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Spectral gap for quantum graphs and their edge connectivity

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Abstract
The spectral gap for Laplace operators on metric graphs and the relation between the graph's edge connectivity is investigated, in particular what happens to the gap if an edge is added to (or deleted from) a graph. It is shown that, in contrast to discrete graphs, the connection between the connectivity and the spectral gap is not one-to-one. The size of the spectral gap depends not only on the topology of the metric graph but on its geometric properties as well. It is shown that adding sufficiently large edges as well as cutting away sufficiently small edges leads to a decrease of the spectral gap. Corresponding explicit criteria are given.

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1. Introduction

The current paper is devoted to spectral properties of Laplace operators on metric graphs also known as *quantum graphs* [1, 6, 10, 14–17]. Our studies were inspired by classical results going back to Fiedler [11] on the second eigenvalue of discrete graphs and by a recent paper by Exner and Jex on the ground state for quantum graphs with delta-coupling [9]. Fiedler proposed to call the second lowest (the first excited) eigenvalue of the discrete Laplacian the *algebraic connectivity* of the corresponding discrete graph. This naming proposal is explained by the close relation between the algebraic connectivity and standard vertex and edge connectivities. For example the algebraic connectivity is zero if and only if the (discrete) graph is not connected. Laplace operators on metric graphs with standard matching conditions at the vertices preserve this property: the ground state is degenerate if and only if the graph is not connected. Exner and Jex investigated the behaviour of the ground state for (continuous) Laplacians on metric graphs as one of the edges is shortened or extended. It was shown that the bound state may increase as the length of an edge is also increasing and the opposite behaviour is expected.

Our goal is to study the behaviour of the first excited eigenvalue when edges are either deleted or added to a metric graph. Bearing in mind that the ground state for standard Laplacians is zero, the first excited eigenvalue gives us the spectral gap (of course, provided the graph is connected). Spectral properties of quantum graphs, especially with equilateral lengths of edges, are closely related to spectral properties of the corresponding discrete Laplacians. Therefore one might expect that the qualitative behaviour of eigenvalues for discrete and continuous Laplacians just coincides. However, it has been shown that the spectral gap for discrete and continuous Laplacians may behave differently as edges are added or deleted without altering the vertex set. This is connected to the fact that adding an edge to a discrete graph does not change the volume of the graph (which is the number of vertices in this case), while adding an edge to a metric graph enlarges the corresponding volume (which is the sum of lengths of the edges).

Adding or deleting an edge without changing the vertex set changes the graph's Euler characteristic. Our studies show how the spectral gap behaves as edges are added or removed. One may wonder whether something may be said about the Euler characteristic by looking at the spectral gap alone. It has been proven in [13, 18, 19], that the Euler characteristic is determined by spectral asymptotics and therefore cannot be recovered from the first a few eigenvalues alone, unless the metric graph consists of edges that are integer multiples of a basic length.

We would like to investigate the spectral gap in relation to the graph's edge connectivity and geometry. A graph is called k-edge-connected if it remains connected after removing fewer than k edges. It has already been proven in [12, 20] that the graph formed by just one edge (or a chain of edges) has the lowest spectral gap among all quantum graphs having the same total length. Therefore it is natural to expect that the spectral gap increases with edge connectivity. Adding an edge to a graph increases its edge connectivity, but the total length $\mathcal{L}$ increases as well. An increase of the total length may lead to a decrease of the spectral gap, since in accordance to Weyl's law the eigenvalues satisfy the asymptotics $\lambda_n \sim \left(\frac{\pi}{\mathcal{L}}\right)^2 n^2$. Similarly, deleting an edge may lead to both a decrease and an increase of the spectral gap. We study these phenomena in detail starting with the addition of edges.

This paper is organized as follows. We prove first a few elementary classical facts about the spectral gap for discrete Laplacians. We continue then with quantum graphs and study the behaviour of the spectral gap as two vertices are glued into one or as an edge is added between two already existing vertices. After that, we move to an exploration of the case where an edge is cut at a certain internal point or where a whole edge is deleted. It is shown that the spectral gap may grow even if a whole interval is cut away from the metric graph. Explicit estimates for the length of the edge that can be cut away are obtained.

Note that the dependence of the graph's spectrum on the coupling constant at the vertices and the edge lengths has been investigated in the interesting paper [2] (see also the recent book by the same authors [3]).

2. Discrete graphs (warming up)

Let G be a discrete graph with M vertices and N edges connecting some of the vertices. Then the corresponding Laplace operator $L(G)$ is defined on the finite dimensional space $\ell_2(G) = \mathbb{C}^M$ by the following formula [7, 8, 21]

$$(L(G)\psi)(m) = \sum_{n\sim m}(\psi(m) - \psi(n)), \tag{1}$$

where the sum is taken over all neighbouring vertices. The Laplace operator can also be defined using the connectivity matrix $C = \{c_{nm}\}$

$$c_{nm} = \begin{cases} 1, & \text{the vertices } n \text{ and } m \text{ are neighbours,} \\ & \text{i.e. connected by an edge;} \\ 0, & \text{otherwise,} \end{cases}$$

and the valence matrix $V = \operatorname{diag}\{v_1, v_2, \ldots, v_M\}$, where v_m are the valencies (degrees) of the corresponding vertices

$$L(G) = V - C,$$

which corresponds to the matrix realization of the operator $L(G)$ in the canonical basis given by the vertices. In the literature one may find other definitions for discrete Laplacians. In [4], [5] one essentially uses the congruent matrix

$$\mathbb{L}(G) = V^{-1/2}L(G)V^{-1/2} = I - V^{-1/2}CV^{-1/2}. \tag{2}$$

Such a definition of the Laplacian matrix is consistent with the analysis of eigenvalues in spectral geometry. The following Laplacian matrix connected with the averaging operator is similar to (2):

$$\mathbf{L}(G) = V^{-1}L(G) = V^{-1/2}\mathbb{L}(G)V^{1/2}. \tag{3}$$

Equation (3) is important for studies of quantum graphs, since its eigenvalues are closely related to the spectrum of the corresponding equilateral graphs.

We now briefly discuss spectral properties of the standard (sometimes called combinatorial) Laplace matrix $L(G)$ given by (1), first of all in relation to the set of edges.

Since the Laplace operator is uniquely defined by the discrete graph G, its eigenvalues $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_{M-1}$ are usually referred to as the eigenvalues of G. The ground state corresponding to $\lambda_0 = 0$ has eigenfunction $\psi_0 = 1$, where $1 \in \mathbb{C}^M$ denotes the vector built up of ones on G. The multiplicity of the ground state coincides with the number of connected components in G. In order to avoid artificial complications only connected graphs will be considered in the sequel. Then the *spectral gap* $\lambda_1 - \lambda_0$ for the discrete Laplacian coincides with the first excited eigenvalue λ_1.

