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INVERSE PROBLEMS FOR QUANTUM TREES II: RECOVERING MATCHING CONDITIONS FOR STAR GRAPHS

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Dedicated to Jan Boman - great person and mathematician on the occasion of his 75th birthday.

Abstract. The inverse problem for the Schrödinger operator on a star graph is investigated. It is proven that such Schrödinger operator, *i.e.* the graph, the real potential on it and the matching conditions at the central vertex, can be reconstructed from the Titchmarsh-Weyl matrix function associated with the graph boundary. The reconstruction is also unique if the spectral data include not the whole Titchmarsh-Weyl function but its principal block (the matrix reduced by one dimension). The same result holds true if instead of the Titchmarsh-Weyl function the dynamical response operator or just its principal block is known.

1. **Introduction.** Differential operators on geometric graphs have been studied from the beginning of 80-ies [10, 13], but recent interest in nano-structures has led to enormous interest in mathematical studies of the problem [17, 19, 20, 22]. In this article we discuss the possibility to reconstruct the matching (boundary) conditions at the unique internal vertex of a star graph from the corresponding Titchmarsh-Weyl matrix function or the dynamical response operator. These operators are also known as Dirichlet-to-Neumann map and the dynamical Dirichlet-to-Neumann map respectively. These operators allow one to solve the inverse problem [18]. We

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are interested in the case where not the whole Titchmarsh-Weyl matrix function but just its principal block is known.[1] Solution to this inverse problem can be used as the main building brick for solving inverse problems for arbitrary trees as it has been already done in [1] for Schrödinger operators with standard matching conditions at the vertices. Our main focus this time is on the recovering of matching conditions, but we solve also the problem to recover the unknown potential. We use the Boundary Control method [5, 3], more precisely, its version proposed in [1] (see also [2] where an inverse problem for the two-velocity wave equation on a tree was studied). Our approach allows one to solve the inverse problem "locally" — the recovery of the parameters of a subgraph requires only the data related to that subgraph.

The problem of reconstructing the Schrödinger operator on a star graph was first discussed by N.I. Gerasimenko and B.S. Pavlov [13, 14] using the Gelfand-Levitan-Marchenko method. The inverse spectral and scattering problems for trees have intensively been studied in recent years by M. Belishev, M. Brown, R. Carlson, G. Freiling, A. Vakulenko, R. Weikard, V. Yurko, and the authors [1, 4, 6, 7, 8, 9, 11, 12, 23, 24, 25]. It has been proven that the knowledge of the Titchmarsh-Weyl matrix function allows one to calculate the potential for standard boundary conditions at the vertices. The case of more general matching conditions has been discussed in [12], but the whole family of matching conditions has not been investigated yet.

In the current article we consider the most general family of irreducible (see the definition in Section 2), in other words, properly connecting matching conditions leading to a self-adjoint operator. We propose to parameterize such conditions by the vertex scattering matrix S^k say for the energy parameter $k^2 = 1$, *i.e.* by the matrix S^1. It appears that irreducibility of the limiting scattering matrix $S^\infty = \lim_{k\to\infty} S^k$ plays a very important role (see formula (3.9) below). Our parametrization is similar to the one developed by M. Harmer [15, 16].

We find the matching conditions and potential on the edges. Our approach provides also an effective algorithm for solving the inverse problem, which is straightforward and more simple compared to the spectral mapping approach used in [12].

The article is organized as follows. We start with investigating the star graph with all edges having the same length. This assumption allows us to use vector notations which simplify many formulas and clarify the ideas behind the Boundary Control method. All necessary notations are introduced in Section 2. We discuss also the expected arbitrariness in the solution of the inverse problem using the principal blocks of the Titchmarsh-Weyl matrix or of the dynamical response operator. In Section 3 the inverse problem for the Laplace operator with general matching conditions is solved. Consideration of the Laplace operator allows us to concentrate our attention on recovering matching conditions, which is the main subject of the present paper. The inverse problem for the Schrödinger operator is solved in Section 4. A star graph with arbitrary lenghts of edges is considered in Section 5. Note that the vector notations are not appropriate in that case and we use standard notations developed for Schrödinger operators on metric graphs in [1].[2]

[1] Under the **principal block** of any $m \times m$ matrix M we understand the $(m-1)\times(m-1)$ matrix obtained from M by deleting the last row and last column.

[2] We would like to use this opportunity to clarify a few statements in [1]. The *reduced* $(m-1)\times(m-1)$ *scattering matrix* $\mathbf{S}_{m-1}(k)$ appearing in Theorem 4.1 is not determined by equation (4.2) but rather is the scattering matrix for the graph Γ with one extra Dirichlet boundary condition introduced at the vertex γ_m. Similarly, by the *back scattering coefficient* corresponding

2. **Notations and preliminary discussions.** We begin with investigating the inverse problem for the simplest star graph Γ formed by m identical intervals $\Delta_j = [x_{2j-1}, x_{2j}],\ j = 1, 2, \dots, m$ all having the same length $\ell = x_{2j} - x_{2j-1}$. The methods developed here can be applied to star graphs with edges having different lengths, but the corresponding formulas get a more complicated form which makes them less transparent. On the other hand all necessary generalizations are rather straightforward and can be carried out without any principal difficulties. The case of general star graph is considered in Section 5.

We assume that these intervals are joined together at the **central vertex** $v_0 = \{x_1, x_3, \dots, x_{2m-1}\}$. The graph **boundary** $\partial\Gamma$ is formed by the vertices $v_j = \{x_{2j}\}$ all having valence 1. Without loss of generality we may assume that $x_{2j-1} = 0$. This assumption will allow us to introduce vector notations (2.1) leading to simplification of many formulas.

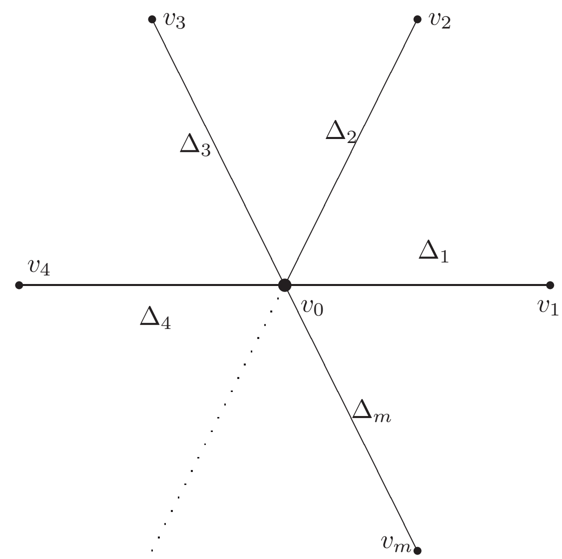

Fig. 1 The star graph Γ.

It is convenient to identify the corresponding Hilbert space $\mathcal{H} = L_2(\Gamma) = \oplus \sum_{j=1}^{m} L_2(\Delta_j)$ with the space of vector valued functions

$$\mathcal{H} = L_2([0, \ell]; \mathbb{C}^m)$$

with the elements

$$\vec{w}(x) = (w_j(x))_{j=1}^{m}, \quad x \in [0, \ell]. \tag{2.1}$$

Our main focus will be on recovering matching conditions at the central vertex, therefore we introduce the Laplace operator with general matching conditions. Such Laplace operator $L = -\frac{d^2}{dx^2}$ in $\mathcal{H}$ is defined on the functions from the Sobolev space

to a boundary vertex γ_n we mean the scattering coefficient corresponding to the graph Γ with extra Dirichlet boundary conditions introduced at all other boundary vertices $\gamma_l, l \neq n$.

