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Pavel Kurasov

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Inverse scattering for lasso graph

Pavel Kurasov[a)]
Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden

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The inverse problem for the magnetic Schrödinger operator on the lasso graph with different matching conditions at the vertex is investigated. It is proven that the Titchmarsh-Weyl function known for different values of the magnetic flux through the cycle determines the unique potential on the loop, provided the entries of the vertex scattering matrix S parametrizing matching conditions satisfy $s_{12}s_{23}s_{31} \neq s_{13}s_{21}s_{32}$. This is in contrast to numerous examples showing that the potential on the loop cannot be reconstructed from the boundary measurements. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4799034]

I. INTRODUCTION

Differential operators on metric graphs are a well-established area of modern mathematical physics with direct applications in nano-physics, wave-propagation, chemistry, and other natural sciences.[13] The corresponding models combine rich geometric properties with exact solvability – spectral and scattering characteristics of such operators can often be calculated analytically which makes them an important tool in theoretical studies of spectral and chaotic phenomena.

We are interested in the inverse problem for the (magnetic) Schrödinger operator with the input data formed by the Titchmarsh-Weyl (matrix) function (TW-function) associated with graph's boundary. If the metric graph is known, then the Schrödinger operator is determined by the real magnetic and electric potentials (see (2.1)) and the matching conditions at the vertices. The TW-function is a Nevanlinna matrix function of the energy parameter and is a natural generalization of the Dirichlet-to-Neumann map. Moreover it is in one-to-one correspondence with the dynamical response operator and the scattering matrix,[1] i.e., it has a clear physical interpretation and can be obtained in experiments, in contrast to the alternative approach based on spectral mappings.[24,25] The corresponding inverse problem for trees is solved completely now[1–3,10,17] following pioneering works.[11,12,8,4,5,20] It is proven that the TW-function determines the metric graph, the matching conditions at the vertices (up to a certain unitary equivalence) and the electric potential on the edges, provided certain explicit conditions are satisfied. Appearance of the magnetic potential does not lead to any new effects, since such potential can be eliminated in the case of trees. On the contrary, for graphs with cycles the TW-function depends on the integrals of the magnetic potential along the cycles (called fluxes) and precisely this dependence can be used to reveal the electric potential. Assume that the metric graph is known and the matching conditions at the vertices are standard (the function is continuous and the sum of normal derivatives is zero), then the TW-function known for different values of the fluxes determines the unique (electric) potential, provided certain non-resonance conditions are satisfied and the graph contains no loops.[14,18] The last requirement is necessary, since even for the lasso graph formed by one loop and one external edge, the potential on the loop in general is not determined by the TW-function in the case of standard matching conditions[23,14] unless it is identically zero (Ambarzumian theorem and generalizations).[9] It was conjectured that non-uniqueness in this inverse problem is related to the symmetric character of matching conditions. Therefore a rather general family of matching conditions was considered in Ref. 15, but to our surprise the non-uniqueness of the potential remained.

[a)]E-mail: pak@math.su.se

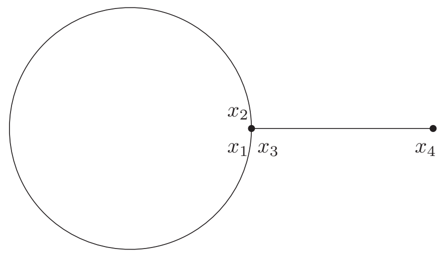

FIG. 1. Lasso graph Γ.

The present article is devoted to the solution of the inverse problem for the lasso graph with most general matching conditions at the internal vertex. All such matching conditions are described by 3×3 irreducible unitary matrices S. It appears that unless the matrix S possesses symmetry (6.9)

$$s_{12}s_{23}s_{31} \neq s_{13}s_{21}s_{32},$$

the electric potential is unique and explicit algorithm to solve the inverse problem is given. The intuition behind this condition is discussed in Sec. VII.

II. TITCHMARSH-WEYL FUNCTION FOR A LASSO GRAPH

Consider the lasso graph depicted in Figure 1.

The magnetic Schrödinger operator

$$L_{q,a} = \left(i\frac{d}{dx} + a(x) \right)^2 + q(x) \tag{2.1}$$

is defined on the functions from the Sobolev space $W_2^2(\Gamma \setminus \{x_1, x_2, x_3, x_4\})$ satisfying certain matching conditions at the internal vertex $V_1 = \{x_1, x_2, x_3\}$ and boundary conditions at the boundary vertex $V_2 = \{x_4\}$.

The electric (q) and magnetic (a) potentials are real and satisfy the assumptions

$$q \in L_2(\Gamma), \quad a \in C(\Gamma). \tag{2.2}$$

The most general matching conditions at the vertex V_1 can be written as[19]

$$i(S - I)\begin{pmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \end{pmatrix} = (S + I)\begin{pmatrix} u'(x_1) \\ -u'(x_2) \\ u(x_3)' \end{pmatrix}, \tag{2.3}$$

where $S = \{s_{ij}\}_{i,j=1}^3$ is an irreducible unitary matrix. A unitary matrix S is irreducible if it cannot be put into the block-diagonal form by permutation of the coordinates. Irreducibility of S is necessary to ensure that matching conditions are properly connecting, i.e., the vertex cannot be divided in two so that conditions connect together boundary values from the new vertices separately), unitarity of S is required to make L symmetric.

The boundary condition at V_2 will be Dirichlet and the corresponding TW-function is defined by the relation

$$M^\Gamma(\lambda) = -\frac{u'(x_4)}{u(x_4)}, \tag{2.4}$$

where u is a solution to the differential equation $Lu = \lambda u, \ \lambda = k^2$, satisfying the matching conditions at the internal vertex, but not necessarily the boundary conditions at V_2. The function M^Γ is a Nevanlinna function of λ and depends on the potential q on the graph. The magnetic potential can be removed by the unitary transformation

$$u(x) \mapsto \hat{u}(x) = e^{i\int_{x_1}^{x} a(y)dy} u(x), \tag{2.5}$$

which implies that the TW-functions depend only on the flux $\Phi = \int_{x_1}^{x_2} a(x)dx$ of the magnetic field through the loop, but not on the particular form of the magnetic potential. In what follows we are going to use notation $M^{\Gamma}(\lambda, z)$, $z = e^{i\Phi}$ indicating the dependence of M on the magnetic flux.

