(x,\mu,\mu))_{12}},\\ (R^{(2)}(x,\mu,\mu))_{22} &= \overline{(R^{(2)}(x,\mu,\mu))_{11}}.\end{aligned}\right\} \tag{3.21}$$

Proof. The first assertion follows directly from lemma 3.1.

The property of matrices from the second assertion is preserved under summation and multiplication of such matrices, as well as taking the inverse. We can see from (3.5) and (3.12) that $R^{(1)}(x,\mu)$ and $Q(x,\mu,\mu)$ possess this conjugation property. Therefore,

$$Q'(x,\mu,\mu), \qquad \mathrm{e}^{Q(x,\mu,\mu)}, \qquad (\mathrm{e}^{Q(x,\mu,\mu)})'$$

and, finally, $R^{(2)}(x,\mu,\mu)$ (from (3.20)) also possess this property. It should be taken into account that $W\{\psi_+(\mu),\psi_-(\mu)\}$ is purely imaginary. □

Let us define

$$\nu(x,\lambda) := -\frac{\mathrm{i}k(\lambda)}{a} - \frac{c\sin(2\omega x+\delta)p_+(x,\lambda)p_-(x,\lambda)}{(x+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}.$$

Finally, we obtain a system of the Levinson form

$$\tilde v'(x) = \left(\begin{pmatrix}\nu(x,\lambda) & 0\\ 0 & -\nu(x,\lambda)\end{pmatrix} + R^{(2)}(x,\lambda,\mu)\right)\tilde v(x). \tag{3.22}$$

4. A Levinson-type theorem for 2 × 2 systems

In this section, we prove two statements that give a uniform estimate and asymptotics of solutions to certain 2×2 differential systems. The approach is the same as for the Levinson theorem [8], but the difference is that we are interested in properties of solution with a given initial condition. Note that concrete values of all appearing constants are important, since they will be needed to prove the uniform asymptotics. In this sense, we can say that in these lemmas the uniform estimates and asymptotics are proven.

Consider the system

$$u_1'(x) = \left(\begin{pmatrix}\lambda(x) & 0\\ 0 & -\lambda(x)\end{pmatrix} + R(x)\right)u_1(x) \tag{4.1}$$

for $x \geqslant 0$, where $u_1(x)$ is a two-dimensional vector function and $R(x)$ is a 2×2 matrix with complex entries.

Lemma 4.1. *Assume that*

$$\int_0^\infty \|R(t)\|\,\mathrm{d}t < \infty \tag{4.2}$$

and that there exists a constant M such that for every $x \leqslant y$ it holds that

$$\int_x^y \operatorname{Re}\lambda(t)\,\mathrm{d}t \geqslant -M.$$

Then every solution u_1 to (4.1) satisfies the estimate

$$\|u_1(x)\| \leqslant \|u_1(0)\| \exp\left(\int_0^x \operatorname{Re}\lambda(t)\,\mathrm{d}t\right) \\ \times \sqrt{1+\mathrm{e}^{4M}} \exp\left(\sqrt{1+\mathrm{e}^{4M}} \int_0^\infty \|R(t)\|\,\mathrm{d}t\right). \quad (4.3)$$

Proof. First transform system (4.1) by variation of parameters. Define

$$\Lambda(x) := \begin{pmatrix} \lambda(x) & 0 \\ 0 & -\lambda(x) \end{pmatrix}$$

and take

$$u_1(x) = \exp\left(\int_0^x \Lambda(t)\,\mathrm{d}t\right) u_2(x) \quad \text{or} \quad u_2(x) := \exp\left(-\int_0^x \Lambda(t)\,\mathrm{d}t\right) u_1(x). \quad (4.4)$$

After substitution, (4.1) becomes

$$u_2'(x) = \exp\left(-\int_0^x \Lambda(t)\,\mathrm{d}t\right) R(x) u_1(x). \quad (4.5)$$

Integrating this from 0 to x and returning to the function u_1, on the left-hand side we get

$$u_1(x) = \exp\left(\int_0^x \Lambda(t)\,\mathrm{d}t\right) u_1(0) + \int_0^x \exp\left(\int_t^x \Lambda(s)\,\mathrm{d}s\right) R(t) u_1(t)\,\mathrm{d}t. \quad (4.6)$$