$W_2^2(\Gamma \setminus \{v_0\})$ satisfying the following matching conditions at the central vertex

$$i(S^1 - I)\vec{w}(0) = (S^1 + I)\vec{w}'(0), \tag{2.2}$$

and Dirichlet boundary conditions on the graph boundary

$$\vec{w}(\ell) = 0. \tag{2.3}$$

The $m \times m$ unitary matrix S^1 appearing in (2.2) uniquely parameterizes all possible matching conditions [21]. Throughout the paper we assume that S^1 is **irreducible**, *i.e.* the corresponding matching conditions are properly connecting.[3] If the matrix S^1 is reducible, then the vertex can be chopped into two vertices so that the matching conditions connect together just boundary values from each of the vertices. Such matching conditions are not properly connecting.

The linear operators corresponding to both reducible and irreducible matrices S^1 are self-adjoint in $\mathcal{H}$, but if S^1 is reducible, then the operator L may be represented as an orthogonal sum $L = \oplus \sum_{n=1}^{N} L_n$, where L_n are Laplace operators on star graphs $\Gamma_n,\ n = 1, 2, \ldots, N$, having m_n edges, $m = \sum_{n=1}^{N} m_n$. One may assume that the corresponding matrices S_n^1 are irreducible and the inverse problem can be studied for each operator L_n separately.

The unitary matrix S^1 appearing in (2.2) is precisely the vertex scattering matrix S^k for the energy $E = k^2$ with $k = 1$, and this parametrization of boundary conditions is one-to-one [21].

Matching conditions (2.2) can easily be generalized for arbitrary metric graphs. For any vertex V, we consider the vector $\vec{w}(V)$ of the limit values of the function w along the edges connected at V and the corresponding vector of normal derivatives $\partial_n \vec{w}(V)$ — derivatives in the direction away from the vertex (outgoing direction). The two vectors so defined have dimension equal to the valency of the vertex V and are independent of the direction the edges are parametrized. Then the coordinate free form of (2.2) is given by

$$i(S^1 - I)\vec{w}(V) = (S^1 + I)\partial_n \vec{w}(V). \tag{2.4}$$

Let $\vec{q} \in L_2(\Gamma)$ be real valued, then the corresponding Schrödinger operator A in $\mathcal{H}$ is determined by

$$A = L + Q, \tag{2.5}$$

where Q is the $m \times m$ matrix,

$$Q = \operatorname{diag}\{q_1, q_2, \ldots, q_m\}.$$

This operator is defined on the same domain $\operatorname{Dom}(L)$.

Thus the Schrödinger operator in $L_2(\Gamma)$ is uniquely determined by the potential q and unitary matrix S^1 appearing in the matching conditions (2.2), whereas the Laplace operator is determined by the matrix S^1 alone.

Consider the unique function $\vec{u} \in L_2(\Gamma)$ which solves the differential equation

$$-\vec{u}'' + Q\vec{u} = \lambda\vec{u}, \quad \Im\lambda > 0, \tag{2.6}$$

satisfies matching conditions (2.2) and has prescribed boundary values $\vec{u}(\ell)$. Such a solution is unique, since otherwise the self-adjoint Schrödinger operator A would have non-real eigenvalues. Let us denote by $\partial_n \vec{u}$ the vector of normal derivatives at all boundary vertices. The direction of the derivatives is pointing inside the star

[3] A square matrix is called **reducible** if there exists a permutation of the coordinates leading to a block-diagonal matrix.

graph. Then the **Titchmarsh-Weyl** matrix function (or simply TW-function) $M(\lambda)$ is uniquely determined by the equality

$$\partial_n \vec{u}(\ell) = M(\lambda)\vec{u}(\ell) \Leftrightarrow \vec{u}'(\ell) = -M(\lambda)\vec{u}(\ell). \tag{2.7}$$

The TW-function is analogous to the Dirichlet-to-Neumann map widely used in inverse problems.

We consider also the corresponding wave equation in $L_2(\Gamma)$. Let $\vec{w}(x,t)$ be the solution to the differential equation

$$\frac{\partial^2}{\partial t^2}\vec{w}(x,t) - \frac{\partial^2}{\partial x^2}\vec{w}(x,t) + Q\vec{w}(x,t) = 0, \;\; x \in (0,\ell),\, t \in (0,T), \tag{2.8}$$

subject to the matching conditions (2.2) for all $t \in (0,T)$, the boundary condition

$$\vec{w}(\ell,t) = \vec{f}(t), \tag{2.9}$$

and satisfying zero initial data

$$\left\{ \begin{array}{rcl} \vec{w}(x,0) & = & 0, \\ \frac{\partial}{\partial t}\vec{w}(x,0) & = & 0. \end{array} \right. \tag{2.10}$$

The function f is referred to as the (Dirichlet) boundary control. We denote the solution of (2.8), (2.2), (2.9), and (2.10) by $\vec{w}^{\vec{f}}$. The **dynamical response operator** R^T is defined in $\mathcal{F}^T = L_2([0,T];\mathbb{C}^m)$ by the equality

$$\begin{array}{c} \left(R^T\vec{f}\right)(t) = \partial_n \vec{w}^{\vec{f}}(\ell,t) = -\dfrac{d}{dx}\vec{w}^{\vec{f}}(\ell,t), \;\; t \in [0,T], \\ \mathrm{Dom}\,(R^T) = \left\{\vec{f} \in W_2^1([0,T];\mathbb{C}^m),\; \vec{f}(0) = 0\right\}. \end{array} \tag{2.11}$$

The initial boundary value problem (2.8), (2.2), (2.9), (2.10) has a classical solution for smooth potentials and boundary controls $\vec{f}$ from the space

$$\left\{\vec{f} \in C^2([0,T];\mathbb{C}^m),\; \vec{f}(0) = \vec{f}'(0) = 0\right\}.$$

The response operator R^T originally defined on this space can be extended to the domain described in (2.11) (see, e.g. representations (3.11) and (4.6) below).

The response operator R^T is an integral operator with a generalized $m \times m$ matrix kernel [1].

The response operator R^T and TW-function $M(\lambda)$ are connected with each other by the Fourier–Laplace transform (see, e.g. [3]; [1, Section 3.2]). Knowledge of $M(\lambda)$ allows finding R^T for all $T > 0$, and knowledge of R^T for all $T > 0$ allows finding $M(\lambda)$. The response operator known for some finite T carries less information than does $M(\lambda)$. We will demonstrate that this data (R^T for a finite T) is sufficient for solving our inverse problems and specify those T.

It has been shown in [1] that the dynamical response operator determines the potential on all boundary edges, *i.e.* on all edges with one of the endpoints belonging to the graph boundary. For the star graph all edges are boundary edges, and therefore R^T (for $T \geq 2\ell$) determines the potential. Then it is natural to expect that R^T determines also the matching conditions at the central vertex. Hence our aim here is investigating the possibility to recover the potential and matching conditions from the principal $(m-1) \times (m-1)$ block of the response operator. In [1] similar result for the Schrödinger operator with standard matching conditions was used to solve the inverse problem for arbitrary trees. The solution of the inverse problem for star graphs serves as a main building brick in that solution. The problem we address here is more difficult, since one needs to recover not only the potential but matching conditions as well. Therefore let us discuss first whether the solution to such inverse problem is unique and what kind of freedom one can expect.

We introduce the $m \times m$ matrix $\mathcal{R}_\theta$:

$$\mathcal{R}_\theta = \operatorname{diag}\{1,1,...,1,e^{i\theta}\} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & e^{i\theta} \end{pmatrix}, \;\; \theta \in [0, 2\pi). \tag{2.12}$$

This matrix determines the unitary transformation of the Hilbert space $\mathcal{H}$,

$$\vec{w} \mapsto \mathcal{R}_\theta \vec{w}. \tag{2.13}$$

In what follows we identify the matrix $\mathcal{R}_\theta$ and the corresponding unitary Hilbert space transformation.