In fact the TW-function for the kernel of Γ formed by the loop with the contact point x_3 will be more important in our studies

$$M^{\mathrm{Ker}\,\Gamma}(\lambda, z) = -\frac{u'(x_3)}{u(x_3)}, \;\; z = e^{i\Phi}. \tag{2.6}$$

We study first whether the potential on the loop is determined by $M^{\mathrm{Ker}\Gamma}(\lambda, z)$ and return back to $M^{\Gamma}(\lambda, z)$ in the main Theorem 6.2.

The transfer matrix T for the (magnetic potential free) Schrödinger equation $-u'' + q(x)u = k^2u$ on the interval $[x_1, x_2]$ will also be used

$$T(k) : \begin{pmatrix} u(x_1) \\ u'(x_1) \end{pmatrix} \mapsto \begin{pmatrix} u(x_2) \\ u'(x_2) \end{pmatrix}, \qquad T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}. \tag{2.7}$$

All entries of the transfer matrix are analytic functions of exponential type having special asymptotics,[22] the transfer matrix has unit determinant

$$\det T(k) = t_{11}(k)t_{22}(k) - t_{12}(k)t_{21}(k) = 1. \tag{2.8}$$

In order to reconstruct the potential q on the loop one needs to know the functions t_{12} and t_{22} (or any other pair (t_{11}, t_{12}), (t_{11}, t_{21}), or (t_{21}, t_{22})).[7,22] The zeroes of these functions determine the Dirichlet-Dirichlet and Dirichlet-Neumann spectra for the Schrödinger operator on the interval $[x_1, x_2]$.[21]

III. CALCULATION OF THE M-FUNCTION

Our immediate goal is to calculate $M(\lambda, z) := M^{\mathrm{Ker}\Gamma}(\lambda, z)$. The values of the function on the loop are related via (2.7) with $T(k)$ substituted with $zT(k)$ in order to take into account the magnetic potential. The boundary values $u(x_3)$ and $u'(x_3)$ are connected via the TW-function $M(\lambda, z)$ (2.6). Substituting these equalities into the matching conditions (2.3) we get the following linear equation

$$\left[i(S-I)\begin{pmatrix} 1 & 0 & 0 \\ zt_{11} & zt_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} + (S+I)\begin{pmatrix} 0 & -1 & 0 \\ zt_{21} & zt_{22} & 0 \\ 0 & 0 & M(\lambda, z) \end{pmatrix} \right] \begin{pmatrix} u(x_1) \\ u'(x_1) \\ u(x_3) \end{pmatrix} = 0, \tag{3.1}$$

which has a nontrivial solution only if the determinant of the matrix is equal to zero. The zero determinant equation allows one to calculate $M(\lambda, z)$, but the corresponding formula is rather involved. We are interested in dependence of $M(\lambda, z)$ upon the phase Φ which suggests us to write this formula in the form

$$M(\lambda, z) = -\frac{a_0 + a_1 z + a_2 z^2}{b_0 + b_1 z + b_2 z^2}, \quad z = e^{i\Phi}. \tag{3.2}$$

Let A be a square matrix, then the following notations will be used:

$$dA = \det A, \quad \text{and} \quad dA_{ij} = \det A_{ij}, \tag{3.3}$$

where A_{ij} are matrices obtained from A by deleting ith row and jth column.

Calculation of the coefficients a_j and b_j

$$\begin{aligned}
a_0 &= \det \begin{pmatrix} s_{11}-1 & s_{11}+1 & s_{13} \\ s_{21} & s_{21} & s_{23} \\ s_{31} & s_{31} & s_{33}-1 \end{pmatrix} \\
&= 2\,(-dS_{12}+s_{21})\,; \\
a_1 &= -i \det \begin{pmatrix} it_{11}s_{12}+t_{21}s_{12} & s_{11}+1 & s_{13} \\ it_{11}(s_{22}-1)+t_{21}(s_{22}+1) & s_{21} & s_{23} \\ it_{11}s_{32}+t_{21}s_{32} & s_{31} & s_{33}-1 \end{pmatrix} \\
&\quad - \det \begin{pmatrix} s_{11}-1 & it_{12}s_{12}+t_{22}s_{12} & s_{13} \\ s_{21} & it_{12}(s_{22}-1)+t_{22}(s_{22}+1) & s_{23} \\ s_{31} & it_{12}s_{32}+t_{22}t_{32} & s_{33}-1 \end{pmatrix} \\
&= t_{11}\,(-dS-1-dS_{11}-s_{11}+dS_{22}+s_{22}+dS_{33}+s_{33}) \\
&\quad + t_{22}\,(-dS-1+dS_{11}+s_{11}-dS_{22}-s_{22}+dS_{33}+s_{33}) \\
&\quad + it_{12}\,(-dS+1+dS_{11}-s_{11}+dS_{22}-s_{22}+dS_{33}-s_{33}) \\
&\quad + it_{21}\,(dS-1+dS_{11}-s_{11}+dS_{22}-s_{22}-dS_{33}+s_{33})\,; \\
a_2 &= i \det \begin{pmatrix} it_{11}s_{12}+t_{21}s_{12} & it_{12}s_{12}+t_{22}s_{12} & s_{13} \\ it_{11}(s_{22}-1)+t_{21}(s_{22}+1) & it_{12}(s_{22}-1)+t_{22}(s_{22}+1) & s_{23} \\ it_{11}s_{32}+t_{21}s_{32} & it_{12}s_{32}+t_{22}s_{32} & s_{33}-1 \end{pmatrix} \\
&= 2\,(-dS_{21}+s_{12})\,.
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
b_0 &= -i \det \begin{pmatrix} s_{11}-1 & s_{11}+1 & s_{13} \\ s_{21} & s_{21} & s_{23} \\ s_{31} & s_{31} & s_{33}+1 \end{pmatrix} \\
&= -2i\,(-dS_{12}-s_{21})\,; \\
b_1 &= -\det \begin{pmatrix} it_{11}s_{12}+t_{21}s_{12} & s_{11}+1 & s_{13} \\ it_{11}(s_{22}-1)+t_{21}(s_{22}+1) & s_{21} & s_{23} \\ it_{11}s_{32}+t_{21}s_{32} & s_{31} & s_{33}+1 \end{pmatrix} \\
&\quad + i \det \begin{pmatrix} s_{11}-1 & it_{12}s_{12}+t_{22}s_{12} & s_{13} \\ s_{21} & it_{12}(s_{22}-1)+t_{22}(s_{22}+1) & s_{23} \\ s_{31} & it_{12}s_{32}+t_{22}t_{32} & s_{33}+1 \end{pmatrix} \\
&= it_{11}\,(dS-1+dS_{11}-s_{11}-dS_{22}+s_{22}+dS_{33}-s_{33}) \\
&\quad + it_{22}\,(dS-1-dS_{11}+s_{11}+dS_{22}-s_{22}+dS_{33}-s_{33}) \\
&\quad + t_{12}\,(-dS-1+dS_{11}+s_{11}+dS_{22}+s_{22}-dS_{33}-s_{33}) \\
&\quad + t_{21}\,(dS+1+dS_{11}+s_{11}+dS_{22}+s_{22}+dS_{33}+s_{33})\,; \\
b_2 &= \det \begin{pmatrix} it_{11}s_{12}+t_{21}s_{12} & it_{12}s_{12}+t_{22}s_{12} & s_{13} \\ it_{11}(s_{22}-1)+t_{21}(s_{22}+1) & it_{12}(s_{22}-1)+t_{22}(s_{22}+1) & s_{23} \\ it_{11}s_{32}+t_{21}s_{32} & it_{12}s_{32}+t_{22}s_{32} & s_{33}+1 \end{pmatrix} \\
&= -2i\,(-dS_{21}-s_{12})\,.
\end{aligned} \tag{3.5}$$