The spectral gap of a discrete graph is a monotonously increasing function of the set of edges. In other words, adding an edge always causes the increase of the second eigenvalue or keeps it unchanged, provided we have the same set of vertices.

Proposition 1. *Let G be a connected discrete graph and let G' be a discrete graph obtained from G by adding one edge between the vertices m_1 and m_2. Let L denote the discrete Laplacian defined by (1). Then the following holds:*

1. *The first excited eigenvalues satisfy the inequality:*

$$\lambda_1(G) \leqslant \lambda_1(G').$$

2. *The equality $\lambda_1(G) = \lambda_1(G')$ holds if and only if the second eigenfunction ψ_1^G on the graph G may be chosen attaining equal values at the vertices m_1 and m_2*

$$\psi_1^G(m_1) = \psi_1^G(m_2).$$

Proof. The first statement follows from the fact that

$$L(G') - L(G) = \begin{pmatrix} & \vdots & & \vdots & \\ \dots & 1 & \dots & -1 & \dots \\ & \vdots & & \vdots & \\ \dots & -1 & \dots & 1 & \dots \\ & \vdots & & \vdots & \end{pmatrix} \tag{4}$$

is a matrix with just four nonzero entries. It is easy to see that the matrix is positive semi-definite, since the eigenvalues are 0 (with the multiplicity $M-1$) and 2 (simple eigenvalue) and therefore $L(G') - L(G) \geqslant 0$ which implies the first statement.

To prove the last assertion let us recall that $\lambda_1(G')$ can be calculated using the Rayleigh quotient

$$\lambda_1(G') = \min_{\psi \perp 1} \frac{\langle \psi, L(G')\psi \rangle}{\langle \psi, \psi \rangle} \geqslant \min_{\psi \perp 1} \frac{\langle \psi, L(G)\psi \rangle}{\langle \psi, \psi \rangle} = \lambda_1(G).$$

Hence the trial function ψ should be chosen orthogonal to the ground state, i.e. with the mean value zero. We have equality in the last formula if and only if ψ minimizing the first and the second quotients can be chosen such that $(L(G') - L(G))\psi = 0$, i.e. $\psi(m_1) = \psi(m_2)$. □

Next we are interested in what happens if we add a pending edge, i.e. an edge connected to the graph at one already existing node.

Proposition 2. *Let G be a connected discrete graph and let G' be another graph obtained from G by adding one vertex and one edge between the new vertex and the vertex m_1. Then the following hold:*

1. *The first excited eigenvalues satisfy the following inequality:*
$$\lambda_1(G) \geqslant \lambda_1(G').$$
2. *The equality $\lambda_1(G) = \lambda_1(G')$ holds if and only if every eigenfunction ψ_1^G corresponding to $\lambda_1(G)$ is equal to zero at m_1*
$$\psi_1^G(m_1) = 0.$$

Proof. Let us define the following vector on G':

$$\varphi(n) := \begin{cases} \psi_1^G(n), & \text{on } G, \\ \psi_1^G(m_1) & \text{on } G' \backslash G. \end{cases}$$

This vector is not orthogonal to the zero energy eigenfunction $1 \in \mathbb{C}^{M+1}$, where we keep the same notation 1 for the vector build up of ones now on G'. Therefore consider the nonzero vector γ shifted by a constant c

$$\gamma(n) := \varphi(n) + c.$$

Here c is chosen so that the orthogonality condition in $l_2(G') = \mathbb{C}^{M+1}$ holds

$$0 = \langle \gamma, 1 \rangle_{l_2(G')} = \underbrace{\langle \psi_1^G, 1 \rangle_{l_2(G)}}_{=0} + \psi_1^G(m_1) + cM',$$

where $M' = M + 1$ is the number of vertices in G'. This implies

$$c = -\frac{\psi_1^G(m_1)}{M'}.$$

Using this vector the following estimate on the first eigenvalue may be obtained:

$$\lambda_1(G') \leqslant \frac{\langle L(G')\gamma, \gamma\rangle_{l_2(G')}}{\|\gamma\|^2_{l_2(G')}} = \frac{\langle L(G)\psi_1^G, \psi_1^G\rangle_{l_2(G)}}{\|\psi_1^G\|^2_{l_2(G)} + c^2M + |\psi_1^G(m_1) + c|^2} \leqslant \lambda_1(G). \tag{5}$$

The last inequality follows from the fact that

$$\langle L(G)\psi_1^G, \psi_1^G\rangle_{l_2(G)} = \lambda_1(G)\|\psi_1^G\|^2,$$

and

$$\|\psi_1^G\|^2_{l_2(G)} + c^2M + |\psi_1^G(m_1) + c|^2 \geqslant \|\psi_1^G\|^2_{l_2(G)}.$$

Note that we have equality if and only if $c = 0$ and $|\psi_1^G(m_1) + c|^2 = 0$ which implies $\psi_1^G(m_1) = 0$. If there exists a ψ_1^G, such that $\psi_1^G(m_1) \neq 0$, then the inequality in (5) is strict and we get

$$\lambda_1(G) > \lambda_1(G').$$

□

We see that the first excited eigenvalue has a tendency to decrease if a pending edge is attached to a graph. It is clear from the proof that gluing of any connected graph (instead of one edge) would lead to the same result, provided there is just one contact vertex. If the number of contact vertices is larger, then the spectral gap may increase as shown in proposition 1.

Note that a different proof of the first part of proposition 1 may be found in [11], corollary 3.2. In the same paper, a slightly weaker claim related to the first part of proposition 2 is provided as property 3.3.

3. Quantum graphs: definitions

Let Γ be a metric graph formed by $N = N(\Gamma)$ compact edges $E_n = [x_{2n-1}, x_{2n}]$, $n = 1, 2, \ldots, N$ (identified with intervals on $\mathbb{R}$) joined together at $M = M(\Gamma)$ vertices (nodes). The free Laplace operator $L^{\mathrm{st}}(\Gamma) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is defined in the Hilbert space $L_2(\Gamma) = \bigoplus_{n=1}^N L_2(E_n)$ on functions $u \in \bigoplus_{n=1}^N W_2^2(E_n)$ satisfying standard matching conditions at the vertices V_m, $m = 1, 2, \ldots, M$

$$\begin{cases} u \text{ is continuous at } V_m, \\ \text{the sum of normal derivatives is zero.} \end{cases} \tag{6}$$

The free Laplacian is self-adjoint in $L_2(\Gamma)$ and is uniquely determined by the metric graph Γ. The quadratic form of L^{st} is defined on the domain $\overset{c}{W}{}_2^1(\Gamma)$ consisting of all functions from $\bigoplus_{n=1}^N W_2^1(E_n)$ which are also continuous at the vertices[3]. Since the domain of the operator is invariant under complex conjugation, the corresponding eigenfunction may be chosen to be real. Therefore, in order to simplify our presentation we assume that the eigenfunctions are real.