Consider the self-adjoint Schrödinger operator

$$A_\theta = \mathcal{R}_\theta A \mathcal{R}_{-\theta}.$$

It is defined by the same differential expression as the operator A, but by different matching conditions. The matching conditions for the operator A_θ are defined as in the formula (2.2) but with the unitary matrix

$$S^1_\theta = \mathcal{R}_\theta S^1 \mathcal{R}_{-\theta} \tag{2.14}$$

instead of S^1. It is clear that the principal $(m-1)\times(m-1)$ blocks of the corresponding TW-functions for the operators A and A_θ coincide. The same is true for the dynamical response operators. Hence one can expect to reconstruct the matching conditions up to the transformation described by (2.14) (if the inverse data consist of the principal block of the TW-function or dynamical response operator). Our goal is to prove that this reconstruction is possible and no other arbitrariness occurs. Moreover, we give an explicit algorithm for this reconstruction.

We use the upper index $m-1$ to denote m-dimensional vectors with zero last components: $\vec{a} = \vec{a}^{m-1} \Rightarrow \left(\vec{a}^{m-1}\right)_m = 0$.

3. **Reconstruction of matching conditions for Laplacians.** We start with solving the inverse problem for Laplace operators, *i.e.* we suppose in this section that $\vec{q} \equiv 0$. Consider the following differential equation

$$\frac{\partial^2}{\partial t^2}\vec{w}(x,t) - \frac{\partial^2}{\partial x^2}\vec{w}(x,t) = 0, \;\; x \in (0,\ell),\, t \in (0,T), \tag{3.1}$$

with the matching conditions (2.2), the boundary control acting at the first $(m-1)$ boundary points

$$\vec{w}(\ell,t) = \vec{f}^{m-1}(t), \;\; (i.e.\, (\vec{f}^{m-1}(t))_m = 0), \tag{3.2}$$

and with zero initial data (2.10).

It is clear that solutions to the differential equation can be written as a combination of d'Alembert waves

$$\vec{w}(x,t) = \vec{b}(t+x) + \vec{a}(t-x), \tag{3.3}$$

where $\vec{b}$ and $\vec{a}$ denote, respectively, the waves going toward the central vertex and coming from it. The boundary control initiates waves on the intervals Δ_j, $j = 1,2,\dots,m-1$, which reach the central vertex at the time $t = \ell$ as earliest. Therefore for sufficiently small t ($t < \ell$) the solution is equal to zero on Δ_m and is given by just one travelling wave

$$\vec{w}(x,t) = \vec{f}^{m-1}(t+x-\ell), \;\; t < \ell.$$

For t slightly larger than ℓ ($\ell < t < 2\ell$) the solution on Δ_m contains only a wave going away from the central vertex

$$\vec{w}(x,t) = \vec{f}^{m-1}(t+x-\ell) + \vec{a}(t-x). \tag{3.4}$$

Our immediate aim is to calculate the function $\vec{a}$ using matching conditions (2.2). These vector conditions can be written using the spectral subspaces $N_{-1} = \ker(S^1 + I)$ and $N_{-1}^{\perp}$ for the unitary matrix S^1. The corresponding orthogonal projectors in $\mathbb{C}^m$ are denoted by $P_{N_{-1}}$ and $P_{N_{-1}}^{\perp}$ respectively.

Let us substitute (3.4) into the matching conditions (2.2)

$$i(S^1 - I)\left(\vec{f}^{m-1}(t-\ell) + \vec{a}(t)\right) = (S^1+I)\left(\frac{d}{dt}\vec{f}^{m-1}(t-\ell) - \frac{d}{dt}\vec{a}(t)\right).$$

Projecting to the subspace N_{-1} we get an easily solvable equation

$$P_{N_{-1}}\vec{f}^{m-1}(t-\ell) + P_{N_{-1}}\vec{a}(t) = 0 \Rightarrow P_{N_{-1}}\vec{a}(t) = -P_{N_{-1}}\vec{f}^{m-1}(t-\ell). \tag{3.5}$$

Projection onto the orthogonal complement to N_{-1} gives the following differential equation

$$H\left(P_{N_{-1}}^{\perp}\vec{f}^{m-1}(t-\ell) + P_{N_{-1}}^{\perp}\vec{a}(t)\right) = \frac{d}{dt}P_{N_{-1}}^{\perp}\vec{f}^{m-1}(t-\ell) - \frac{d}{dt}P_{N_{-1}}^{\perp}\vec{a}(t),$$

where

$$H = i\frac{S^1 - I}{S^1 + I}P_{N_{-1}}^{\perp} \tag{3.6}$$

is a Hermitian matrix in $N_{-1}^{\perp}$. The solution to this differential equation is given by the formula

$$P_{N_{-1}}^{\perp}\vec{a}(t) = P_{N_{-1}}^{\perp}\vec{f}^{m-1}(t-\ell) - 2He^{-Ht}\int_0^t e^{H\tau}P_{N_{-1}}^{\perp}\vec{f}^{m-1}(\tau-\ell)d\tau, \tag{3.7}$$

where we put $\vec{f}(t) = 0$ for $t \leq 0$.

Combining (3.5) and (3.7) we obtain solution to the wave equation satisfying matching conditions for $t < 2\ell$

(3.8)
$$\begin{aligned}\vec{w}(x,t) &= \vec{f}^{m-1}(t+x-\ell) - P_{N_{-1}}\vec{f}^{m-1}(t-x-\ell) + P_{N_{-1}}^{\perp}\vec{f}^{m-1}(t-x-\ell) \\ &\quad -2He^{-H(t-x)}\int_0^{t-x} e^{H\tau}P_{N_{-1}}^{\perp}\vec{f}^{m-1}(\tau-\ell)d\tau \\ &= \vec{f}^{m-1}(t+x-\ell) + S^{\infty}\vec{f}^{m-1}(t-x-\ell) \\ &\quad -2He^{-H(t-x-\ell)}\int_0^{t-x-\ell} e^{H\tau}P_{N_{-1}}^{\perp}\vec{f}^{m-1}(\tau)d\tau,\end{aligned}$$

where

$$S^{\infty} = -P_{N_{-1}} + P_{N_{-1}}^{\perp} \tag{3.9}$$

is the large energy limit of the vertex scattering matrix:

$$S^{\infty} = \lim_{k\to\infty} S^k = \lim_{k\to\infty}\frac{(k+1)S^1 + (k-1)I}{(k-1)S^1 + (k+1)I}.$$

We find now the solution to the system (3.1), (2.2), (3.2), (2.10) for $t \in (2\ell, 3\ell)$. This solution can be obtained from (3.8) by adding the wave reflected from the boundary (Dirichlet conditions),

(3.10)
$$\begin{array}{rcl} \vec{w}(x,t) & = & \vec{f}^{m-1}(t+x-\ell) \\ & & +S^\infty \vec{f}^{m-1}(t-x-\ell) - 2He^{-H(t-x-\ell)} \displaystyle\int_0^{t-x-\ell} e^{H\tau} P_{N_{-1}}^\perp \vec{f}^{m-1}(\tau) d\tau \\ & & -S^\infty \vec{f}^{m-1}(t+x-3\ell) \\ & & +2He^{-H(t+x-3\ell)} \int_0^{t+x-3\ell} e^{H\tau} P_{N_{-1}}^\perp \vec{f}^{m-1}(\tau) d\tau. \end{array}$$

This implies the formula for the principal block of the dynamical response operator for $T \in (0, 3\ell)$:

(3.11)
$$\begin{array}{rcl} \left(R^T \vec{f}^{m-1} \right)(t) & = & \partial_n \vec{w}(\ell, t) = -\dfrac{\partial}{\partial x} \vec{w}(\ell, t) \\ & = & -\dfrac{d}{dt} \vec{f}^{m-1}(t) + 2S^\infty \dfrac{d}{dt} \vec{f}^{m-1}(t-2\ell) - 4HP_{N_{-1}}^\perp \vec{f}^{m-1}(t-2\ell)) \\ & & +4H^2 e^{-H(t-2\ell)} \displaystyle\int_{-\infty}^{t-2\ell} e^{H\tau} P_{N_{-1}}^\perp \vec{f}^{m-1}(\tau) d\tau. \end{array}$$

The response operator can be seen as an integral operator with the generalized kernel $r(t-\tau)$ where

$$r(t) = -\delta'(t) + 2S^\infty \delta'(t-2\ell) - 4HP_{N_{-1}}^\perp \delta(t-2\ell) + 4H^2 e^{-H(t-2\ell)} P_{N_{-1}}^\perp \theta(t-2\ell). \tag{3.12}$$

The kernel $4H^2 e^{-H(t-2\ell)} P_{N_{-1}}^\perp \theta(t-2\ell)$ is locally L_2, therefore we have proven the following lemma:

Lemma 3.1. *The principal $(m-1)\times(m-1)$ block of the response operator R^T, $T > 2\ell$ determines the principal $(m-1)\times(m-1)$ blocks of the limit scattering matrix S^∞*

$$P_m^\perp S^\infty P_m^\perp$$

and of the H-operator

$$P_m^\perp P_{N_{-1}}^\perp H P_{N_{-1}}^\perp P_m^\perp,$$

where $P_m^\perp$ denotes the orthogonal projector on the $m-1$ dimensional subspace of $\mathbb{C}^m$ spanned up by the first $m-1$ basis vectors.