Let us just note that the coefficients a_0, a_2, b_0, and b_2 do not depend on λ, while a_1 and b_1 are analytic functions of λ.

IV. FLUX DEPENDENCE AND IRREDUCIBILITY

In order to solve the inverse problem one needs to determine the elements of the transfer matrix from the kernel TW-function $M(\lambda, z)$. In general we need to reconstruct two functions of one variable ($a_1(\lambda)$ and $b_1(\lambda)$) from one function of two variables ($M(\lambda, z)$).

Theorem 4.1. *The kernel TW-function $M(\lambda, z) = M^{\ker\Gamma}(\lambda, z)$, $z = e^{i\Phi}$ is independent of the magnetic flux Φ if and only if the unitary matrix S is reducible.*

Proof. If S is reducible, then $M(\lambda, z)$ is independent of z for one of the two reasons:

(1) the interval $[x_4, x_3]$ is not connected to the loop,
(2) the magnetic potential can be removed by the standard transformation (2.5).

To prove that $M(\lambda, z)$ does not depend on z only if S is reducible, let us assume that

- $M(\lambda, z)$ is independent of z,
- S is irreducible.

We shall arrive to a contradiction, proving our assertion.

The zero determinant equation corresponding to (3.1) can be written using vector notations as follows

$$\det\left[i\begin{pmatrix} s_{11}-1 \\ s_{21} \\ s_{31}\end{pmatrix} + iz\begin{pmatrix} s_{12} \\ s_{22}-1 \\ s_{32}\end{pmatrix} t_{11}(\lambda) + z\begin{pmatrix} s_{12} \\ s_{22}+1 \\ s_{32}\end{pmatrix} t_{21}(\lambda); \right.$$
$$-\begin{pmatrix} s_{11}+1 \\ s_{21} \\ s_{31}\end{pmatrix} + iz\begin{pmatrix} s_{12} \\ s_{22}-1 \\ s_{32}\end{pmatrix} t_{12}(\lambda) + z\begin{pmatrix} s_{12} \\ s_{22}+1 \\ s_{32}\end{pmatrix} t_{22}(\lambda); \tag{4.1}$$
$$\left. i\begin{pmatrix} s_{13} \\ s_{23} \\ s_{33}-1\end{pmatrix} + M(\lambda, z)\begin{pmatrix} s_{13} \\ s_{23} \\ s_{33}+1\end{pmatrix}\right] = 0.$$

Case I Assume that $b_0 \neq 0$.

Since $M(\lambda, z)$ does not depend on $z = 0$, it can be calculated by putting $z = 0$ in (4.1) $M(\lambda, z) = -\frac{a_0}{b_0}$. It follows also that the vectors

$$\begin{pmatrix} s_{11}-1 \\ s_{21} \\ s_{31}\end{pmatrix}, \begin{pmatrix} s_{11}+1 \\ s_{21} \\ s_{31}\end{pmatrix}, \text{ and } i\begin{pmatrix} s_{13} \\ s_{23} \\ s_{33}-1\end{pmatrix} + M\begin{pmatrix} s_{13} \\ s_{23} \\ s_{33}+1\end{pmatrix}$$

are linear dependent. Since the first two vectors are not collinear, unless $s_{21} = s_{31} = 0$ which is impossible for irreducible matrices S, the three vectors are linear dependent only if there exist complex numbers x, y such that

$$i\begin{pmatrix} s_{13} \\ s_{23} \\ s_{33}-1\end{pmatrix} + M\begin{pmatrix} s_{13} \\ s_{23} \\ s_{33}+1\end{pmatrix} = x\begin{pmatrix} s_{11}-1 \\ s_{21} \\ s_{31}\end{pmatrix} + y\begin{pmatrix} s_{11}+1 \\ s_{21} \\ s_{31}\end{pmatrix}. \tag{4.2}$$

Substituting the last vector in (4.1) using the last equality and calculating the determinant in (4.1) we get formally a second order polynomial in z. This polynomial is identically equal to zero, hence all its coefficients are zero as well. Consider the first order term in z which is a linear combination of the functions $t_{ij}(\lambda)$

$$0 = \tau_{11} t_{11}(\lambda) + \tau_{12} t_{12}(\lambda) + \tau_{21} t_{21}(\lambda) + \tau_{22} t_{22}(\lambda), \qquad \tau_{ij} \neq \tau_{ij}(\lambda).$$

Let us calculate the corresponding coefficients τ_{ij}. For example, using (4.2) the coefficient τ_{11} can be expressed as

$$\begin{aligned} \tau_{11} &= -i \det \left[\begin{pmatrix} s_{12} \\ s_{22}-1 \\ s_{32} \end{pmatrix}, \begin{pmatrix} s_{11}+1 \\ s_{21} \\ s_{31} \end{pmatrix}, x \begin{pmatrix} s_{11}-1 \\ s_{21} \\ s_{31} \end{pmatrix} + y \begin{pmatrix} s_{11}+1 \\ s_{21} \\ s_{31} \end{pmatrix} \right] \\ &= -2ix \det \begin{bmatrix} s_{21} & s_{22}-1 \\ s_{31} & s_{32} \end{bmatrix}. \end{aligned}$$