The spectrum of the Laplacian is discrete $\lambda_0 = 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and will be referred to as the spectrum of Γ. If Γ is connected, then the ground state $\lambda_0 = 0$ has multiplicity one and the corresponding eigenfunction is $\psi_0^\Gamma = 1$. Since only connected graphs will be considered, the spectral gap $\lambda_1(\Gamma) - \lambda_0(\Gamma)$ coincides with the energy of the first excited state λ_1.

[3] Any function from the Sobolev space $W_2^1(\Gamma)$ is continuous inside each edge, but such functions are not necessarily continuous at the vertices.

4. Increasing edge connectivity: gluing vertices together

As already mentioned, the spectral gap has been extensively investigated for discrete graphs. Our goal here is to study the spectral gap for Laplacians on metric graphs especially in relation to the edge connectivity of the underlying metric graphs. More precisely, our aim is to prove the analogues of propositions 1 and 2 for quantum graphs. Our original idea was to study the behaviour of the spectral gap when a new edge is added to the original metric graph. But this procedure increases the total length of the graph and it is therefore not surprising that the spectral gap has tendency to decrease in contrast to proposition 1 (see theorem 3 below). Hence let us start our studies by presenting a direct analogue of proposition 1 for quantum graphs. The corresponding theorem answers the following question: what happens to the spectral gap if two vertices in a metric graph are joined into one common vertex? This procedure does not change the set of edges and therefore the total length of the graph is also preserved, but increases the graph's edge connectivity instead.

Theorem 1. *Let Γ be a connected metric graph and let Γ' be another metric graph obtained from Γ by joining together two of its vertices, say V_1 and V_2. Then the following holds:*

1. *The spectral gap satisfies the inequality*

$$\lambda_1(\Gamma) \leqslant \lambda_1(\Gamma'). \tag{7}$$

2. *The equality $\lambda_1(\Gamma) = \lambda_1(\Gamma')$ holds if and only if the eigenfunction ψ_1 corresponding to the first excited state can be chosen such that*

$$\psi_1(V_1) = \psi_1(V_2). \tag{8}$$

Proof. The first excited state can be calculated by minimizing the Rayleigh quotient $\frac{\|u'\|^2}{\|u\|^2}$ corresponding to the standard Laplacian over the set of functions from the domain of the quadratic form which are in addition orthogonal to the ground state eigenfunction $\psi_0 = 1$. For the original graph Γ the domain of the quadratic form consists of all $W_2^1(\Gamma)$ functions which are continuous at all vertices of Γ. The corresponding set for Γ' is characterized by one additional condition $u(V_1) = u(V_2)$—continuity of the function at the new vertex $V_1 \cup V_2$. Inequality (7) for the corresponding minima follows.

To prove the second statement we first note that if the minimizing function ψ_1 for Γ satisfies in addition (8), then the same function is a minimizer for Γ' and the corresponding eigenvalues coincide. It is clear since the domain of the quadratic form keeps only the continuity of functions at the vertices. Conversely if $\lambda_1(\Gamma) = \lambda_1(\Gamma')$, then the eigenfunction for $L^{\mathrm{st}}(\Gamma')$ is also a minimizer for the Rayleigh quotient for Γ and therefore is an eigenfunction for $L^{\mathrm{st}}(\Gamma)$ satisfying in addition (8). □

Proposition 1 and theorem 1 appear to be rather similar at first glance. But the reasons for the spectral gap to increase are different. In the case of discrete graphs, the difference between the Laplace operators is a nonnegative matrix. For quantum graphs the differential operators are identical, but inequality (7) is valid due to the fact that the opposite inequality holds for the domains of the quadratic forms.

Corollary 1. *Theorem 1 implies that the flower graph consisting of N loops attached to one vertex has the largest spectral gap among all graphs formed by a given set of edges.* (See figure 1.)

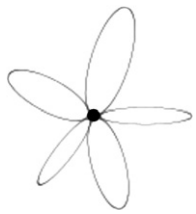

Figure 1. Flower graph.

5. Adding an edge

Our goal in this section is to study behaviour of the spectral gap as an extra edge is added to the metric graph. We start by proving a direct analogue of proposition 2.

Theorem 2. *Let Γ be a connected metric graph and let Γ' be another graph obtained from Γ by adding one vertex and one edge connecting the new vertex with the vertex V_1.*

1. *The first eigenvalues satisfy the following inequality:*
$$\lambda_1(\Gamma) \geqslant \lambda_1(\Gamma').$$
2. *The equality $\lambda_1(\Gamma) = \lambda_1(\Gamma')$ holds if and only if every eigenfunction ψ_1 corresponding to $\lambda_1(\Gamma)$ is equal to zero at V_1*
$$\psi_1(V_1) = 0.$$

Proof. Let us define the following function on Γ':
$$f(x) := \begin{cases} \psi_1(x), & x \in \Gamma, \\ \psi_1(V_1) & x \in \Gamma' \backslash \Gamma. \end{cases}$$
This function is in general not orthogonal to the zero energy eigenfunction $1 \in L_2(\Gamma')$. Therefore consider the nonzero function g differed from f by a constant
$$g(x) := f(x) + c,$$
where c is chosen so that the orthogonality condition in $L_2(\Gamma')$ holds
$$0 = \langle g(x), 1\rangle_{L_2(\Gamma')} = \underbrace{\langle \psi_1, 1\rangle_{L_2(\Gamma)}}_{=0} + \psi_1(V_1)\ell + c\mathcal{L}',$$
where ℓ and $\mathcal{L}'$ are the length of the added edge and the total length of Γ' respectively. This implies
$$c = -\frac{\psi_1(V_1)\ell}{\mathcal{L}'}.$$
Using this vector the following estimate for the first eigenvalue may be obtained:
$$\lambda_1(\Gamma') \leqslant \frac{\langle L^{\mathrm{st}} g, g\rangle_{L_2(\Gamma')}}{\|g\|^2_{L_2(\Gamma')}} = \frac{\langle L^{\mathrm{st}}\psi_1, \psi_1\rangle_{L_2(\Gamma)}}{\|\psi_1\|^2_{L_2(\Gamma)} + c^2\mathcal{L} + |\psi_1(V_1) + c|^2\ell} \leqslant \lambda_1(\Gamma).$$
Here $\mathcal{L}$ denotes the total length of the metric graph Γ. The last inequality follows from the fact that
$$\langle L^{\mathrm{st}}\psi_1, \psi_1\rangle_{L_2(\Gamma)} = \lambda_1(\Gamma)\|\psi_1\|^2,$$

and

$$\|\psi_1\|^2_{L_2(\Gamma)} + c^2\mathcal{L} + |\psi_1(V_1) + c|^2\ell \geqslant \|\psi_1\|^2_{L_2(\Gamma)}.$$

Note that in the last expression the equality holds if and only if $c = 0$ and $|\psi_1(V_1)+c|^2 = 0$ which implies $\psi_1(V_1) = 0$. This proves the second assertion. □

In the proof of the last theorem we did not really use that $\Gamma'\backslash\Gamma$ is an edge. It is straightforward to generalize the theorem for the case where $\Gamma'\backslash\Gamma$ is an arbitrary finite connected graph joined to Γ at one vertex only.