In what follows we discuss whether the matrices S^∞ and H are determined by their principal blocks. It is obvious that this reconstruction is not unique as we have already discussed. We assume that not only the matrix S^1, but also the limit scattering matrix S^∞ is irreducible.

Theorem 3.2. *Consider the set of $m\times m$ irreducible limit vertex scattering matrices S^∞ having the same principal $(m-1)\times(m-1)$ block $P_m^\perp S^\infty P_m^\perp$. This family of matrices can be described using one real phase parameter so that*

$$S_\theta^\infty = \mathcal{R}_\theta S^\infty \mathcal{R}_{-\theta}, \ \theta \in [0, 2\pi), \tag{3.13}$$

where $\mathcal{R}_\theta$ is given by (2.12) and S^∞ is a certain particular member of the family.

Proof. Reconstruction of any unitary matrix from its principal $(m-1)\times(m-1)$ block contains in general two arbitrary phase parameters and can be carried out using the fact that the entries of such matrix satisfy the normalization and orthogonality conditions:

$$\begin{array}{ll} \sum_{j=1}^m |s_{ij}|^2 = 1, & \sum_{i=1}^m |s_{ij}|^2 = 1, \\ \sum_{j=1}^m s_{ij}\overline{s_{lj}} = 0, i \neq l, & \sum_{i=1}^m s_{ij}\overline{s_{il}} = 0, j \neq l. \end{array}$$

Assume that the principal $(m-1)\times(m-1)$ block of the matrix S^∞ is known. Consider the last row in S^∞. The absolute values of $s_{mj} = (S^\infty)_{mj}, j = 1, 2, ..., m-1$ can be calculated from the normalization conditions. At least one of these numbers is different from zero, otherwise the matrix S^∞ is reducible. Consider any such different from zero element, say with the index $m1$. All possible values of this element can be described by one real phase parameter α as follows $s_{m1} = s^0_{m1}e^{i\alpha}$, where s^0_{m1} is any complex number with the prescribed absolute value. Then all other elements $s_{mj}, j = 2, ..., m-1$ can be calculated using the orthogonality conditions. In the same way one may consider the last column and introduce a certain parameter $\beta \in \mathbb{R}$ such that $s_{1m} = s^0_{1m}e^{i\beta}$. Then all elements $s_{jm},\ j = 2, 3, \dots, m$ are uniquely determined.

Let us summarize our calculations by stating the following result: the family of unitary matrices having the same principal $(m-1)\times(m-1)$ block can be described using two real parameters so that

$$S^\infty_{\alpha,\beta} = \mathcal{R}_\alpha S^\infty \mathcal{R}_\beta, \tag{3.14}$$

where S^∞ is a certain particular member of the family.

It is clear that the unitary transformation (2.13) changes the matching conditions in accordance with (2.14). The same relation holds for the limit vertex scattering matrices.

Hence it remains to prove that if S^∞ is a limit scattering matrix, then $S^\infty\mathcal{R}_\beta$ cannot be a limit scattering matrix. Let us recall that the limit scattering matrix is not only unitary but also Hermitian (as follows from (3.9), its eigenvalues are equal to ± 1).

Multiplication of the Hermitian matrix S^∞ by $\mathcal{R}_\beta$ leads to a Hermitian matrix only if all non-diagonal elements in the last column are equal to zero and thus the matrix S^∞ is block-diagonal, and hence reducible, which contradicts our assumptions.

Summing up, all possible matrices S^∞ having the same principal $(m-1)\times(m-1)$ block are described by formula (3.13). ☐

The assumption of Theorem 3.2 that S^∞ is irreducible can be weakened. In fact we used just the fact that $S^\infty_{mm} \neq \pm 1$, in other words that S^∞ is not block-diagonal with $(m-1)\times(m-1)$ and 1×1 blocks.

In the following Lemma we discuss the possibility to reconstruct the unitary matrix S^1.

Lemma 3.3. *Assume that the limit scattering matrix S^∞ is irreducible, then the knowledge of the subspace N_{-1} and of the $(m-1)\times(m-1)$ matrix*

$$P_m^\perp P_{N_{-1}}^\perp H P_{N_{-1}}^\perp P_m^\perp \tag{3.15}$$

determines the unique matching condition, i.e. the unique matrix S^1.

Proof. Consider the $(m-1)\times(m-1)$ Hermitian matrix (3.15). We extend it to the Hermitian $m\times m$ matrix $\hat{H} = H \oplus O_{N_{-1}}$, where $O_{N_{-1}}$ is the zero matrix in the subspace N_{-1}. The kernel of $\hat{H}$ contains the whole subspace N_{-1}. Since S^∞ is irreducible, the subspace N_{-1} is not trivial and contains at least one vector with nonzero m-th component. (Otherwise $S^\infty_{mm}=1$ and S^∞ is reducible.) Then applying the matrix $\hat{H}$ to this vector we should get zero vector. This fact allows us to calculate the elements $\hat{h}_{jm},\ j=1,2,\dots,m-1$ of the last column in $\hat{H}$. Using the fact that $\hat{H}$ is Hermitian we reconstruct the last row except the element $\hat{h}_{mm}$, which again can be calculated using the fact that $\hat{H}$ maps every vector from N_{-1} to the zero vector.

It remains to calculate the unitary matrix S^1:

$$S^1 = \frac{iI_{N_{-1}^\perp}+H}{iI_{N_{-1}^\perp}-H}\oplus(-1)I_{N_{-1}}, \tag{3.16}$$

where $I_{N_{-1}^\perp}$ and $I_{N_{-1}}$ are the identity operators in $N_{-1}^\perp$ and N_{-1} respectively and $(iI_{N_{-1}^\perp}+H)(iI_{N_{-1}^\perp}-H)^{-1}$ is considered as a unitary operator in $N_{-1}^\perp$. □

The last lemma may give an impression that using the knowledge of the principal $(m-1)\times(m-1)$ block of S^∞ and of $P_m^\perp P_{N_{-1}^\perp} H P_{N_{-1}^\perp} P_m^\perp$ allows one to reconstruct unique matching conditions. This is not true, since the principal $(m-1)\times(m-1)$ block of S^∞ allows one to reconstruct S^∞ up to the unitary transformation (3.13), *i.e.* the subspace N_{-1} is determined up to multiplication by R_θ. Choosing different possible subspaces N_{-1} one gets different possible matrices S^1 (described in fact by the same unitary transformation (3.13)).