Now multiply this expression by

$$\exp\left(-\int_0^x \Lambda(s)\,\mathrm{d}s\right)$$

and define

$$u_3(x) := \exp\left(-\int_0^x \Lambda(t)\,\mathrm{d}t\right) u_1(x). \quad (4.7)$$

We get the following equation for u_3 considered in $L_\infty((0,\infty);\mathbb{C}^2)$:

$$u_3(x) = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\int_0^x \lambda(s)\,\mathrm{d}s\right) \end{pmatrix} u_1(0) \\ + \int_0^x \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\int_t^x \lambda(s)\,\mathrm{d}s\right) \end{pmatrix} R(t) u_3(t)\,\mathrm{d}t. \quad (4.8)$$

The norm of the operator V,

$$V\colon u \mapsto \int_0^x \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\int_t^x \lambda(s)\,\mathrm{d}s\right) \end{pmatrix} R(t) u(t)\,\mathrm{d}t,$$

is bounded by

$$\|V\| \leqslant \sqrt{1+\mathrm{e}^{4M}} \int_0^\infty \|R(t)\|\,\mathrm{d}t$$

and the norm of the jth power is bounded by

$$\|V^j\| \leqslant \frac{1}{j!}\biggl(\sqrt{1+\mathrm{e}^{4M}} \int_0^\infty \|R(t)\|\,\mathrm{d}t\biggr)^j.$$

Hence,

$$u_3(x) = (I-V)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \exp(-2\int_0^x \lambda(s)\,\mathrm{d}s)\end{pmatrix} u_1(0)$$

and

$$\|u_3\|_{L_\infty((0,\infty),\mathbb{C}^2)} \leqslant \exp\biggl(\sqrt{1+\mathrm{e}^{4M}} \int_0^\infty \|R(t)\|\,\mathrm{d}t\biggr)\sqrt{1+\mathrm{e}^{4M}}\|u_1(0)\|.$$

Returning to u_1, we arrive at the estimate (4.3). □

The next lemma states the asymptotics of the solution.

Lemma 4.2. *Let all conditions of lemma 4.1 be satisfied. Then the following asymptotics hold:*

(1) if

$$\int_0^\infty \operatorname{Re}\lambda(t)\,\mathrm{d}t < +\infty, \tag{4.9}$$

then every solution u_1 of (4.1) has the asymptotics

$$\begin{aligned} &u_1(x) \\ &\quad= \begin{pmatrix} \exp(\int_0^x \lambda(s)\,\mathrm{d}s) & 0 \\ 0 & \exp(-\int_0^x \lambda(s)\,\mathrm{d}s)\end{pmatrix} \\ &\qquad\times\biggl(u_1(0) + \int_0^\infty \begin{pmatrix} \exp(-\int_0^t \lambda(s)\,\mathrm{d}s) & 0 \\ 0 & \exp(\int_0^t \lambda(s)\,\mathrm{d}s)\end{pmatrix} R(t)u_1(t)\,\mathrm{d}t + o(1)\biggr) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{as } x \to +\infty; \end{aligned}$$

(2) if

$$\int_0^\infty \operatorname{Re}\lambda(t)\,\mathrm{d}t = +\infty, \tag{4.10}$$

then every solution u_1 of (4.1) has the asymptotics

$$\begin{aligned} u_1(x) &= \exp\biggl(\int_0^x \lambda(s)\,\mathrm{d}s\biggr) \\ &\quad\times\biggl(\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\biggl(u_1(0) + \int_0^\infty \exp\biggl(-\int_0^t \lambda(s)\,\mathrm{d}s\biggr)R(t)u_1(t)\,\mathrm{d}t\biggr) + o(1)\biggr) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{as } x \to +\infty. \end{aligned}$$

Proof. Asymptotics 1. Consider the function u_2 given by (4.4) and integrate (4.5):

$$u_2(x) = u_1(0) + \int_0^x \exp\biggl(-\int_0^t \mathit{\Lambda}(s)\,\mathrm{d}s\biggr) R(t)u_1(t)\,\mathrm{d}t. \tag{4.11}$$