Similar calculations give

$$\begin{aligned} \tau_{12} &= -2y \det \begin{bmatrix} s_{21} & s_{22}-1 \\ s_{31} & s_{32} \end{bmatrix}, \\ \tau_{21} &= -2x \det \begin{bmatrix} s_{21} & s_{22}+1 \\ s_{31} & s_{32} \end{bmatrix}, \\ \tau_{22} &= \;\; 2iy \det \begin{bmatrix} s_{21} & s_{22}+1 \\ s_{31} & s_{32} \end{bmatrix}. \end{aligned}$$

Assume first that $x \neq 0 \neq y$. The asymptotics as $\lambda \to \infty$ is dominated by $t_{21} \sim -k \sin k(x_2 - x_1)$ and the corresponding coefficient is zero if and only if

$$\det \begin{bmatrix} s_{21} & s_{22}+1 \\ s_{31} & s_{32} \end{bmatrix} = 0 \tag{4.3}$$

holds. Under this condition the coefficient τ_{22} is zero and the next term in asymptotics is given by $\tau_{11} t_{11}$. The corresponding coefficient is zero if and only if

$$\det \begin{bmatrix} s_{21} & s_{22}-1 \\ s_{31} & s_{32} \end{bmatrix} = 0, \tag{4.4}$$

which also implies that $\tau_{12} = 0$.

If $x = 0$ and $y \neq 0$, then $\tau_{21} = \tau_{11} = 0$ and the asymptotics is dominated by $\tau_{22} t_{22}$. But $\tau_{22} = 0$ if and only if (4.3) holds. Considering coefficient in front of t_{12} we arrive at (4.4).

Similarly if $x \neq 0$ and $y = 0$ we prove that (4.3) and (4.4) hold. The special case $x = y = 0$ will be considered at the end of the proof.

Equations (4.3) and (4.4) satisfied simultaneously imply that $s_{31} = 0$. Then from (4.3) we get that either s_{21} or s_{32} is zero. In the first case we get that $|s_{11}| = 1$ and therefore the matrix S is reducible. In the second case $|s_{33}| = 1$ and again S is reducible.

It remains to study the case $x = y = 0$, i.e., when the last column vector in (4.1) is zero

$$\begin{cases} (i+M) s_{13} & = 0, \\ (i+M) s_{23} & = 0, \\ (i+M) s_{33} + (-i+M) & = 0. \end{cases}$$

If at least one of s_{13}, s_{23} is different from zero, then $M = -i$ and the third equation cannot be satisfied. It follows that $s_{13} = s_{23} = 0$, i.e., the matrix S is reducible.

Case II Assume that $b_0 = 0$.

The TW-function does not depend on z only if $a_0 = 0$ as well. Therefore the entries of S satisfy simultaneously

$$-dS_{12} - s_{21} = 0 \text{ and } dS_{12} + s_{21} = 0 \Rightarrow s_{21} = dS_{12} = 0.$$

Taking into account that $dS_{12} = s_{21} s_{33} - s_{31} s_{23}$ we conclude that either $s_{31} = 0$ or $s_{23} = 0$. In both cases S is reducible. □

Assume that the TW-function $M(\lambda, z)$ is known and that S is irreducible. Fix any complex λ. Then M is a meromorphic non-constant function of z. Taking any two $z \neq 0 \neq z'$ such that $M(\lambda, z) \neq M(\lambda, z')$ we arrive at the following linear system

$$\begin{cases} a_1(\lambda) + M(\lambda, z) b_1(\lambda) = -\frac{1}{z} a_0 - z a_2 - \frac{1}{z} M(\lambda, z) b_0 - z M(\lambda, z) b_2, \\ a_1(\lambda) + M(\lambda, z') b_1(\lambda) = -\frac{1}{z'} a_0 - z' a_2 - \frac{1}{z'} M(\lambda, z')(\lambda) b_0 - z' M(\lambda, z')(\lambda) b_2. \end{cases} \tag{4.5}$$

The determinant of the linear system is $M(\lambda, z') - M(\lambda, z) \neq 0$ and we conclude that the functions $a_1(\lambda)$ and $b_1(\lambda)$ are determined by $M(\lambda, z)$. It is enough to know the TW-function $M(\lambda, z)$ for just two values of the magnetic flux $z = e^{i\Phi}$, for which it attains different values.

V. HERMITIAN PARAMETRIZATION

Almost all matching conditions can be parametrized by Hermitian matrices. More precisely all matching conditions corresponding to S with $\det(S + I) \neq 0$ can be parametrized by Hermitian 3×3 matrices $H = i \frac{S-I}{S+I}$, as follows

$$H \begin{pmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \end{pmatrix} = \begin{pmatrix} u'(x_1) \\ -u'(x_2) \\ u'(x_3) \end{pmatrix}. \tag{5.1}$$

Also this subclass of matching conditions appears to be rather general, it does not contain standard matching conditions – the most widely used family. The case of arbitrary (irreducible) unitary matrices will be considered in Sec. VI, but formulas in the Hermitian case are much more transparent and will help us to understand the main ideas. The corresponding methods are essentially the same.

Let us calculate the TW-function using the Hermitian parametrization. As before (2.6) and (2.7) lead to the following equation

$$\det \left[H \begin{pmatrix} 1 & 0 & 0 \\ z t_{11} & z t_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ z t_{21} & z t_{22} & 0 \\ 0 & 0 & M(\lambda, z) \end{pmatrix} \right] = 0. \tag{5.2}$$

This equation allows one to calculate the TW-function M

$$M(\lambda, z) = -\frac{\det \begin{bmatrix} h_{11} + h_{12} z t_{11} & h_{12} z t_{12} - 1 & h_{13} \\ h_{21} + h_{22} z t_{11} + z t_{21} & h_{22} z t_{12} + z t_{22} & h_{23} \\ h_{31} + h_{32} z t_{11} & h_{32} z t_{12} & h_{33} \end{bmatrix}}{\det \begin{bmatrix} h_{11} + h_{12} z t_{11} & h_{12} z t_{12} - 1 \\ h_{21} + h_{22} z t_{11} + z t_{21} & h_{22} z t_{12} + z t_{22} \end{bmatrix}}. \tag{5.3}$$

This function can be written in the form (3.2)

$$M(\lambda, z) = -\frac{dH_{12} + z\,(dH\, t_{12} + dH_{11}\, t_{11} + dH_{22}\, t_{22} + h_{33}\, t_{21}) + z^2 dH_{21}}{h_{21} + z\,(dH_{33}\, t_{12} + h_{22}\, t_{11} + h_{11}\, t_{22} + t_{21}) + z^2 h_{12}}, \tag{5.4}$$

where we used that $\det T = 1$ as well as notations (3.3).