We summarize our studies in the following

Theorem 3.4. *Let Γ be a star graph formed by m edges of length ℓ connected together at the vertex v_0. Consider the Laplace operator $L=-\frac{d^2}{dx^2}$ defined in $L_2(\Gamma)$ on the domain of functions from $W_2^2(\Gamma\setminus\{0\})=W_2^2([0,\ell],\mathbb{C}^m)$ satisfying matching conditions (2.2) and Dirichlet boundary conditions at all boundary points. Let the limit scattering matrix S^∞ be irreducible. Then the principal $(m-1)\times(m-1)$ block of the response operator R^T, $T>2\ell$ determines the matching conditions at the central vertex up to the unitary transformation (2.14), in other words the family of all possible operators L corresponding to the same principal $(m-1)\times(m-1)$ block of the response operator can be parameterized as follows*

$$L_\theta = \mathcal{R}_\theta L \mathcal{R}_{-\theta},\ \ \theta\in[0,2\pi), \tag{3.17}$$

where L is any particular member of the family.

The theorem is proven under the assumption that S^∞ is irreducible, but it holds true even under the weaker assumption that just S^1 is irreducible. (The corresponding proof is more involved.) However, in more general situation of the next section we still have to assume that S^∞ is irreducible.

4. **Reconstruction of matching conditions for Schrödinger operators.** Let us consider the case where the potential may be different from zero. Our aim is to prove the following theorem, which is one of the main results of this article.

Theorem 4.1. *Assume that:*

- *Γ is a star graph formed by m edges of length ℓ connected together at the vertex v_0;*
- *$A=L+Q$ is a Schrödinger operator in $L_2(\Gamma)=L_2([0,\ell],\mathbb{C}^m)$, where $L=-\frac{d^2}{dx^2}$ is the Laplace operator defined on the domain of functions from the*

space $W_2^2([0,\ell],\mathbb{C}^m)$ satisfying matching conditions (2.2) and Dirichlet boundary conditions at all boundary points, and $Q = \mathrm{diag}\{q_1, q_2, \dots, q_m\}$, $\vec{q} \in L_2([0,\ell],\mathbb{C}^m)$ is a real valued potential;

- *the matrix S^1 parameterizing the matching conditions (2.2) is irreducible as well as the limit scattering matrix S^∞ defined by (3.9).*

Then the principal $(m-1)\times(m-1)$ block of the response operator R^T for any $T \geq 4\ell$ determines the Schrödinger operator A (i.e. the potential $\vec{q}$ and the matrix S^1 from the matching conditions at the central vertex) up to the similarity transformation (2.14), in other words the family of all possible operators A corresponding to the same principal $(m-1)\times(m-1)$ block of the response operator can be parameterized as follows:

$$A_\theta = \mathcal{R}_{-\theta} A \mathcal{R}_\theta, \quad \theta \in [0, 2\pi), \tag{4.1}$$

where A is any particular member of the family.

The Theorem holds even if the principal $(m-1)\times(m-1)$ block of the TW-function $M(\lambda)$ is known. Note that to know $R^T, T > 2\ell$ suffices to determine the matching conditions (up to the described unitary transformation), whereas the knowledge of $R^T, T \geq 4\ell$ is necessary for recovering the potential on Δ_m.

To prove the theorem, we first consider the case where the potential is different from zero only on the edge Δ_m, then we reduce a general problem to this situation.

Lemma 4.2. *Theorem 4.1 is valid under additional assumption that the potential $\vec{q}$ is different from zero only on the edge Δ_m, i.e.*

$$q_j(x) = 0,\ j = 1, 2, \dots, m-1. \tag{4.2}$$

Proof. Assume that the principal $(m-1)\times(m-1)$ block of the TW-function or of the dynamical response operator is known.

Consider the wave equation (2.8), subject to the matching conditions (2.2), the boundary control through the first $m-1$ boundary points (3.2) and with zero initial data (2.10). Since the potential q is different from zero only on the edge Δ_m, any solution of (2.8) on the intervals $\Delta_j, j = 1, 2, \dots, m-1$ can be written as before as a combination of d'Alembert waves:

$$P_m^\perp \vec{w}(x,t) = \vec{b}^{m-1}(t+x) + \vec{a}^{m-1}(t-x). \tag{4.3}$$

The boundary control initiates the following wave travelling toward the central vertex

$$P_m^\perp \vec{w}(x,t) = \vec{f}^{m-1}(t+x-l), \quad t < \ell.$$

It is clear that

$$w_m(x,t) = 0, \quad t < \ell,$$

and the last two formulas can be combined as

$$\vec{w}(x,t) = \vec{f}^{m-1}(t+x-l), \quad t < \ell. \tag{4.4}$$

For slightly larger values of t ($\ell < t < 2\ell$) the solution will contain waves scattered by the central vertex, but just the same incoming wave

$$P_m^\perp \vec{w}(x,t) = \vec{f}^{m-1}(t+x-l) + \vec{a}^{m-1}(t-x), \quad \ell < t < 2\ell. \tag{4.5}$$

The last component $w_m(x,t)$ will be in general different from zero for $t > \ell$. The values of $w_m(0,t)$ and its normal derivative $\partial_n w_m(x,t)|_{x=0} = w_m'(0,t)$ are related via the dynamical response operator R_m^T for the interval Δ_m:

$$(R_m^T w_m(0,\cdot))(t) = w_m'(0,t), \tag{4.6}$$

where

$$(R_m^T g)(t) = -g'(t) + \int_0^t r_m(t-\tau) g(\tau) d\tau, \; t \in (0, 2\ell). \tag{4.7}$$

The kernel $r_m(t)$ belongs to L_1 provided $q \in L_1$ [3, 1].

In what follows it is convenient to use the notation $a_m(t) = w_m(0,t)$. Then the matching conditions give us the following system of integro-differential equations

$$\begin{aligned} &i(S^1 - I)\left(\vec{f}^{m-1}(t-\ell) + \vec{a}(t)\right) \\ &= (S^1 + I)\left(\frac{d}{dt}\vec{f}^{m-1}(t-\ell) - \frac{d}{dt}\vec{a}(t) + \int_0^t r_m(t-\tau) P_m \vec{a}(\tau) d\tau\right), \end{aligned} \tag{4.8}$$

where P_m is the projector on the last basis vector in $\mathbb{C}^m$. Consider this system of equations in the orthogonal decomposition

$$\mathbb{C}^m = N_{-1} \oplus N_{-1}^{\perp}.$$

Projecting the left and right hand sides of the equation on N_{-1} we get the system of easily solvable linear equations

$$P_{N_{-1}} \vec{f}^{m-1}(t-\ell) + P_{N_{-1}} \vec{a}(t) = 0 \Rightarrow P_{N_{-1}} \vec{a}(t) = -P_{N_{-1}} \vec{f}^{m-1}(t-\ell). \tag{4.9}$$

On the other hand, projection onto $N_{-1}^{\perp}$ gives the system of integro-differential equations

$$\begin{aligned} & H\left(P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t-\ell) + P_{N_{-1}^{\perp}} \vec{a}(t)\right) \\ &= \frac{d}{dt} P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t-\ell) - \frac{d}{dt} P_{N_{-1}^{\perp}} \vec{a}(t) + \int_0^t r_m(t-\tau) P_{N_{-1}^{\perp}} P_m \vec{a}(\tau) d\tau \\ &\Rightarrow H P_{N_{-1}^{\perp}} \vec{a}(t) + \frac{d}{dt} P_{N_{-1}^{\perp}} \vec{a}(t) - \int_0^t r_m(t-\tau) P_{N_{-1}^{\perp}} P_m \vec{a}(\tau) d\tau \\ &= -H P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t-\ell) + \frac{d}{dt} P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t-\ell). \end{aligned} \tag{4.10}$$

Multiplying the last equation by $\exp(Ht)$ and integrating from $t=0$ to $t=t'$ gives us

$$\begin{aligned} & \exp(Ht') P_{N_{-1}^{\perp}} \vec{a}(t') - \int_0^{t'} \exp(Ht) \int_0^t r_m(t-\tau) P_{N_{-1}^{\perp}} P_m \vec{a}(\tau) d\tau dt \\ &= \exp(Ht') P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t'-\ell) - 2H \int_0^{t'} \exp(Ht) P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t-\ell) dt. \end{aligned} \tag{4.11}$$

Finally we arrive at the equation

$$\begin{aligned} & P_{N_{-1}^{\perp}} \vec{a}(t) - \int_0^t k(t-\tau) \vec{a}(\tau) d\tau \\ &= P_{N_{-1}^{\perp}} \vec{f}^{m-1}(t-\ell) - 2H \exp(-Ht) \int_0^t \exp(H\tau) P_{N_{-1}^{\perp}} \vec{f}^{m-1}(\tau-\ell) d\tau, \end{aligned} \tag{4.12}$$

with the matrix-valued difference kernel k,

$$k(s)=\exp(-Hs)\int_0^s \exp(Hy)r_m(y)P_{N_{-1}^{\perp}}P_m dy. \tag{4.13}$$

Let us note that the kernel is continuous and equal to zero at the origin; its first derivative belongs to L_1 locally.