We return now to our original goal and investigate the behaviour of the spectral gap when an edge between two vertices is added to a metric graph.

Theorem 3. *Let Γ be a connected metric graph and L^{st}—the corresponding free Laplace operator. Let Γ' be a metric graph obtained from Γ by adding an edge between the vertices V_1 and V_2. Assume that the eigenfunction ψ_1 corresponding to the first excited eigenvalue can be chosen such that*

$$\psi_1(V_1) = \psi_1(V_2). \tag{9}$$

Then the following inequality for the second eigenvalues holds:

$$\lambda_1(\Gamma) \geqslant \lambda_1(\Gamma').$$

Proof. To prove the inequality let us consider the eigenfunction $\psi_1(\Gamma)$ for $L^{\mathrm{st}}(\Gamma)$. We introduce a new function on Γ'

$$f(x) = \begin{cases} \psi_1(x), & x \in \Gamma, \\ \psi_1(V_1) \quad (= \psi_1(V_2)) & x \in \Gamma'\backslash\Gamma. \end{cases}$$

This function is not orthogonal to the constant function. Let us adjust the constant c so that the nonzero function $g(x) = f(x) + c$ is orthogonal to 1 in $L_2(\Gamma')$[4]:

$$0 = \langle g(x), 1\rangle_{L_2(\Gamma')} = \underbrace{\langle \psi_1(x), 1\rangle_{L_2(\Gamma)}}_{=0} + \psi_1(V_1)\ell + c\mathcal{L}' = 0,$$

where ℓ is the length of the added edge and $\mathcal{L}'$ is the total length of the graph Γ', as before. We have used that the eigenfunction ψ_1 has mean value zero, i.e. is orthogonal to the ground state. This implies

$$c = -\frac{\psi_1(V_1)\ell}{\mathcal{L}'}.$$

Now we are ready to get an estimate for $\lambda_1(\Gamma')$ using Rayleigh quotient

$$\lambda_1(\Gamma') \leqslant \frac{\langle L^{\mathrm{st}}(\Gamma')g, g\rangle_{L_2(\Gamma')}}{\|g\|^2_{L_2(\Gamma')}}.$$

The numerator and denominator can be evaluated as follows

$$\langle L^{\mathrm{st}}(\Gamma')g, g\rangle_{L_2(\Gamma')} = \langle L^{\mathrm{st}}(\Gamma)\psi_1, \psi_1\rangle_{L_2(\Gamma)} = \lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)},$$

$$\begin{aligned} \|g\|^2_{L_2(\Gamma')} &= \|\psi_1 + c\|^2_{L_2(\Gamma)} + |\psi_1(V_1) + c|^2\ell \\ &= \|\psi_1\|^2_{L_2(\Gamma)} + c^2\mathcal{L} + |\psi_1(V_1) + c|^2\ell \\ &\geqslant \|\psi_1\|^2_{L_2(\Gamma)}. \end{aligned}$$

[4] In what follows we are going to use the same notation 1 for the functions identically equal to one on both metric graphs Γ and Γ'.

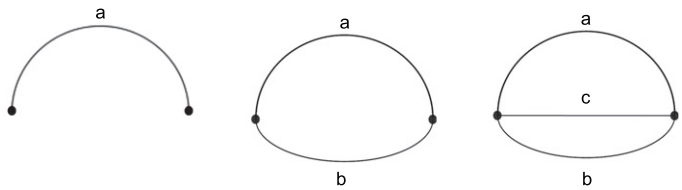

Figure 2. Graphs Γ, Γ', and Γ''.

It follows, that

$$\lambda_1(\Gamma) \geqslant \lambda_1(\Gamma'). \qquad \square$$

Let us illustrate the above theorem with a couple of examples:

Example 1. Let Γ be the graph formed by one edge of length a (see figure 2). The spectrum of $L^{\mathrm{st}}(\Gamma)$ is

$$\sigma(L^{\mathrm{st}}(\Gamma)) = \left\{\left(\frac{\pi}{a}\right)^2 n^2\right\}_{n=0}^{\infty}.$$

All eigenvalues have multiplicity one.

Consider the graph Γ' obtained from Γ by adding an edge of length b, so that Γ' is formed by two intervals of lengths a and b connected in parallel. The graph Γ' is equivalent to the circle of length $a+b$. The spectrum is:

$$\sigma(L^{\mathrm{st}}(\Gamma')) = \left\{\left(\frac{2\pi}{a+b}\right)^2 n^2\right\}_{n=0}^{\infty},$$

where all the eigenvalues except for the ground state have double multiplicity.

Let us study the relation between the first eigenvalues:

$$\lambda_1(\Gamma) = \frac{\pi^2}{a^2}, \ \lambda_1(\Gamma') = \frac{4\pi^2}{(a+b)^2}.$$

Any relation between these values is possible:

$$b > a \ \Rightarrow \ \lambda_1(\Gamma) > \lambda_1(\Gamma'),$$
$$b < a \ \Rightarrow \ \lambda_1(\Gamma) < \lambda_1(\Gamma').$$

Therefore the first eigenvalue is not in general a monotone decreasing function of the set of edges. The spectral gap decreases only if certain additional conditions are satisfied.

Example 2. Consider, in addition to graph Γ' discussed in example 1 , the graph Γ'' obtained from Γ' by adding another one edge of length c between the same two vertices. Hence Γ'' is formed by three parallel edges of lengths a, b and c. The first eigenfunction for $L^{\mathrm{st}}(\Gamma')$ can always be chosen so that its values at the vertices are equal. Then, in accordance with theorem 3, the first eigenvalue for Γ'' is less or equal to the first eigenvalue for Γ':

$$\lambda_1(\Gamma'') \leqslant \lambda_1(\Gamma').$$

This fact can easily be supported by explicit calculations.

The above examples and proved theorems show that the spectral gap has a tendency to decrease when a new sufficiently long edge is added. It is not surprising, since the addition of an edge increases the total length of the graph, but the eigenvalues satisfy Weyl's law and therefore are asymptotically close to $(\pi n)^2/\mathcal{L}^2$. This is in contrast to discrete graphs, for which the addition of an edge does not lead to an increase in the number of vertices.