Writing $M(\lambda, z)$ in the form analogous to (3.2)

$$M(\lambda, z) = -\frac{\alpha_0 + z\alpha_1 + z^2 \alpha_2}{\beta_0 + z\beta_1 + z^2 \beta_2} \tag{5.5}$$

we get

$$\begin{aligned} \alpha_0 &= dH_{12}, \\ \alpha_1(\lambda) &= dH\, t_{12}(\lambda) + dH_{11}\, t_{11}(\lambda) + dH_{22}\, t_{22}(\lambda) + h_{33}\, t_{21}(\lambda), \\ \alpha_2 &= dH_{21}, \\ \beta_0 &= h_{21}, \\ \beta_1(\lambda) &= dH_{33}\, t_{12}(\lambda) + h_{22}\, t_{11}(\lambda) + h_{11}\, t_{22}(\lambda) + t_{21}(\lambda), \\ \beta_2 &= h_{12}. \end{aligned} \tag{5.6}$$

The coefficients α_j, β_j are closely related to a_j and b_j introduced in connection with the unitary parametrization

$$\begin{cases} a_j = \det(S+I)\, \alpha_j, \\ b_j = \det(S+I)\, \beta_j. \end{cases} \quad j = 0, 1, 2, \tag{5.7}$$

Let us note that the functions $\alpha_1(\lambda)$ and $\beta_1(\lambda)$ attain real values for real λ since the transfer matrix coefficients $t_{ij}(\lambda)$ are real for those λ and the matrix H is Hermitian and therefore the coefficients

$$dH, \quad dH_{11}, \quad dH_{22}, \quad dH_{33}, \quad h_{11}, \quad h_{22}, \quad h_{33}$$

are all real as well.

The second and fifth equations in (5.6) can be written in the matrix form

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} dH_{11} & h_{33} \\ h_{22} & 1 \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} + \begin{pmatrix} dH & dH_{22} \\ dH_{33} & h_{11} \end{pmatrix} \begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix}. \tag{5.8}$$

At least one of the two matrices appearing on the right-hand side of the equation is invertible if H is irreducible. Really, the corresponding determinants are equal to $-|h_{23}|^2$ and $-|dH_{23}|^2$, respectively. Assume that

$$h_{23} = dH_{23} = 0,$$

and hence due to Hermiticity $h_{32} = dH_{32} = 0$ as well. It follows that $h_{13}h_{21} = 0$ implying that either h_{13} or h_{12} is also equal to zero. In both cases the corresponding matrix H is reducible (i.e., block diagonal after a permutation of the coordinates), leading to a reducible S. The two cases will be treated using similar methods.

Case A Assume that $h_{23} \neq 0$.

Inverting the first matrix we get

$$\begin{aligned} \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix} = &-\frac{1}{|h_{23}|^2} \begin{pmatrix} 1 & -h_{33} \\ -h_{22} & dH_{11} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \\ &+ \frac{1}{|h_{23}|^2} \begin{pmatrix} dH - h_{33} dH_{33} & -|h_{13}|^2 \\ |dH_{13}|^2 & -h_{11}|h_{23}|^2 + h_{22}|h_{13}|^2 \end{pmatrix} \begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix}. \end{aligned} \tag{5.9}$$

The unit determinant condition (2.8) for the transfer matrix gives the following equation

$$\begin{aligned} &|dH_{13}|^2 t_{12}^2 + |h_{13}|^2 t_{22}^2 - (h_{13} dH_{13} + h_{31} dH_{31})\, t_{12} t_{22} \\ &+ (h_{22}\alpha_1 - dH_{11}\beta_1)\, t_{12} + (\alpha_1 - h_{33}\beta_1) t_{22} = -|h_{23}|^2. \end{aligned} \tag{5.10}$$

The equation may be rearranged as

$$|dH_{13} t_{12} - h_{31} t_{22}|^2 + (h_{22}\alpha_1 - dH_{11}\beta_1)\, t_{12} + (\alpha_1 - h_{33}\beta_1) t_{22} = -|h_{23}|^2.$$

Let us complete to whole squares to get

$$\left| dH_{13} t_{12} - h_{31} t_{22} + \frac{h_{32}\,(-h_{21}\alpha_1 + dH_{12}\beta_1)}{h_{13} dH_{13} - h_{31} dH_{31}} \right|^2 = |h_{23}|^2 \left(\frac{|-h_{21}\alpha_1 + dH_{12}\beta_1|^2}{|h_{13} dH_{13} - h_{31} dH_{31}|^2} - 1 \right). \tag{5.11}$$

The last formula holds only if the (pure imaginary) denominator is different from zero

$$h_{13} dH_{13} - h_{31} dH_{31} \neq 0. \tag{5.12}$$

The last equation can also be written as (5.17).

Case B Assume that $dH_{23} \neq 0$.