We denote the right hand side of (4.12) by $P_{N_{-1}^{\perp}}\vec{a}^0(t)$, since it represents the solution to the differential equation in the case of zero potential (*i.e.* $r_m\equiv 0$). In order to put this equation into Volterra form, let us use the decomposition

$$\vec{a}=P_{N_{-1}^{\perp}}\vec{a}+P_{N_{-1}}\vec{a}$$

to get

$$P_{N_{-1}^{\perp}}\vec{a}(t)-\int_0^t k(t-\tau)P_{N_{-1}^{\perp}}\vec{a}(\tau)d\tau=P_{N_{-1}^{\perp}}\vec{a}^0(t)+\int_0^t k(t-\tau)P_{N_{-1}}\vec{a}(\tau)d\tau. \tag{4.14}$$

Note that the vector valued function $P_{N_{-1}}\vec{a}(\tau)$ is already determined by (4.9). Equation (4.14) can be solved by iterations:

(4.15)

$$\begin{aligned} P_{N_{-1}^{\perp}}\vec{a}(t) \; &= \; P_{N_{-1}^{\perp}}\vec{a}^0(t)+\int_0^t k(t-\tau)P_{N_{-1}}\vec{a}(\tau)d\tau \\ &\quad +\int_0^t p(t-\tau)\left(P_{N_{-1}^{\perp}}\vec{a}^0(\tau)+\int_0^\tau k(\tau-s)P_{N_{-1}}\vec{a}(s)ds\right)d\tau, \end{aligned}$$

where the kernel $p(s)$ has properties similar to those of $k(s)$, *i.e.* p is a continuous function and its derivative is from L_1 locally, moreover $p(0)=0$.

Finally, the function $\vec{a}$ is determined by the formula

(4.16)

$$\begin{aligned} &\vec{a}(t)\\ &=P_{N_{-1}}\vec{a}(t)+P_{N_{-1}^{\perp}}\vec{a}(t)\\ &=-P_{N_{-1}}\vec{f}^{m-1}(t-\ell)+P_{N_{-1}^{\perp}}\vec{f}^{m-1}(t-\ell)\\ &-2H\exp(-Ht)\int_0^t \exp(H\tau)P_{N_{-1}^{\perp}}\vec{f}^{m-1}(\tau-\ell)d\tau-\int_0^t k(t-\tau)P_{N_{-1}}\vec{f}^{m-1}(\tau-\ell)d\tau\\ &+\int_0^t p(t-\tau)\\ &\qquad\left(P_{N_{-1}^{\perp}}\vec{f}^{m-1}(\tau-\ell)-2H\exp(-H\tau)\textstyle\int_0^\tau \exp(Hs)P_{N_{-1}^{\perp}}\vec{f}^{m-1}(s-\ell)ds\right)d\tau\\ &-\int_0^t p(t-\tau)\int_0^\tau k(\tau-s)P_{N_{-1}}\vec{f}^{m-1}(s-\ell)dsd\tau. \end{aligned}$$

Summing up the solution to the system (2.8), (2.2), (3.2), and (2.10) for $\ell<t<2\ell$ on the first $m-1$ edges is given by

(4.17)
$$\begin{aligned}
&P_m^\perp \vec{w}(x,t)\\
&= \vec{f}^{m-1}(t+x-\ell) + P_m^\perp S^\infty \vec{f}^{m-1}(t-x-\ell)\\
&-2P_m^\perp H \exp(-H(t-x)) \int_\ell^{t-x} \exp(H\tau) P_{N_{-1}^\perp} \vec{f}^{m-1}(\tau-\ell)d\tau\\
&-P_m^\perp \int_\ell^{t-x} k(t-x-\tau) P_{N_{-1}} \vec{f}^{m-1}(\tau-\ell)d\tau\\
&+P_m^\perp \int_\ell^{t-x} p(t-x-\tau)\\
&\qquad \left(P_{N_{-1}^\perp} \vec{f}^{m-1}(\tau-\ell) - 2H\exp(-H\tau)\textstyle\int_\ell^\tau \exp(Hs) P_{N_{-1}^\perp} \vec{f}^{m-1}(s-\ell)ds\right) d\tau\\
&-P_m^\perp \int_\ell^{t-x} p(t-x-\tau) \int_\ell^\tau k(\tau-s) P_{N_{-1}} \vec{f}^{m-1}(s-\ell) ds d\tau,
\end{aligned}$$

where we have changed some of the integration limits taking into account that $\vec{f}^{m-1}(t) = \vec{0}$ for $t < 0$. The corresponding dynamical response operator R^T for $t \in (0, 2\ell)$ does not feel the reflection from the central vertex, since the scattered wave $\vec{a}(t-x)$ is equal to zero for $x = \ell$ and $t < 2\ell$. In order to get a nontrivial response one has to consider the values of t which are greater than 2ℓ. Only for such values of t the wave initiated by the boundary control has enough time to get to the central vertex, to be reflected from it and to reach the boundary again.

Consider now the interval $2\ell < t < 3\ell$. In order to calculate the solution for such values of t one has to take into account the reflection of the $\vec{a}$-wave from the boundary. Since we consider the Dirichlet boundary control (3.2), the solution is:

$$P_m^\perp \vec{w}(x,t) = \vec{f}^{m-1}(t+x-\ell) + P_m^\perp \vec{a}(t-x) - P_m^\perp \vec{a}(t+x-2\ell), \tag{4.18}$$

where $\vec{a}$ is the function given by (4.16). To determine the principal $(m-1)\times(m-1)$ block of the response operator we have to calculate

$$\partial_n P_m^\perp \vec{w}(\ell,t) = -\frac{\partial}{\partial x} P_m^\perp \vec{w}(x,t)|_{x=\ell} = -\frac{d}{dt}\vec{f}^{m-1}(t) + 2P_m^\perp \frac{d}{dt}\vec{a}(t-\ell).$$

It follows that

$$\begin{aligned}
&\left(P_m^\perp R^T P_m^\perp \vec{f}^{m-1}\right)(t)\\
= \quad & -\frac{d}{dt}\vec{f}^{m-1}(t) + 2P_m^\perp S^\infty P_m^\perp \frac{d}{dt}\vec{f}^{m-1}(t-2\ell)\\
&-4P_m^\perp P_{N_{-1}^\perp} H P_{N_{-1}^\perp} P_m^\perp \vec{f}^{m-1}(t-2\ell) + \int_\ell^{t-\ell} g(t-\tau)\vec{f}^{m-1}(\tau-\ell)d\tau,
\end{aligned} \tag{4.19}$$

where g is a certain L_1 kernel. The last formula shows that the kernel of the response operator in the presence of a nontrivial potential on Δ_m has the same main singularities as the kernel of the response operator for the Laplacian (corresponding to the same matching conditions at the central vertex).

Formula (4.19) shows that the principal block of the response operator determines the following matrices

$$P_m^\perp S^\infty P_m^\perp \quad \text{and} \quad P_m^\perp P_{N_{-1}}^\perp H P_{N_{-1}}^\perp P_m^\perp,$$

which in accordance with Theorem 3.2 and Lemma 3.3 determine the matrix S^1 up to the transformation (3.13) containing the real parameter θ.