Since for every $x \leqslant y$ we have the estimate

$$\int_x^y \operatorname{Re}\lambda(s)\,\mathrm{d}s \leqslant \int_x^y |\operatorname{Re}\lambda(s)|\,\mathrm{d}s \leqslant \int_0^\infty |\operatorname{Re}\lambda(s)|\,\mathrm{d}s \leqslant \int_0^\infty \operatorname{Re}\lambda(s)\,\mathrm{d}s + 2M,$$

the exponent under the integral in (4.11) is bounded. The solution $u_1(t)$ is also bounded in this case, due to lemma 4.1. Hence the integral in (4.11) converges as $x \to +\infty$ and there exists

$$\lim_{x\to+\infty} u_2(x) = u_1(0) + \int_0^\infty \exp\biggl(-\int_0^t \mathit{\Lambda}(s)\,\mathrm{d}s\biggr) R(t)u_1(t)\,\mathrm{d}t.$$

Returning to u_1, we obtain the answer.

Asymptotics 2. Consider the function u_3 given by (4.7) and the corresponding equation (4.8). It follows from lemma 4.1 that $u_3(t)$ is bounded, and hence Lebesgue's dominated convergence theorem implies that the following limit exists:

$$\lim_{x\to+\infty} u_3(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \biggl(u_1(0) + \int_0^\infty R(t)u_3(t)\,\mathrm{d}t\biggr).$$

This is equivalent to the announced asymptotics for u_1. □

5. Asymptotics for the solution φ and Weyl–Titchmarsh-type formula

In this section, we put together the results obtained earlier to find the asymptotics of the solution $\varphi_\alpha(x,\lambda)$ and prove the Weyl–Titchmarsh-type formula for the operator $\mathcal{L}_\alpha$. Consider the set

$$U(\beta) := \bigcup_{\mu\in\sigma(\mathcal{L}_{\mathrm{per}})\setminus\{\lambda_j,\mu_j,\nu_{j,+},\nu_{j,-},j\geqslant 0\}} U(\beta,\mu),$$

which belongs to $\overline{\mathbb{C}}_+$ and contains

$$\sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j,\mu_j,\nu_{j,+},\nu_{j,-}, j \geqslant 0\}$$

as part of its boundary. The value of β is arbitrary here.

THEOREM 5.1. *Let $2a\omega/\pi \notin \mathbb{Z}$ and $q_1 \in L_1(\mathbb{R}_+)$, then the solution φ_α of the Cauchy problem*

$$-\varphi_\alpha''(x,\lambda) + \biggl(q(x) + \frac{c\sin(2\omega x+\delta)}{(x+1)^\gamma} + q_1(x)\biggr)\varphi_\alpha(x,\lambda) = \lambda\varphi_\alpha(x,\lambda), \quad x \in \mathbb{R}_+,$$
$$\varphi_\alpha(0,\lambda) = \sin\alpha, \qquad \varphi_\alpha'(0,\lambda) = \cos\alpha$$

has the following asymptotics: for every $\lambda \in U(\beta)$ there exists $A_\alpha(\lambda)$ such that

(1) *if* $\lambda \in \mathbb{C}_+ \cap U(\beta)$, *then*

$$\begin{aligned}\varphi_\alpha(x,\lambda) &= A_\alpha(\lambda)\psi_-(x,\lambda) + o(\mathrm{e}^{\operatorname{Im} k(\lambda)x/a}),\\ \varphi'_\alpha(x,\lambda) &= A_\alpha(\lambda)\psi'_-(x,\lambda) + o(\mathrm{e}^{\operatorname{Im} k(\lambda)x/a})\end{aligned}$$

as $x \to +\infty$,

(2) *if* $\lambda \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, \nu_{j,+}, \nu_{j,-}, j \geqslant 0\}$, *then*

$$\begin{aligned}\varphi_\alpha(x,\lambda) &= A_\alpha(\lambda)\psi_-(x,\lambda) + \overline{A_\alpha(\lambda)}\psi_+(x,\lambda) + o(1),\\ \varphi'_\alpha(x,\lambda) &= A_\alpha(\lambda)\psi'_-(x,\lambda) + \overline{A_\alpha(\lambda)}\psi'_+(x,\lambda) + o(1)\end{aligned}$$

as $x \to +\infty$.