Condition (9) in theorem 3 is not easy to check for non-trivial graphs and therefore it might be interesting to obtain another explicit sufficient conditions. In what follows we would like to discuss one such geometric condition ensuring that the spectral gap drops as a new edge is added to a graph. The main idea is to compare the length ℓ of the new edge with the total length of the original graph $\mathcal{L}(\Gamma)$. It appears that if $\ell > \mathcal{L}(\Gamma)$, then the spectral gap always decreases. We have already observed this phenomenon when discussing example 1, where the behaviour of λ_1 depended on the ratio between the lengths a and b. If $b \equiv \ell > a \equiv \mathcal{L}(\Gamma)$, then the gap decreases. It is surprising that the same explicit condition holds for arbitrary connected graphs Γ.

Theorem 4. *Let Γ be a connected finite compact metric graph of length $\mathcal{L}(\Gamma)$ and let Γ' be a graph constructed from Γ by adding an edge of length ℓ between certain two vertices. If*

$$\ell > \mathcal{L}(\Gamma), \tag{10}$$

then the eigenvalues of the corresponding free Laplacians satisfy the estimate

$$\lambda_1(\Gamma) \geqslant \lambda_1(\Gamma'). \tag{11}$$

Proof. Let ψ_1 be any eigenfunction corresponding to the first excited eigenvalue $\lambda_1(\Gamma)$ of $L^{\mathrm{st}}(\Gamma)$. It follows that the minimum of the Rayleigh quotient is attained at ψ_1:

$$\lambda_1(\Gamma) = \min_{u \in \overset{c}{W}{}_2^1(\Gamma): u \perp 1} \frac{\|u'\|^2_{L_2(\Gamma)}}{\|u\|^2_{L_2(\Gamma)}} = \frac{\|\psi_1'\|^2_{L_2(\Gamma)}}{\|\psi_1\|^2_{L_2(\Gamma)}},$$

where $\overset{c}{W}{}_2^1(\Gamma)$ denotes the set of continuous on graph Γ W_2^1-functions. Let us denote by V_1 and V_2 the vertices in Γ, where the new edge E of length ℓ is attached.

The eigenvalue $\lambda_1(\Gamma')$ can again be estimated using the Rayleigh quotient

$$\lambda_1(\Gamma') = \min_{u \in \overset{c}{W}{}_2^1(\Gamma'): u \perp 1} \frac{\|u'\|^2_{L_2(\Gamma')}}{\|u\|^2_{L_2(\Gamma')}} \leqslant \frac{\|g'\|^2_{L_2(\Gamma)}}{\|g\|^2_{L_2(\Gamma)}}, \tag{12}$$

where $g(x)$ is any function in $\overset{c}{W}{}_2^1(\Gamma')$ orthogonal to constant function in $L_2(\Gamma')$. Let us choose the trial function g of the form $g(x) = f(x) + c$ where

$$f(x) := \begin{cases} \psi_1(x), & x \in \Gamma, \\ \gamma_1 + \gamma_2 \sin\left(\frac{\pi x}{\ell}\right) & x \in \Gamma' \backslash \Gamma = E = [-\ell/2, \ell/2], \end{cases} \tag{13}$$

with $\gamma_1 = (\psi_1(V_1) + \psi_1(V_2))/2$ and $\gamma_2 = (\psi_1(V_2) - \psi_1(V_1))/2$. Here we assumed that the left end point of the interval is connected to V_1 and the right end point to V_2. The function f obviously belongs to $\overset{c}{W}{}_2^1(\Gamma')$, since it is continuous at V_1 and V_2 , but it is not necessarily orthogonal to the ground state eigenfunction 1. The constant c is adjusted in order to ensure the orthogonality

$$\langle g, 1\rangle_{L_2(\Gamma')} = 0$$

holds. The constant c can easily be calculated

$$0 = \langle g, 1\rangle_{L_2(\Gamma')} = c\mathcal{L}' + \underbrace{\langle \psi_1, 1\rangle_{L_2(\Gamma)}}_{=0} + \int_{-\ell/2}^{\ell/2} \left(\gamma_1 + \gamma_2 \sin\left(\frac{\pi x}{\ell}\right)\right) \mathrm{d}x = c\mathcal{L}' + \gamma_1 \ell$$

$$\Rightarrow c = -\frac{\gamma_1 \ell}{\mathcal{L}'}. \tag{14}$$

The function g can be used as a trial function in (12) to estimate the spectral gap. Let us begin by computing the denominator using the fact that g is orthogonal to 1

$$\begin{aligned}\|g\|^2_{L_2(\Gamma')} &= \|f+c\|^2_{L_2(\Gamma')} = \|f\|^2_{L_2(\Gamma')} - \|c\|^2_{L_2(\Gamma')} \\ &= \|\psi_1\|^2_{L_2(\Gamma)} + \int_{-\ell/2}^{\ell/2}\left(\gamma_1+\gamma_2\sin\left(\frac{\pi x}{\ell}\right)\right)^2 \mathrm{d}x - c^2\mathcal{L}' \\ &= \|\psi_1\|^2_{L_2(\Gamma)} + \ell\gamma_1^2 + \frac{\ell}{2}\gamma_2^2 - c^2\mathcal{L}'.\end{aligned} \tag{15}$$

The numerator yields

$$\begin{aligned}\|g'\|^2_{L_2(\Gamma')} &= \|f'\|^2_{L_2(\Gamma')} = \|\psi_1'\|^2_{L_2(\Gamma)} + \int_{-\ell/2}^{\ell/2}\left(\gamma_2^2\frac{\pi^2}{\ell^2}\cos^2\left(\frac{\pi x}{\ell}\right)\right)\mathrm{d}x \\ &= \lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)} + \gamma_2^2\frac{\pi^2}{2\ell}.\end{aligned} \tag{16}$$

After plugging (15) and (16) into (12) we obtain

$$\lambda_1(\Gamma') \leqslant \frac{\lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)} + \gamma_2^2\frac{\pi^2}{2\ell}}{\|\psi_1\|^2_{L_2(\Gamma)} + \ell\gamma_1^2 + \frac{\ell}{2}\gamma_2^2 - c^2\mathcal{L}'}.$$

Using (14) the last estimate can be written as

$$\lambda_1(\Gamma') \leqslant \frac{\lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)} + \gamma_2^2\frac{\pi^2}{2\ell}}{\|\psi_1\|^2_{L_2(\Gamma)} + \ell\gamma_1^2\left(1-\frac{\ell}{\mathcal{L}'}\right) + \frac{\ell}{2}\gamma_2^2} \leqslant \frac{\lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)} + \gamma_2^2\frac{\pi^2}{2\ell}}{\|\psi_1\|^2_{L_2(\Gamma)} + \frac{\ell}{2}\gamma_2^2}, \tag{17}$$

where we used that $\ell < \mathcal{L}' = \mathcal{L} + \ell$. It remains to take into account the following estimate for λ_1 proven in [12, 20]

$$\lambda_1(\Gamma) \geqslant \left(\frac{\pi}{\mathcal{L}}\right)^2. \tag{18}$$

Then taking into account (10) estimate (17) can be written as

$$\lambda_1(\Gamma') \leqslant \frac{\lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)} + \lambda_1(\Gamma)\gamma_2^2\ell/2}{\|\psi_1\|^2_{L_2(\Gamma)} + \gamma_2^2\ell/2} = \lambda_1(\Gamma). \tag{19}$$

This proves the theorem. □

Estimate (18) was crucial for our proof. It relates the spectral gap and the total length of the metric graph, i.e. geometric and spectral properties of quantum graphs. It might be interesting to prove an analogue of the last theorem for discrete graphs. Proposition 1 states that the spectral gap increases if one edge is added to a discrete graph. Adding a long edge should correspond to adding a chain of edges to a discrete graph.