Inverting the second matrix in (5.9) we get

$$\begin{aligned} \begin{pmatrix} t_{12} \\ t_{22} \end{pmatrix} = & -\frac{1}{|dH_{23}|^2} \begin{pmatrix} h_{11} & -dH_{22} \\ -dH_{33} & dH \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \\ & + \frac{1}{|dH_{23}|^2} \begin{pmatrix} -h_{11}|h_{23}|^2 + h_{22}|h_{13}|2 & |h_{13}|^2 \\ -|dH_{13}|^2 & dH - h_{33}dH_{33} \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{21} \end{pmatrix}. \end{aligned} \tag{5.13}$$

The unit determinant condition (2.8) gives

$$\begin{aligned} & |dH_{13}|^2 t_{11}^2 + |h_{13}|^2 t_{21}^2 - (h_{13}dH_{13} + h_{31}dH_{31})\, t_{11}t_{12} \\ & + (-dH_{33}\alpha_1 + dH\beta_1)t_{11} + (-h_{11}\alpha_1 + dH_{22}\beta_1)t_{21} = -|dH_{23}|^2. \end{aligned} \tag{5.14}$$

As before we perform quadratic completion in two stages, provided (5.17) holds

$$|dH_{13}t_{11} - h_{31}t_{21}|^2 + (-dH_{33}\alpha_1 + dH\beta_1)t_{11} + (-h_{11}\alpha_1 + dH_{22}\beta_1)t_{21} = -|dH_{23}|^2 \tag{5.15}$$

and

$$\begin{aligned} & \left| dH_{13}t_{11} - h_{31}t_{21} + \frac{dH_{23}}{h_{13}dH_{13} - h_{31}dH_{31}}(h_{21}\alpha_1 - dH_{12}\beta_1) \right|^2 \\ & = |dH_{23}|^2 \left(\frac{|^2 h_{21}\alpha_1 - dH_{12}\beta_1|^2}{|h_{13}dH_{13} - h_{31}dH_{31}|^2} - 1 \right). \end{aligned} \tag{5.16}$$

The last formula holds only if (5.12), or equivalently (5.17), is satisfied.

Theorem 5.1. *Let L be the magnetic Schrödinger operator (2.1) on the lasso graph with the domain defined by Hermitian irreducible matching conditions (5.1) at the vertex. Assume that the Hermitian matrix H is known and its entries satisfy the following condition*

$$h_{13}h_{21}h_{32} \notin \mathbb{R}. \tag{5.17}$$

Then the kernel Titchmarsh-Weyl function $M(\lambda, z)$ $(\equiv M^{\mathrm{Ker}\Gamma}(\lambda, z))$ *determines the unique potential q on the loop of the lasso graph* Γ.

Proof. The kernel Titchmarsh-Weyl function known as a function of the magnetic flux allows one to determine the functions $\alpha_1(\lambda)$ and $\beta_1(\lambda)$. Assume first that $h_{23} \neq 0$ (Case A). Under the formulated assumptions formula (5.11) allows one to reconstruct the analytic function

$$dH_{13}t_{12} - h_{31}t_{22} + \frac{h_{32}\,(-h_{21}\alpha_1 + dH_{12}\beta_1)}{h_{13}dH_{13} - h_{31}dH_{31}} \tag{5.18}$$

up to a phase, which in turn can be determined from the asymptotics. The asymptotics of the function in (5.18) is just $\frac{|h_{23}|^2 h_{31}}{h_{13}dH_{13} - h_{31}dH_{31}} k \sin k\ell_1,\ \ell_1 = x_2 - x_1$. The coefficient $h_{23}h_{31}$ is different from zero due to (5.17). It follows that the function $dH_{13}t_{12} - h_{31}t_{22}$ is determined. Then condition (5.17) assures that the the coefficients dH_{13} and h_{13} do not have equal phases. Really assume the opposite, i.e., that $\mathbb{R} \ni \frac{dH_{13}}{h_{31}} = \frac{h_{21}h_{32}}{h_{31}} - h_{22}$. Then $\frac{h_{21}h_{32}}{h_{31}} \in \mathbb{R} \Rightarrow h_{21}h_{32}h_{13} \in \mathbb{R}$ and we get a contradiction to (5.17). Then the (real valued) functions $t_{12}(\lambda)$ and $t_{22}(\lambda)$ are determined by considering the real and imaginary parts of $dH_{13}t_{12} - h_{31}t_{22}$ for $\lambda \in \mathbb{R}$. The two functions t_{12} and t_{22} determine the unique potential q on the loop.[7,21,22]

The Case B ($dH_{23} \neq 0$) is treated in a similar way. □

We have shown that for all matching conditions parameterized by 3 × 3 Hermitian matrices H subject to (5.17) the kernel Titchmash-Weyl matrix $M^{\mathrm{her}\Gamma}(\lambda, z)$ determines the unique potential on the loop. Already this result is surprising compared to what was proven in Refs. 14 and 15 for standard and real unitary S. In order to accomplish our studies we need to consider the whole set of (irreducible) matching conditions and therefore return back to parametrization via unitary matrices S.

VI. COMPLETE SOLUTION OF THE INVERSE PROBLEM

This section is devoted to the solution of the inverse problem for most general matching conditions at the vertex. All such conditions are described by irreducible unitary matrices S via (2.3).

Equations (3.4) and (3.5) for a_1 and b_1 can be written in the matrix form similar to (5.8)

$$\begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = A \begin{pmatrix} t_{11}(\lambda) \\ t_{21}(\lambda) \end{pmatrix} + B \begin{pmatrix} t_{12}(\lambda) \\ t_{22}(\lambda) \end{pmatrix} \tag{6.1}$$

with

$$\begin{aligned} a_{11} &= -dS - 1 - dS_{11} - s_{11} + dS_{22} + s_{22} + dS_{33} + s_{33} \\ a_{12} &= i(dS - 1 + dS_{11} - s_{11} + dS_{22} - s_{22} - dS_{33} + s_{33}) \\ a_{21} &= i(dS - 1 + dS_{11} - s_{11} - dS_{22} + s_{22} + dS_{33} - s_{33}) \\ a_{22} &= dS + 1 + dS_{11} + s_{11} + dS_{22} + s_{22} + dS_{33} + s_{33} \\ b_{11} &= i(-dS + 1 + dS_{11} - s_{11} + dS_{22} - s_{22} + dS_{33} - s_{33}) \\ b_{12} &= -dS - 1 + dS_{11} + s_{11} - dS_{22} - s_{22} + dS_{33} + s_{33} \\ b_{21} &= -dS - 1 + dS_{11} + s_{11} + dS_{22} + s_{22} - dS_{33} - s_{33} \\ b_{22} &= i(dS - 1 - dS_{11} + s_{11} + dS_{22} - s_{22} + dS_{33} - s_{33}). \end{aligned} \tag{6.2}$$

Our first step is to prove that the matrices A and B do not have zero determinant simultaneously, i.e., at least one of these two matrices is always invertible.

Lemma 6.1. Assume that the unitary matrix S is irreducible, then at least one of the matrices A and B given by (6.2) is invertible.