The function A_α *is analytic in the interior of* $U(\beta)$ *and has boundary values on*

$$\sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, \nu_{j,+}, \nu_{j,-}, j \geqslant 0\}.$$

Proof. We are going to omit the index α here since the value of the boundary parameter is fixed throughout this proof. According to (3.2) and (3.4) we write

$$\begin{pmatrix}\varphi(x)\\ \varphi'(x)\end{pmatrix} = \begin{pmatrix}\psi_-(x,\lambda) & \psi_+(x,\lambda)\\ \psi'_-(x,\lambda) & \psi'_+(x,\lambda)\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}k(\lambda)x/a} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}k(\lambda)x/a}\end{pmatrix} v_\varphi(x,\lambda). \tag{5.1}$$

This is the definition of v_φ, a solution of (3.6) corresponding to φ. Let us fix the point

$$\mu \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, \nu_{j,+}, \nu_{j,-}, j \geqslant 0\}$$

and consider $\lambda \in U(\beta,\mu)$. The function

$$\tilde{v}_\varphi(x,\lambda,\mu) := \mathrm{e}^{-Q(x,\lambda,\mu)} v_\varphi(x,\lambda)$$

is a solution to (3.22) corresponding to φ. Let us see that the conditions of lemma 4.1 are satisfied for system (3.22) uniformly with respect to $\lambda \in U(\beta,\mu)$. First of all we have estimate (4.2) from lemma 3.4 and

$$\operatorname{Re}\nu(x,\lambda) = \frac{\operatorname{Im} k(\lambda)}{a} - \operatorname{Re}\left(\frac{c\sin(2\omega x+\delta)p_+(x,\lambda)p_-(x,\lambda)}{(x+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\right).$$

Estimating the second term in the same way as in lemma 3.1 we have

$$\begin{aligned}&\left|\int_x^y \operatorname{Re}\frac{c\sin(2\omega t+\delta)p_+(t,\lambda)p_-(t,\lambda)}{(t+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\,\mathrm{d}t\right|\\ &\qquad\leqslant \frac{|c|a}{\pi|W\{\psi_+(\lambda),\psi_-(\lambda)\}|}\\ &\qquad\quad\times\left(\sum_{n=-\infty}^{\infty}|\tilde{b}_n(\lambda)|\left(\frac{1}{|2a\omega/\pi+2n|}+\frac{1}{|2a\omega/\pi-2n|}\right)\right)\frac{1}{(x+1)^\gamma},\end{aligned}$$

where

$$\tilde{b}_n(\lambda) := \frac{1}{a}\int_0^a p_+(x,\lambda)p_-(x,\lambda)\mathrm{e}^{-2\pi\mathrm{i}nx/a}\,\mathrm{d}x$$

are Fourier coefficients for $p_+(\cdot,\lambda)p_-(\cdot,\lambda)$. Analogously to (3.19) we have

$$|\tilde{b}_n(\lambda)| \leqslant \frac{a}{4\pi^2 n^2}\int_0^a |(\psi_+(x,\lambda)\psi_-(x,\lambda))''|\,\mathrm{d}x.$$

So there exists $c_6(\beta,\mu)$ such that for every $\lambda\in U(\beta,\mu)$ and $n\neq 0$

$$|\tilde{b}_n(\lambda)| \leqslant \frac{c_6(\beta,\mu)}{n^2},$$

while

$$|\tilde{b}_0(\lambda)| \leqslant c_6(\beta,\mu).$$

Finally, there exists $c_7(\beta,\mu)$ such that

$$\left|\int_x^y \operatorname{Re}\frac{c\sin(2\omega t+\delta)p_+(t,\lambda)p_-(t,\lambda)}{(t+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\,\mathrm{d}t\right| \leqslant c_7(\beta,\mu)$$

for every $0\leqslant x\leqslant y$ and $\lambda\in U(\beta,\mu)$. Thus, we can take

$$M(\lambda)\equiv c_7(\beta,\mu)$$

for these values of λ. Lemma 4.1 gives the estimate

$$\|\tilde{v}_\varphi(x,\lambda,\mu)\| \leqslant \|\tilde{v}_\varphi(0,\lambda,\mu)\|\mathrm{e}^{\operatorname{Im}k(\lambda)x/a}c_8(\beta,\mu), \tag{5.2}$$

where

$$c_8(\beta,\mu) := \sqrt{1+\mathrm{e}^{4c_7(\beta,\mu)}}\exp\left(\sqrt{1+\mathrm{e}^{4c_7(\beta,\mu)}}\max_{\lambda\in U(\beta,\mu)}\int_0^\infty \|R^{(2)}(t,\lambda)\|\,\mathrm{d}t\right).$$