The previous theorem gives us a sufficient geometric condition for the spectral gap to decrease. Let us now study the case where the spectral gap is increasing. Similarly, as we proved that adding one edge that is *long enough* always makes the spectral gap smaller (theorem 4), we claim that an edge that is *short enough* makes it grow. We have already seen in theorem 1 that adding an edge of zero length (joining two vertices into one) may lead to an increase of the spectral gap. It appears that the criterion for a gap to decrease can be formulated explicitly in terms of the eigenfunction on the larger graph. Therefore let us change our point of view and study the behaviour of the spectral gap as an edge is deleted.

6. Decreasing edge connectivity: cutting edges

In the following section we will study the spectral gap's behaviour when one of the edges is deleted. The result of such a procedure is not obvious, since the deletion of an edge decreases the total length of the metric graph and one expects that the first excited eigenvalue increases. On the other hand deleting an edge decreases the graph's edge connectivity and therefore the spectral gap is expected to decrease. It is easy to construct examples when one of these two tendencies prevails: example 1 shows that the spectral gap may both decrease and increase when an edge is deleted.

Let us discuss first what happens when one of the edges is cut in a certain internal point. Let Γ^* be a connected metric graph obtained from a metric graph Γ by cutting one of the edges, say $E_1 = [x_1, x_2]$ at a point $x^* \in (x_1, x_2)$. It will be convenient to denote by x_1^* and x_2^* the points on the two sides of the cut. In other words, the graph Γ^* has precisely the same set of edges and vertices as Γ except that the edge $[x_1, x_2]$ is substituted by two edges $[x_1, x_1^*]$ and $[x_2^*, x_2]$ and two new vertices $V_1^* = \{x_1^*\}$ and $V_2^* = \{x_2^*\}$ are added to the set of vertices.

The spectral gap for the graphs Γ and Γ^* can be calculated by minimizing the same Rayleigh quotient over the set of W_2^1-functions with zero average. The only difference is that the functions used to calculate $\lambda_1(\Gamma)$ are necessarily continuous at x^*

$$u(x_1^*) = u(x_2^*)$$

(as functions from $W_2^1[x_1, x_2]$). The functions used in calculating $\lambda_1(\Gamma^*)$ do not necessarily attain the same values at the points x_1^* and x_2^*. It follows that $\lambda_1(\Gamma^*) \leqslant \lambda_1(\Gamma)$, since the set of admissible functions is larger for Γ^*. If the minimizing function for Γ^* has the same values at x_1^* and x_2^*, then it is also an eigenfunction for $L^{\mathrm{st}}(\Gamma)$ and therefore $\lambda_1(\Gamma^*) = \lambda_1(\Gamma)$. Moreover, if the spectral gap for the graphs is the same, then every function minimizing the quotient for Γ minimizes the quotient for Γ^* as well and therefore satisfies Neumann condition at x^*. It follows that every eigenfunction for $L^{\mathrm{st}}(\Gamma)$ corresponding to λ_1 is also an eigenfunction for $L^{\mathrm{st}}(\Gamma^*)$. This proves the following theorem.

Theorem 5. *Let Γ be a connected metric graph and let Γ^* be another graph obtained from Γ by cutting one of the edges at an internal point x^* producing two new vertices V_1^* and V_2^*.*

1. *The first excited eigenvalues satisfy the following inequality*

$$\lambda_1(\Gamma^*) \leqslant \lambda_1(\Gamma). \tag{20}$$

2. *If $\lambda_1(\Gamma^*) = \lambda_1(\Gamma)$ then every eigenfunction of $L^{\mathrm{st}}(\Gamma)$ corresponding to $\lambda_1(\Gamma)$ satisfies Neumann condition at the cut point x^*: $\psi_1'(x^*) = 0$. If at least one of the eigenfunctions on Γ^* satisfies $\psi_1^*(V_1^*) = \psi_1^*(V_2^*)$, then $\lambda_1(\Gamma^*) = \lambda_1(\Gamma)$.*

This theorem is a certain reformulation of theorem 1 and implies that the spectral gap has a tendency to decrease when an edge is cut at an internal point. Note that this time the total length of the graph is preserved.

7. Deleting an edge

Let us now study what happens if an edge is deleted, or if a whole interval is cut away from an edge (without gluing the remaining intervals together). Let Γ be a connected metric graph as before and let Γ^* be a graph obtained from Γ by deleting one of the edges.

The following theorem proves a sufficient condition that guarantees that the spectral gap is decreasing as one of the edges is deleted.

Theorem 6. *Let Γ be a connected finite compact metric graph of the total length $\mathcal{L}$ and let Γ^* be another connected metric graph obtained from Γ by deleting one edge of length ℓ between certain vertices V_1 and V_2. Assume in addition that*

$$\left(\max_{\psi_1:L^{\mathrm{st}}(\Gamma)\psi_1=\lambda_1\psi_1}\frac{(\psi_1(V_1)-\psi_1(V_2))^2}{(\psi_1(V_1)+\psi_1(V_2))^2}\cot^2\frac{k_1\ell}{2}-1\right)\frac{k_1}{2}\cot\frac{k_1\ell}{2}\geqslant(\mathcal{L}-\ell)^{-1}, \tag{21}$$

where $\lambda_1(\Gamma)=k_1^2,\ k_1>0$ is the first excited eigenvalue of $L^{\mathrm{st}}(\Gamma)$, then

$$\lambda_1(\Gamma^*)\leqslant\lambda_1(\Gamma). \tag{22}$$

Proof. It will be convenient to denote the edge to be deleted by $E=\Gamma\backslash\Gamma^*$ as well as to introduce notation $\mathcal{L}^*=\mathcal{L}-\ell$ for the total length of Γ^*.