Proof. If $\det(S + I) \neq 0$, then the matching conditions can be written using Hermitian matrix H and the corresponding system (5.8) possesses this property. Hence it remains to study the case where

$$\det(S + I) = 0 \Leftrightarrow dS + 1 + dS_{11} + s_{11} + dS_{22} + s_{22} + dS_{33} + s_{33} = 0.$$

Equation (6.1) transforms as

$$\begin{aligned} &\begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} 2(dS_{22} + s_{22} + dS_{33} + s_{33}) & 2i(dS + dS_{11} + dS_{22} + s_{33}) \\ 2i(dS + dS_{11} + s_{22} + dS_{33}) & 0 \end{pmatrix} \begin{pmatrix} t_{11}(\lambda) \\ t_{21}(\lambda) \end{pmatrix} \\ &+ \begin{pmatrix} 2i(1 + dS_{11} + dS_{22} + dS_{33}) & 2(dS_{11} + s_{11} + dS_{33} + s_{33}) \\ 2(dS_{11} + s_{11} + dS_{22} + s_{22}) & 2i(dS + s_{11} + dS_{22} + dS_{33}) \end{pmatrix} \begin{pmatrix} t_{12}(\lambda) \\ t_{22}(\lambda) \end{pmatrix}. \end{aligned} \tag{6.3}$$

Putting the determinants of the matrices A and B equal to zero we get the linear system

$$\begin{cases} \frac{1}{4} \det A = (dS_{22} + s_{33})(s_{22} + dS_{33}) - (1 + s_{11})(dS + dS_{11}) = 0, \\ \frac{1}{4} \det B = (dS_{22} - s_{33})(s_{22} - dS_{33}) - (1 - s_{11})(dS - dS_{11}) = 0, \end{cases}$$

implying that

$$\begin{cases} s_{23} dS_{23} + s_{32} dS_{32} = 0, \\ dS_{32} dS_{23} + s_{23} s_{32} = 0. \end{cases} \tag{6.4}$$

Taking into account that S is a unitary matrix, the first equation in the last system can be written as

$$-\det S \left(|s_{23}|^2 + |s_{32}|^2 \right) = 0$$

implying that

$$s_{23} = s_{32} = 0.$$

It follows from the second equation in (6.4) that either dS_{32} or dS_{23} is zero, which in turn implies that S is reducible. Really, assuming that $dS_{23} = 0$ one gets $s_{11}s_{32} - s_{31}s_{12} = 0$, but $s_{32} = 0$ and it follows that either s_{31} or s_{12} is zero and S is reducible. The case $dS_{32} = 0$ is treated in a similar way. □

Case A Assume that det $A \neq 0$.

Inverting the matrix A in (6.1) we get equation analogous to (5.9). Substituting t_{ij} into the unit determinant condition (2.8) and completing to whole squares one gets the equation

$$\begin{aligned} &\left| f_A(\lambda) + \frac{1}{4} \frac{s_{11}s_{32} - s_{12}s_{31} + s_{32}}{s_{12}s_{23}s_{31} - s_{13}s_{32}s_{21}} g(\lambda) \right|^2 \\ &= |s_{11}s_{23} - s_{13}s_{21} + s_{23}|^2 \left(\frac{1}{16} \frac{|g(\lambda)|^2}{|s_{12}s_{23}s_{31} - s_{13}s_{32}s_{21}|^2} - 1 \right). \end{aligned} \tag{6.5}$$

with

$$f_A(\lambda) := (s_{22}s_{31} - s_{21}s_{32} - s_{31})\, t_{12}(\lambda) - i\,(s_{22}s_{31} - s_{21}s_{32} + s_{31})\, t_{22}(\lambda) \tag{6.6}$$

and

$$g(\lambda) := i\,(s_{23}s_{31} - s_{21}s_{33} - s_{21})\, a_1(\lambda) + (s_{23}s_{31} - s_{21}s_{33} + s_{21})\, b_1(\lambda).$$

We have to assume that the denominator is different from zero (assumption (6.9)). The function $g(\lambda)$ is known and (6.5) allows to determine the function $f_A(\lambda)$ taking into account standard asymptotics of the functions t_{12} and t_{22}.

Case B Assume that det $B \neq 0$.

This case is completely similar to Case A. We need to invert the matrix B instead of A. Substituting as before t_{ij} into the unit determinant condition (2.8) leads to

$$\begin{aligned} &\left| f_B(\lambda) - \frac{i}{4} \frac{s_{11}s_{32} - s_{12}s_{31} - s_{32}}{s_{12}s_{23}s_{31} - s_{13}s_{32}s_{21}} g(\lambda) \right|^2 \\ &= |s_{12}s_{31} - s_{11}s_{32} + s_{32}|^2 \left(\frac{1}{16} \frac{|g(\lambda)|^2}{|s_{12}s_{23}s_{31} - s_{13}s_{32}s_{21}|^2} - 1 \right), \end{aligned} \tag{6.7}$$

with

$$f_B(\lambda) := (s_{22}s_{31} - s_{21}s_{32} - s_{31})\, t_{11}(\lambda) - i\,(s_{22}s_{31} - s_{21}s_{32} + s_{31})\, t_{21}(\lambda). \tag{6.8}$$

We need to introduce condition (6.9) to guarantee that the determinant is different from zero. The function $g(\lambda)$ is known and (6.7) allows to determine the function $f_B(\lambda)$ taking into account standard asymptotics of the functions t_{11} and t_{21}.

Theorem 6.2. *Let $L_{q,\,a}$ be the magnetic Schrödinger operator on the lasso graph Γ given by (2.1) and (2.3) under the assumption (2.2). Let the metric graph Γ and the unitary irreducible 3×3 matrix S be known.*

Assume in addition that the Titchmarsh-Weyl function $M^\Gamma(\lambda, z)$ is known for two different values $z \neq z'$ of the magnetic flux $z = \exp(i \int_{x_1}^{x_2} a(x)dx)$, such that $M^\Gamma(\cdot, z) \neq M^\Gamma(\cdot, z')$, then the potential q is determined uniquely, provided

$$s_{12}s_{23}s_{31} \neq s_{13}s_{21}s_{32} \tag{6.9}$$

holds.