The conditions of lemma 4.2 are also satisfied; (4.9) holds for $\lambda\in\mathbb{R}\cap U(\beta,\mu)$ and (4.10) holds for $\lambda\in\mathbb{C}_+\cap U(\beta,\mu)$. So, lemma 4.2 gives the following asymptotics:

(i) for $\lambda\in\mathbb{C}_+\cap U(\beta,\mu)$,

$$\begin{aligned}
&\tilde{v}_\varphi(x,\lambda,\mu)\\
&\quad=\exp\left(-\mathrm{i}k(\lambda)\frac{x}{a}-\int_0^x\frac{c\sin(2\omega t+\delta)p_+(t,\lambda)p_-(t,\lambda)\,\mathrm{d}t}{(t+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\right)\\
&\qquad\times\Bigg(\begin{pmatrix}1&0\\0&0\end{pmatrix}\Bigg(\tilde{v}_\varphi(0,\lambda,\mu)\\
&\qquad\qquad+\int_0^\infty\exp\left(\mathrm{i}k(\lambda)\frac{t}{a}+\int_0^t\frac{c\sin(2\omega s+\delta)p_+(s,\lambda)p_-(s,\lambda)\,\mathrm{d}s}{(s+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\right)\\
&\qquad\qquad\qquad\times R^{(2)}(t,\lambda,\mu)\tilde{v}_\varphi(t,\lambda,\mu)\,\mathrm{d}t\Bigg)+o(1)\Bigg);
\end{aligned}$$

(ii) for $\lambda = \mu$,

$$\begin{aligned}
&\tilde{v}_\varphi(x,\mu,\mu)\\
&= \left(\begin{matrix} \exp\left(-\mathrm{i}k(\mu)\dfrac{x}{a} - \displaystyle\int_0^x \dfrac{c\sin(2\omega t+\delta)p_+(t,\mu)p_-(t,\mu)\,\mathrm{d}t}{(t+1)^\gamma W(\psi_+(\mu),\psi_-(\mu))}\right) & 0 \\ 0 & \exp\left(\mathrm{i}k(\mu)\dfrac{x}{a} + \displaystyle\int_0^x \dfrac{c\sin(2\omega t+\delta)p_+(t,\mu)p_-(t,\mu)\,\mathrm{d}t}{(t+1)^\gamma W(\psi_+(\mu),\psi_-(\mu))}\right) \end{matrix}\right)\\
&\quad\times \Bigg(\tilde{v}_\varphi(0,\mu,\mu)\\
&\qquad + \int_0^\infty \left(\begin{matrix} \exp\left(\mathrm{i}k(\mu)\dfrac{t}{a} + \displaystyle\int_0^t \dfrac{c\sin(2\omega s+\delta)p_+(s,\mu)p_-(s,\mu)\,\mathrm{d}s}{(s+1)^\gamma W(\psi_+(\mu),\psi_-(\mu))}\right) & 0 \\ 0 & \exp\left(-\mathrm{i}k(\mu)\dfrac{t}{a} - \displaystyle\int_0^t \dfrac{c\sin(2\omega s+\delta)p_+(s,\mu)p_-(s,\mu)\,\mathrm{d}s}{(s+1)^\gamma W(\psi_+(\mu),\psi_-(\mu))}\right) \end{matrix}\right)\\
&\qquad\qquad \times R^{(2)}(t,\mu,\mu)\tilde{v}_\varphi(t,\mu,\mu)\,\mathrm{d}t + o(1)\Bigg).
\end{aligned} \tag{5.3}$$