It remains to prove that the potential on Δ_m is also uniquely determined by the principal $(m-1)\times(m-1)$ block of the response operator R^T, $T \geq 4\ell$. Equation

(4.8) can be written as
(4.20)
$$\begin{aligned} i(S^1-I)\left(\vec f^{m-1}(t-\ell)+\vec a^{m-1}(t)\right)-(S^1+I)\frac{d}{dt}\left(\vec f^{m-1}(t-\ell)-\vec a^{m-1}(t)\right)\\ =-ia_m(t)\,(S^1-I)\vec e_m+(R_m^Ta_m)(t)\,(S^1+I)\vec e_m,\end{aligned}$$

where $\vec e_m$ is the m-th basic vector in $\mathbb{C}^m$ and $\vec a^{m-1}=P_m^\perp\vec a$. We assume that the matrix S^1 is known, then the last equation allows one to calculate both a_m - the control function for the interval Δ_m and $R_m^Ta_m$ - the corresponding response. We need to show that any admissible function $a_m\in W_2^1[\ell,3\ell], a_m(\ell)=0$, can be obtained. (The reflected wave a_m is equal to zero for $t<\ell$.)

Since the matrix S^1 is irreducible, the subspace N_{-1} contains a vector $\vec u\in N_{-1}$ with $u_m\neq 0$, then it holds

$$(S^1+I)P_m^\perp\vec u=-u_m(S^1+I)\vec e_m.$$

It follows in particular that the two ranges coincide

$$\mathcal{R}(S^1+I)P_m^\perp=\mathcal{R}(S^1+I)\ \left(=\mathbb{C}^m\ominus N_{-1}\right).$$

Similarly the space $\mathbb{C}^m\ominus N_{-1}$ contains a vector $\vec w\in\mathbb{C}^m\ominus N_{-1}$ with $w_m\neq 0$, which implies that $(S^1-I)\vec w\perp N_{-1}$. Then there exists a certain vector $\vec v^{m-1}$ such that

$$(S^1-I)\vec w=(S^1+I)\vec v^{m-1}.$$

The last equation implies that

$$(S^1-I)P_m^\perp\vec w-(S^1+I)\vec v^{m-1}=-w_m(S^1-I)\vec e_m$$

i.e. the equation

$$(S^1-I)\vec w^{m-1}-(S^1+I)\vec v^{m-1}=(S^1-I)\vec e_m$$

has a solution.

Summing up one can always find a solution to (4.20), *i.e.* the functions

$$\vec f^{m-1}(t-\ell)+\vec a^{m-1}(t)$$

and

$$\frac{d}{dt}\left(\vec f^{m-1}(t-\ell)-\vec a^{m-1}(t)\right),$$

allowing to calculate the control $\vec f^{m-1}$ and the reflected wave $\vec a^{m-1}$ leading to any given a_m. To reconstruct potential on Δ_m we need to know the response operator R_m^T for $T\geq 3\ell$, which in turn requires knowledge of the principal $(m-1)\times(m-1)$ block of the reponse operator R^T for $T\geq 4\ell$. The potential on Δ_m is independent of the parameter θ in the matrix S^1. $\square$

We return now to the proof of the theorem formulated at the beginning of this section.

Proof of Theorem 4.1. The principal block of the dynamical response operator for $T=2\ell$ determines the potential on the first $m-1$ edges [1]. It was proven that the diagonal element of dynamical response operator known for $T=2d$ determines the potential on the corresponding boundary edge on the distance less or equal to d from the boundary (provided d is less than the length of the boundary edge). The reconstruction procedure is local and can be applied to trees with arbitrary matching conditions at the vertices. Hence the principal block of the dynamical response operator for A known for $T=2\ell$ determines the potential on the first

$m-1$ edges. It remains to prove that the matching conditions and the potential on Δ_m can be recovered.

For readers convenience we present here a proof based on TW-function instead of the dynamical response operator. Let us consider two Schrödinger operators A and $\hat{A}$ both acting in $L_2(\Gamma)$ and defined by the same matching condition at the central vertex, but by different potentials:

$$\begin{array}{ll} A = L + Q, & Q = \operatorname{diag}\{q_1, q_2, \ldots, q_m\}, \\ \hat{A} = L + \hat{Q}, & \hat{Q} = \operatorname{diag}\{0, 0, \ldots, 0, q_m\}, \end{array}$$

where L is a Laplacian defined by matching conditions (2.2). Let $M(\lambda)$ and $\hat{M}(\lambda)$ denote the corresponding TW-functions. To prove Theorem 4.1 it is enough to calculate the principal block of $\hat{M}(\lambda)$ from the principal block of $M(\lambda)$. Consider any vector $\vec{v}^{m-1}$ and the corresponding solution u to equation (2.6) subject to matching conditions (2.2) and boundary conditions

$$\vec{u}(\ell) = \vec{v}^{m-1}.$$

The normal derivatives of u can be calculated

$$P_m^{\perp} \partial_n \vec{u}(\ell) = P_m^{\perp} M(\lambda) P_m^{\perp} \vec{v}^{m-1}. \tag{4.21}$$

The knowledge of $u_j(\ell), u_j'(\ell)$ and of the corresponding potential $q_j(x)$ allows one to calculate $u_j(0), u_j'(0)$, $j = 1, 2, \ldots, m-1$. Consider now solutions $\hat{u}$ to the Schrödinger equation with zero potential,

$$-\hat{u}_j''(x) = k^2 \hat{u}_j(x), \tag{4.22}$$

satisfying the same initial data

$$\hat{u}_j(0) = u_j(0), \;\; \hat{u}_j'(0) = u_j'(0), \quad j = 1, 2, \ldots, m-1. \tag{4.23}$$

Let us introduce the notations

$$\hat{v}_j = \hat{u}_j(\ell), \;\; \partial_n \hat{v}_j = -\hat{u}_j'(\ell).$$

It is clear that the vectors $\vec{\hat{v}}^{m-1}$ and $\partial_n \vec{\hat{v}}^{m-1}$ are uniquely determined by $\vec{v}^{m-1}$. Moreover the mapping

$$\vec{v}^{m-1} \mapsto \vec{\hat{v}}^{m-1}$$

is one-to-one, since the vector $\vec{v}^{m-1}$ is uniquely determined by $\vec{\hat{v}}^{m-1}$. (The transformation is similar to the one just described.) Hence the vectors $\vec{\hat{v}}^{m-1}$ span the $m-1$-dimensional subspace of $\mathbb{C}^m$ and the matrix $P_m^{\perp} \hat{M}(\lambda) P_m^{\perp}$ connecting $\vec{\hat{v}}^{m-1}$ and $\partial_n \vec{\hat{v}}^{m-1}$ is unique

$$\partial_n \vec{\hat{v}}^{m-1} = P_m^{\perp} \hat{M}(\lambda) P_m^{\perp} \vec{\hat{v}}^{m-1}.$$

Summing up $P_m^{\perp} M(\lambda) P_m^{\perp}$ determines $P_m^{\perp} \hat{M}(\lambda) P_m^{\perp}$. Using the principal block of $\hat{M}(\lambda)$, one can reconstruct the potential on Δ_m and the matching conditions up to the transformation described by (4.1).