Proof. The TW-function M^Γ defined by (2.4) is in one-to-one correspondence with the dynamical response operator which allows to use the boundary control method[6] to recover the electric potential on the boundary edge $[x_3, x_4]$ and therefore to reduce the inverse problem to recovering of the potential on the loop from the kernel TW-function $M^{\ker\Gamma}$ defined by (2.6). This procedure is described in detail in Refs. 1, 2, and 14. It is clear that $M^{\ker\Gamma}(\lambda, z) \neq M^{\ker\Gamma}(\lambda, z')$. Hence the first step is to determine $M^{\ker\Gamma}(\lambda, z)$ from $M^{\Gamma}(\lambda, z)$.

The second step is to calculate the analytic functions $a_1(\lambda)$ and $b_1(\lambda)$ from $M^{\ker\Gamma}(\lambda, z)$ given by (3.2). Since $M(\lambda, z) \neq M(\lambda, z')$ the functions a_1 and b_1 can be calculated by solving the linear system (4.5).

The third step is to calculate the entries $t_{12}(\lambda)$, $t_{22}(\lambda)$ or $t_{11}(\lambda)$, $t_{21}(\lambda)$ of the transfer matrix. This is done by considering the linear system (6.1) and inverting either the matrix A or the matrix B and substituting the result into the unit determinant condition (2.8). Depending on which matrix A or B is invertible we recover the function $f_A(\lambda)$ or $f_B(\lambda)$ given by (6.6) and (6.8), respectively. The functions $t_{ij}(\lambda)$ can be calculated from $f_{A,B}$ taking into account that these functions are real valued for $\lambda \in \mathbb{R}$, provided the coefficients in front of these functions do not have the same phase.

Let us prove by contradiction that the coefficients $s_{22}s_{31} - s_{21}s_{32} - s_{31}$ and $-i(s_{22}s_{31} - s_{21}s_{32} + s_{31})$ have different phases. Assume the opposite or equivalently that

$$(s_{22}s_{31} - s_{21}s_{32} - s_{31})\,\overline{(s_{22}s_{31} - s_{21}s_{32} + s_{31})}$$

is pure imaginary. It is equivalent to

$$|s_{22}s_{31} - s_{21}s_{32}|^2 = |s_{31}|^2 \Leftrightarrow |dS_{13}| = |s_{31}|.$$

Taking into account that S is unitary and its determinant has unit absolute value, we conclude that (7.2) holds. (Unitarity of S implies that $\overline{s_{13}} = \frac{1}{\det S} dS_{13}$ and hence $|s_{13}| = |dS_{13}|$.) We get a contradiction implying that the coefficients have different phases.

It follows that either the functions t_{12} and t_{22} can be calculated from f_A, or t_{11} and t_{21} from f_B. Then the potential q on the interval $[x_1, x_2]$ is recovered using classical methods.[7,21,22] □

The proposed method not only proves the uniqueness, but provides an explicit algorithm to recover the potential first on the interval $[x_3, x_4]$ (Boundary control) and then on the loop (Levitan-Gasymov procedure).

VII. ON THE SUFFICIENT CONDITION

The following lemma gives several equivalent reformulations of the sufficient condition (6.9).

Lemma 7.1. Let S be irreducible unitary 3×3 *matrix, then the following conditions are equivalent*

$$s_{12}s_{23}s_{31} = s_{13}s_{21}s_{32} \tag{7.1}$$

$$|s_{12}| = |s_{21}| \; (\Leftrightarrow |s_{13}| = |s_{31}| \Leftrightarrow |s_{23}| = |s_{32}|)\,. \tag{7.2}$$

$$|s_{32}s_{13}| = |s_{31}s_{23}|. \tag{7.3}$$

Proof. We note first that the three conditions in (7.2) are equivalent. For example $|s_{12}| = |s_{21}| \Leftrightarrow |s_{13}| = |s_{31}|$ follows from the normalization of the first row and the first column in S.

Let us prove now that (7.1) ⇔ (7.2). Assume that (7.1) holds with all s_{ij}, $i \neq j$ different from zero. If in addition $|s_{12}| < |s_{21}|$, then the normalization of the first row and the first column implies that $|s_{31}| < |s_{13}|$. Normalization of the second row and second column leads to $|s_{23}| < |s_{32}|$. Hence $|s_{12}|\,|s_{23}|\,|s_{31}| < |s_{21}|\,|s_{32}|\,|s_{13}|$, which contradicts (7.1).

Assume now that (7.1) holds and one of s_{ij} is zero, say $s_{12} = 0$. Then (7.1) implies that at least one of s_{13}, s_{21}, or s_{32} is equal to zero. s_{13} and s_{32} cannot be equal to zero, since S is irreducible. It follows that $s_{21} = 0 = s_{12}$ and (7.2) holds.

To prove the opposite implication assume that (7.2) holds. The orthogonality of the first two rows and the first two columns in S gives

$$s_{11} + \frac{s_{12}s_{21}}{|s_{21}|^2}\overline{s_{22}} + \frac{s_{13}\overline{s_{23}}s_{21}}{|s_{21}|^2} = 0 \text{ and } s_{11} + \frac{s_{21}s_{12}}{|s_{12}|^2}\overline{s_{22}} + \frac{s_{31}\overline{s_{32}}s_{12}}{|s_{12}|^2} = 0,$$

implying $s_{13}\overline{s_{12}s_{23}} = s_{31}\overline{s_{32}s_{21}}$ and finally (7.1).

The proof of (7.1) ⇔ (7.3) is similar. □

It follows that the sufficient condition (6.9) can be formulated as

$$|s_{32}s_{13}| \neq |s_{31}s_{23}|. \tag{7.4}$$

Consider the wave evolution on the lasso graph. Assume that an observer is sending waves along the outgrowth and tries to determine the potential on the graph. To determine the potential on the loop one needs to study waves coming back after passing along the loop in one or the other direction. There are precisely two such (shortest) trajectories. Crossing the internal vertex V_1 these waves are multiplied by the scattering coefficients $s_{32}s_{13}$ and $s_{31}s_{23}$. Condition (7.4) implies that the corresponding amplitudes are different and one may distinguish between the waves coming after having passed the loop in different directions. A similar effect was observed in Ref. 16, where the loop with two intervals attached was considered. If the unitary matrix determining matching conditions is chosen so that one may distinguish between the waves passing the loop in different directions, then the potential on the loop is uniquely reconstructible.

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