Since $Q(x,\lambda,\mu) = O(1/(x+1)^\gamma)$, we can define, for $\lambda \in U(\beta,\mu)$, that

$$\begin{aligned}
A(\lambda,\mu) := \Bigg\langle \begin{pmatrix}1\\0\end{pmatrix}, &\exp\left(-\int_0^\infty \frac{c\sin(2\omega t+\delta)p_+(t,\lambda)p_-(t,\lambda)\,\mathrm{d}t}{(t+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\right)\\
&\times \Bigg(\mathrm{e}^{-Q(0,\lambda,\mu)} v_\varphi(0,\lambda)\\
&\quad + \int_0^\infty \exp\left(\mathrm{i}k(\lambda)\frac{t}{a} + \int_0^t \frac{c\sin(2\omega s+\delta)p_+(s,\lambda)p_-(s,\lambda)\,\mathrm{d}s}{(s+1)^\gamma W\{\psi_+(\lambda),\psi_-(\lambda)\}}\right)\\
&\qquad \times R^{(2)}(t,\lambda,\mu)\mathrm{e}^{Q(t,\lambda,\mu)} v_\varphi(t,\lambda)\,\mathrm{d}t\Bigg)\Bigg\rangle
\end{aligned}$$

(where $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{C}^2$), which yields

$$\lim_{x\to+\infty} v_\varphi(x,\lambda)\mathrm{e}^{\mathrm{i}k(\lambda)x/a} = \begin{pmatrix} A(\lambda,\mu)\\ 0\end{pmatrix}. \tag{5.4}$$

From this, we see that the coefficient $A(\lambda,\mu)$ does not depend on μ, so we will denote it by $A(\lambda)$. Relation (5.1) can be written as

$$v_\varphi(x,\lambda) = \frac{1}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\begin{pmatrix}\psi_+'(x,\lambda)\varphi(x,\lambda)-\psi_+(x,\lambda)\varphi'(x,\lambda)\\ \varphi'(x,\lambda)\psi_-(x,\lambda)-\varphi(x,\lambda)\psi_-'(x,\lambda)\end{pmatrix}, \qquad (5.5)$$

so $v_\varphi(x,\cdot)$ is analytic in $\mathbb{C}_+$ and continuous up to

$$\sigma(\mathcal{L}_{\text{per}})\setminus\{\lambda_j,\mu_j,j\geqslant 0\}.$$

From the estimate (5.2) and properties of $Q(x,\lambda,\mu)$ and $R^{(2)}(x,\lambda,\mu)$ given by lemmas 3.1 and 3.4 it follows that $A(\lambda)$ is continuous in $U(\beta,\mu)$ and analytic in its interior. Thus, A is analytic in the interior of $U(\beta)$ having non-tangential boundary limits on

$$\sigma(\mathcal{L}_{\text{per}})\setminus\{\lambda_j,\mu_j,\nu_{j,+},\nu_{j,-},j\geqslant 0\}$$

that coincide with its values on this set.

The solution $\varphi(x,\lambda)$ and its derivative are real if λ is real. Thus, (5.5) shows that the upper and the lower components of the vector $v_\varphi(x,\lambda)$ are complex conjugate for $\lambda\in\sigma(\mathcal{L}_{\text{per}})\setminus\{\lambda_j,\mu_j,j\geqslant 0\}$. This property is preserved if we multiply the vector by a matrix X such that

$$X_{21}=\bar{X}_{12},\qquad X_{22}=\bar{X}_{11},$$

as in (3.21). It follows from lemma 3.4 that the upper and lower components of the vectors in the equality (5.3) are complex conjugate to each other. Hence for $\lambda=\mu$ we have

$$v_\varphi(x,\mu)=\begin{pmatrix}A(\mu)\mathrm{e}^{-\mathrm{i}k(\mu)x/a}\\ \overline{A(\mu)}\mathrm{e}^{\mathrm{i}k(\mu)x/a}\end{pmatrix}+o(1)\quad\text{as } x\to+\infty. \qquad (5.6)$$

The asymptotics of the solution φ and its derivative follows from (5.1), (5.4) and (5.6). □

Using the obtained asymptotics both on the spectrum and in $\mathbb{C}_+$ we now prove the Weyl–Titchmarsh-type formula.

Theorem 5.2. *Let $2a\omega/\pi\notin\mathbb{Z}$ and $q_1\in L_1(\mathbb{R}_+)$, then for almost all $\lambda\in\sigma(\mathcal{L}_{\text{per}})$ the spectral density of the operator $\mathcal{L}_\alpha$, defined by (1.1), is given by*

$$\rho_\alpha'(\lambda)=\frac{1}{2\pi|W\{\psi_+(\lambda),\psi_-(\lambda)\}||A_\alpha(\lambda)|^2},$$

where A_α is the same as in theorem 5.1.