Let us consider any real eigenfunction ψ_1 on Γ corresponding to the eigenvalue $\lambda_1(\Gamma)$. We then define the function $g\in\overset{c}{W}{}_2^1(\Gamma^*)$ by

$$g=\psi_1|_{\Gamma^*}+c,$$

where the constant c is to be adjusted so that g has mean value zero on Γ^*:

$$\langle g,1\rangle_{L_2(\Gamma^*)}=0. \tag{23}$$

Straightforward calculations lead to

$$0=\langle\psi_1,1\rangle_{L_2(\Gamma^*)}+c\mathcal{L}^*=-\langle\psi_1,1\rangle_{L_2(E)}+c\mathcal{L}^*$$

$$\Rightarrow c=\frac{\int_E\psi_1(x)\mathrm{d}x}{\mathcal{L}^*}. \tag{24}$$

The function g can then be used to estimate the second eigenvalue $\lambda_1(\Gamma^*)$:

$$\lambda_1(\Gamma^*)=\min_{u\in\overset{c}{W}{}_2^1(\Gamma^*):u\perp1}\frac{\|u'\|^2_{L_2(\Gamma^*)}}{\|u\|^2_{L_2(\Gamma^*)}}\leqslant\frac{\|g'\|^2_{L_2(\Gamma^*)}}{\|g\|^2_{L_2(\Gamma^*)}}. \tag{25}$$

Bearing in mind that $\langle\psi_1,1\rangle_{L_2(\Gamma)}=0$ and using (24) we evaluate the denominator in (25) first:

$$\begin{aligned}\|g\|^2_{L_2(\Gamma^*)}=\|\psi_1+c\|^2_{L_2(\Gamma^*)}&=\int_\Gamma(\psi_1+c)^2\,\mathrm{d}x-\int_E(\psi_1+c)^2\,\mathrm{d}x\\&=\|\psi_1\|^2_{L_2(\Gamma)}-\int_E{\psi_1}^2\,\mathrm{d}x-\frac{1}{\mathcal{L}^*}\left(\int_E\psi_1\,\mathrm{d}x\right)^2.\end{aligned} \tag{26}$$

The numerator similarly yields

$$\|g'\|^2_{L_2(\Gamma^*)}=\int_\Gamma(\psi_1')^2\,\mathrm{d}x-\int_E(\psi_1')^2\,\mathrm{d}x=\lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)}-\int_E(\psi_1')^2\,\mathrm{d}x. \tag{27}$$

Plugging (26) and (27) into (25) we arrive at

$$\lambda_1(\Gamma^*)\leqslant\frac{\lambda_1(\Gamma)\|\psi_1\|^2_{L_2(\Gamma)}-\int_E(\psi_1')^2\,\mathrm{d}x}{\|\psi_1\|^2_{L_2(\Gamma)}-\int_E{\psi_1}^2\,\mathrm{d}x-\frac{1}{\mathcal{L}^*}\left(\int_E\psi_1\,\mathrm{d}x\right)^2}. \tag{28}$$

Let us evaluate the integrals appearing in (28) taking into account that ψ_1 is a solution to Helmholtz equation on the edge E which can be parameterized as $E=[-\ell/2,\ell/2]$ so that $x=-\ell/2$ belongs to V_1 and $x=\ell/2$ to V_2

$$\psi_1|_E(x)=\alpha\sin(k_1x)+\beta\cos(k_1x), \tag{29}$$

where

$$\alpha = -\frac{\psi_1(V_1) - \psi_1(V_2)}{2\sin(k_1\ell/2)}, \quad \beta = \frac{\psi_1(V_1) + \psi_1(V_2)}{2\cos(k_1\ell/2)}. \tag{30}$$

Direct calculations imply

$$\int_E \psi_1(x)\mathrm{d}x = \frac{2\beta}{k_1}\sin\left(\frac{k_1\ell}{2}\right);$$
$$\int_E (\psi_1(x))^2\mathrm{d}x = \frac{\alpha^2+\beta^2}{2}\ell - \frac{\alpha^2-\beta^2}{2}\frac{\sin(k_1\ell)}{k_1};$$
$$\int_E (\psi_1'(x))^2\mathrm{d}x = k_1^2\left(\frac{\alpha^2+\beta^2}{2}\ell + \frac{\alpha^2-\beta^2}{2}\frac{\sin(k_1\ell)}{k_1}\right).$$

Inserting calculated values into (28) we get

$$\lambda_1(\Gamma^*) \leqslant \lambda_1(\Gamma)\frac{\|\psi_1\|^2_{L_2(\Gamma)} - \dfrac{\alpha^2+\beta^2}{2}\ell - \dfrac{\alpha^2-\beta^2}{2}\dfrac{\sin(k_1\ell)}{k_1}}{\|\psi_1\|^2_{L_2(\Gamma)} - \dfrac{\alpha^2+\beta^2}{2}\ell + \dfrac{\alpha^2-\beta^2}{2}\dfrac{\sin(k_1\ell)}{k_1} - \dfrac{1}{\mathcal{L}^*}\dfrac{4\beta^2}{\lambda_1(\Gamma)}\sin^2\left(\dfrac{k_1\ell}{2}\right)}. \tag{31}$$

To guarantee that the quotient is not greater than 1 and therefore $\lambda_1(\Gamma^*) \leqslant \lambda_1(\Gamma)$ it is enough that

$$\frac{\alpha^2-\beta^2}{2}\frac{\sin(k_1\ell)}{k_1} \geqslant -\frac{\alpha^2-\beta^2}{2}\frac{\sin(k_1\ell)}{k_1} + \frac{1}{\mathcal{L}^*}\frac{4\beta^2}{\lambda_1(\Gamma)}\sin^2\left(\frac{k_1\ell}{2}\right)$$

$$\iff \frac{k_1}{2}\left(\frac{\alpha^2}{\beta^2} - 1\right)\cot\left(\frac{k_1\ell}{2}\right) \geqslant (\mathcal{L}^*)^{-1}. \tag{32}$$

Using (30) the last inequality can be written as

$$\left(\frac{(\psi_1(V_1) - \psi_1(V_2))^2}{(\psi_1(V_1) + \psi_1(V_2))^2}\cot^2\left(\frac{k_1\ell}{2}\right) - 1\right)\frac{k_1}{2}\cot\left(\frac{k_1\ell}{2}\right) \geqslant (\mathcal{L}^*)^{-1}.$$

Remembering that the eigenfunction ψ_1 could be chosen arbitrary we arrive at (21). □

Roughly speaking, the condition (21) means that the length ℓ is sufficiently small. Indeed, for small ℓ the cotangent term is of order $1/\ell$. Therefore the right-hand side of (21) is of order $1/\ell^3$ and thus growing to infinity as ℓ decreases.