Proof. In addition to φ_α consider another solution of (3.1), to be denoted by $\theta_\alpha:=\varphi_{\alpha+\pi/2}$, satisfying the initial conditions

$$\theta_\alpha(0,\lambda)=\cos\alpha,\qquad \theta_\alpha'(0,\lambda)=-\sin\alpha.$$

The Wronskian of φ_α and θ_α is equal to 1. Theorem 5.1 yields, for $\lambda\in U(\beta)\cap\mathbb{C}_+$,

$$\theta_\alpha(x,\lambda)=A_{\alpha+\pi/2}(\lambda)\psi_-(x,\lambda)+o(\mathrm{e}^{\operatorname{Im}k(\lambda)x/a})\quad\text{as } x\to+\infty.$$

Since the operator $\mathcal{L}_\alpha$ is in the limit point case, the combination

$$\theta_\alpha + m_\alpha \varphi_\alpha$$

belongs to $L_2(0,\infty)$ (where m_α is the Weyl function for $\mathcal{L}_\alpha$). It has the asymptotics

$$\theta_\alpha(x,\lambda) + m_\alpha(\lambda)\varphi_\alpha(x,\lambda) = (A_{\alpha+\pi/2}(\lambda) + m_\alpha A_\alpha(\lambda))\psi_-(x,\lambda) + o(\mathrm{e}^{\operatorname{Im} k(\lambda)x/a}).$$

Therefore,

$$m_\alpha(\lambda) = -\frac{A_{\alpha+\pi/2}(\lambda)}{A_\alpha(\lambda)}$$

for $\lambda \in U(\beta) \cap \mathbb{C}_+$ and

$$m_\alpha(\lambda + \mathrm{i}0) = -\frac{A_{\alpha+\pi/2}(\lambda)}{A_\alpha(\lambda)}$$

for $\lambda \in \sigma(\mathcal{L}_{\mathrm{per}}) \setminus \{\lambda_j, \mu_j, \nu_{j,+}, \nu_{j,-}, j \geqslant 0\}$. It follows from subordinacy theory [10] that the spectrum of $\mathcal{L}_\alpha$ on this set is purely and absolutely continuous and

$$\rho'_\alpha(\lambda) = \frac{1}{\pi}\operatorname{Im} m_\alpha(\lambda+\mathrm{i}0) = \frac{A_\alpha(\lambda)\overline{A_{\alpha+\pi/2}(\lambda)} - \overline{A_\alpha(\lambda)}A_{\alpha+\pi/2}(\lambda)}{2\pi\mathrm{i}|A_\alpha(\lambda)|^2}. \tag{5.7}$$

Theorem 5.1 yields, for these values of λ,

$$\theta_\alpha(x,\lambda) = A_{\alpha+\pi/2}(\lambda)\psi_-(x,\lambda) + \overline{A_{\alpha+\pi/2}(\lambda)}\psi_+(x,\lambda) + o(1),$$
$$\theta'_\alpha(x,\lambda) = A_{\alpha+\pi/2}(\lambda)\psi'_-(x,\lambda) + \overline{A_{\alpha+\pi/2}(\lambda)}\psi'_+(x,\lambda) + o(1),$$

as $x \to +\infty$. Substituting these asymptotics and the asymptotics of φ_α and φ'_α into the expression for the Wronskian, we get

$$1 = (\overline{A_\alpha(\lambda)}A_{\alpha+\pi/2}(\lambda) - A_\alpha(\lambda)\overline{A_{\alpha+\pi/2}(\lambda)})W\{\psi_+(\lambda), \psi_-(\lambda)\}$$

(the term $o(1)$ cancels, since both sides are independent of x). Combining with (5.7) we have

$$\rho'_\alpha(\lambda) = \frac{1}{-2\pi\mathrm{i}W\{\psi_+(\lambda),\psi_-(\lambda)\}|A_\alpha(\lambda)|^2} = \frac{1}{2\pi|W\{\psi_+(\lambda),\psi_-(\lambda)\}||A_\alpha(\lambda)|^2},$$

which completes the proof. □

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