Let us apply the above theorem to obtain an estimate for the length of the piece that can be cut from an edge so that the spectral gap still decreases. It appears that such estimate can be given in terms of an eigenfunction ψ_1 corresponding to the first excited eigenvalue. Consider any edge in Γ, say $E_1 = [x_1, x_2]$ and choose an arbitrary internal point $x^* \in (x_1, x_2)$. Assume that we cut away an interval of length ℓ centred at x^*. Of course the length ℓ should satisfy the obvious geometric condition: $x_1 \leqslant x^* - \ell/2$ and $x^* + \ell/2 \leqslant x_2$. We assume in addition that

$$\ell < \frac{\pi}{2k_1} \tag{33}$$

guaranteeing in particular that the cotangent function in (21) is positive.

The function ψ_1 on the edge E_1 can be written in a form similar to (29)

$$\psi_1(x) = \alpha\sin k_1(x - x^*) + \beta\cos k_1(x - x^*). \tag{34}$$

Then formula (32) implies that the spectral gap decreases as the interval $[x^*-\ell/2, x^*+\ell/2]$ is cut away from the graph if

$$|\alpha| > |\beta|. \tag{35}$$

and the following estimate is satisfied

$$\cot\left(\frac{k_1\ell}{2}\right) \geqslant \frac{2}{k_1\mathcal{L}^*\left(\frac{\alpha^2}{\beta^2}-1\right)}. \tag{36}$$

Using the fact that under condition (33) we have $\cot\left(\frac{k_1\ell}{2}\right) \geqslant \frac{\pi}{2k_1\ell}$ the following explicit estimate on ℓ can be obtained

$$\ell \leqslant \frac{\pi}{4}(\mathcal{L}-\ell)\left(\frac{\alpha^2}{\beta^2}-1\right), \tag{37}$$

of course under condition (35). For the spectral gap not to increase it is enough that estimate (37) is satisfied for at least one eigenfunction ψ_1:

$$\ell \leqslant \min\left\{\frac{\pi}{2k_1},\ \frac{\pi}{4}(\mathcal{L}-\ell)\max_{\psi_1:L^{\mathrm{st}}(\Gamma)\psi_1=\lambda_1\psi_1}\left(\frac{\alpha^2}{\beta^2}-1\right)\right\}, \tag{38}$$

where we have taken into account (33).

We see that if the eigenfunction ψ_1 is sufficiently asymmetric with respect to the point x^* (i.e. (35) is satisfied), then a certain sufficiently small interval can be cut from the edge ensuring that the spectral gap decreases despite the total length decreasing. Additional condition (35) was expected, since if ψ_1 is symmetric with respect to x^*, then the spectral gap may increase for any ℓ. Really, one may imagine that the deletion of the interval is performed in two stages. One cuts the edge E_1 at the point x^* first. Then one deletes the intervals $[x^*-\ell/2, x_1^*]$ and $[x_2^*, x^*+\ell/2]$. If $\alpha = 0$ (symmetric function), then the spectral gap may be preserved in accordance to theorem 5. Deleting the pending edges (intervals $[x^*-\ell/2, x_1^*]$ and $[x_2^*, x^*+\ell/2]$) may only lead to an increase of the spectral gap due to theorem 3.

8. Summary

We have shown that deleting not so long edges or cutting away short intervals from edges may lead to a decrease of the spectral gap despite the fact that total length of the graph decreases. This effect reminds us of the phenomena discovered in [9], where the behaviour of the spectral gap under extension of edges was discussed. It appeared that the ground state may decrease with the increase of edge lengths, provided graphs are of complicated topology.

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References

[1] Berkolaiko G, Carlson R, Fulling S A and Kuchment P (ed) 2006 *Quantum Graphs and Their Applications: Proc. AMS-IMS-SIAM Joint Summer Research Conf. (Snowbird, UT, 19–23 June 2005) (Contemporary Mathematics* vol 415*)* (Providence, RI: American Mathematical Society)

[2] Berkolaiko G and Kuchment P 2012 Dependence of the spectrum of a quantum graph on vertex conditions and edge lengths *Spectral Geometry (Proceedings of Symposia Pure Mathematics* vol 84*)* (Providence, RI: American Mathematical Society) pp 117–37

[3] Berkolaiko G and Kuchment P 2013 *Introduction to Quantum Graphs (Mathematical Surveys and Monographs* vol 186*)* (Providence, RI: American Mathematical Society)
[4] Cattaneo C 1997 The spectrum of the continuous Laplacian on a graph *Monatsh. Math.* **124** 215–35
[5] Chung F 1997 *Spectral Graph Theory (CBMS Regional Conference Series in Mathematics* vol 92*)* (Providence, RI: American Mathematical Society)
[6] Colin de Verdière Y 1998 *Spectres de graphes* (Paris: Société Mathématiques de France)
[7] Cvetković D, Doob M and Sachs H 1980 *Spectra of Graphs. Theory and Application* (New York: Academic)
[8] Cvetković D, Rowlinson P and Simić S 2010 *An Introduction to the Theory of Graph Spectra* (Cambridge: Cambridge University Press)
[9] Exner P and Jex M 2012 On the ground state of quantum graphs with attractive δ-coupling *Phys. Lett.* A **376** 713–7
[10] Exner P and Šeba P 1989 Free quantum motion on a branching graph *Rep. Math. Phys.* **28** 7–26
[11] Fiedler M 1973 Algebraic connectivity of graphs *Czech. Math. J.* **23** 298–305
[12] Friedlander L 2005 Extremal properties of eigenvalues for a metric graph *Ann. Inst. Fourier* **55** 199–211
[13] Fulling S A, Kuchment P and Wilson J H 2007 Index theorems for quantum graphs *J. Phys. A: Math. Theor.* **40** 14165–80
[14] Gerasimenko N I and Pavlov B S 1988 Scattering problems on noncompact graphs *Teor. Mat. Fiz.* **74** 345–59
Gerasimenko N I and Pavlov B S 1988 Scattering problems on noncompact graphs *Theor. Math. Phys.* **74** 230–40
[15] Kostrykin V and Schrader R 1999 Kirchoff's rule for quantum wires *J. Phys. A: Math. Gen.* **32** 595–630
[16] Kuchment P 2004 Quantum graphs: I. Some basic structures. Special section on quantum graphs *Waves Random Media* **14** S107–28
[17] Kuchment P 2005 Quantum graphs: II. Some spectral properties of quantum and combinatorial graphs *J. Phys. A: Math. Gen.* **38** 4887–900
[18] Kurasov P 2008 Graph Laplacians and topology *Ark. Mat.* **46** 95–111
[19] Kurasov P 2008 Schrödinger operators on graphs and geometry: I. Essentially bounded potentials *J. Funct. Anal.* **254** 934–53
[20] Kurasov P and Naboko S 2013 On Rayleigh theorem for quantum graphs *J. Spectral Theory* at press
[21] Mohar B 1991 The Laplacian spectrum of graphs *Graph Theory, Combinatorics, and Applications* vol 2 (New York: Wiley) pp 871–98