To accomplish the proof we have to show that the knowledge of $P_m^{\perp} R^T P_m^{\perp}$ for $T = 4\ell$ is enough. This can be done by contradiction. We are going to use once more the ideas developed in the proof of Lemma 4.2. Assume that two different Schrödinger operators on Γ determine the dynamical response operators R^T and $\tilde{R}^T$ such that

$$P_m^{\perp} R^T P_m^{\perp} = P_m^{\perp} \tilde{R}^T P_m^{\perp} \;\text{ for }\; T = 4\ell, \tag{4.24}$$

but

$$P_m^\perp R^T P_m^\perp \neq P_m^\perp \tilde{R}^T P_m^\perp, \tag{4.25}$$

in general. The expression (4.19) has to be modified, but will contain the same main singularities allowing to reconstruct the matching conditions, up to the parameter θ of course. Hence the two Schrödinger operators may be different only if the potentials on Δ_m are different and therefore if the corresponding response operators R_m^T and $\tilde{R}_m^T$ are different already for $T = 3\ell$[4], *i.e.* there exists a certain a_m, such that $R_m^T a_m \neq \tilde{R}_m^T a_m$. Equation (4.20) can be written as follows

$$\begin{aligned} & i(S^1 - I)\vec{w}^{m-1}(0,t) - (S^1 + I)\frac{\partial}{\partial x}\vec{w}^{m-1}(0,t) \\ &= -ia_m(t)(S^1 - I)\vec{e}_m + (R_m^T a_m)(t)(S^1 + I)\vec{e}_m, \end{aligned} \tag{4.26}$$

where $\vec{w}(x,t)$ is a solution to the corresponding wave equation. The vector functions $\vec{w}^{m-1}(0,t)$ and $\frac{\partial}{\partial x}\vec{w}^{m-1}(0,t)$ for $t \in [0, 3\ell]$ are uniquely determined by the control function $\vec{f}^{m-1}(t) = \vec{w}^{m-1}(\ell,t)$ and the corresponding response $-\frac{\partial}{\partial x}\vec{w}(0,t) = (P_m^\perp R^T P_m^\perp \vec{f}^{m-1})(t)$ for $t \in [0, 4\ell]$. On the other hand as in the proof of Lemma 4.2, equation (4.26) shows that the vector functions $\vec{w}^{m-1}(0,t)$ and $\frac{\partial}{\partial x}\vec{w}^{m-1}(0,t)$ determine unique a_m and $R_m^T a_m$, which implies that $R_m^T a_m = \tilde{R}_m^T a_m$ and we get a contradiction. To see that every admissble function a_m is possible, consider again (4.26) allowing from a_m and $R_m^T a_m$ to find (non unique) vector functions $\vec{w}^{m-1}(0,t)$ and $\frac{\partial}{\partial x}\vec{w}^{m-1}(0,t)$, which in turn allow to calculate control $\vec{f}^{m-1}(t) = \vec{w}^{m-1}(\ell,t)$ leading to given a_m. ☐

5. Arbitrary star graphs: Generalizations and discussions.

Developed methods can be easily applied to arbitrary star graphs, *i.e.* the restriction that the lengths of the edges are all equal is not essential. Vector notations for the potential q and the function u are not appropriate anymore. Nevertheless we do not need to reformulate all definitions (for the TW-function, the dynamical response operator, *etc.*), since all necessary changes are completely obvious (see, e.g. [1]).

Theorem 5.1. *Assume that:*

- *Γ is a star graph formed by m edges $\Delta_j = [x_{2j-1}, x_{2j}]$, $j = 1, 2, \ldots, m$ connected together at the vertex $v_0 = \{x_1, x_3, \ldots, x_{2m-1}\}$;*
- *$L = -\frac{d^2}{dx^2}$ is a Laplace operator in $L_2(\Gamma)$ defined on the domain of functions from $w \in W_2^2(\Gamma \setminus \{v_0\})$ satisfying the matching conditions at the vertex v_0*

$$i(S^1 - I)\begin{pmatrix} w(x_1) \\ w(x_3) \\ \vdots \\ w(x_{2m-1}) \end{pmatrix} = (S^1 + I)\begin{pmatrix} w'(x_1) \\ w'(x_3) \\ \vdots \\ w'(x_{2m-1}) \end{pmatrix} \tag{5.1}$$

and Dirichlet boundary conditions at the vertices $v_j = \{x_{2j}\}$

$$w(x_{2j}) = 0, \quad j = 1, 2, \ldots, m; \tag{5.2}$$

- *$A = L + q$ is a Schrödinger operator in $L_2(\Gamma)$ with a certain real potential $q \in L_2(\Gamma)$;*
- *the matrix S^1 parameterizing the matching conditions (5.1) is irreducible as well as the limiting scattering matrix S^∞ defined by (3.9);*

[4] The corresponding control functions vanish for $t < \ell$.

- *$M(\lambda)$ and R^T are the TW-function and the dynamical response operator associated with the graph boundary $\partial\Gamma = \{x_2, x_4, \dots, x_{2m}\}$.*

Then the reduced spectral data consisting either of the principal $(m-1)\times(m-1)$ block of the TW-function or of the response operator R^T , $T \geq 2\max_{j=1,2,\dots,m-1}\{\ell_j+\ell_m\}$, where $\ell_j = x_{2j} - x_{2j-1}$, determine the Schrödinger operator A, more precisely the potential q and the matrix S^1 from the matching conditions at the central vertex, up to the similarity transformation

$$A \mapsto \rho_{-\theta} A \rho_\theta, \tag{5.3}$$

where ρ_θ is the unitary operator of multiplication by the function

$$\rho_\theta = \left\{ \begin{array}{ll} 1, & x \in \Delta_j,\ j = 1, 2, \dots, m-1; \\ e^{i\theta}, & x \in \Delta_m; \end{array} \right. \qquad \theta \in [0, 2\pi)$$

All possible operators A corresponding to the same reduced spectral data can be parameterized as follows

$$A_\theta = \rho_{-\theta} A \rho_\theta,\ \ \theta \in [0, 2\pi) \tag{5.4}$$

where A is any particular member of the family.

In other words, the reduced spectral data determine the unique real potential q and the matching conditions at the central vertex up to the similarity transformation

$$S^1 \mapsto \mathcal{R}_\theta S^1 \mathcal{R}_{-\theta},\ \ \theta \in [0, 2\pi).$$

Proof. The proof follows the main lines of proof of Theorem 4.1. The diagonal part of the principal block of the dynamical response operator R^T determines the potential on the edges $\Delta_j, j = 1, 2, \dots, m-1$.

Together with the graph Γ let us consider another star graph $\tilde{\Gamma}$ formed by $m-1$ edges all having the same length ℓ and the edge Δ_m. Consider the Schrödinger operator $\tilde{A}$ in $L_2(\tilde{\Gamma})$ determined by the same matching conditions at the central vertex and the potential

$$\tilde{q}(x) = \left\{ \begin{array}{ll} 0, & x \notin \Delta_m, \\ q_m(x), & x \in \Delta_m. \end{array} \right.$$

The corresponding TW-function and the dynamical response operator will be denoted by $\tilde{M}(\lambda)$ and $\tilde{R}^T$ respectively.

Let us show how to calculate the principal block of $\tilde{M}(\lambda)$ from the principal block of $M(\lambda)$. This transformation is similar to one used in the proof of Theorem 4.1. Consider any vector $\vec{v}^{m-1}$ and the corresponding solutions u to the equation

$$-u'' + q(x)u = \lambda u,\ \ x \in \Delta_j,\ j = 1, 2, \dots, m,$$

to matching conditions (2.2) and boundary conditions

$$u(x_{2j}) = v_j,\ u(x_{2m}) = 0.$$

The derivatives of u at the points $x_{2j}, j = 1, 2, \dots, m-1$ can be calculated

$$P_m^\perp \begin{pmatrix} u'(x_2) \\ u'(x_4) \\ \vdots \\ u'(x_{2m-2}) \\ 0 \end{pmatrix} = -P_m^\perp M(\lambda) P_m^\perp \vec{v}^{m-1}. \tag{5.5}$$

The knowledge of $u(x_{2j}), u'(x_{2j})$ and of the corresponding potential $q(x),\ x \in \Delta_j$ allows one to calculate $u(x_{2j-1}), u'(x_{2j-1}),\ j = 1, 2, \ldots, m-1$. Consider now solutions $\tilde{u}_j$ to the Schrödinger equation with zero potential,

$$-\tilde{u}_j'' = \lambda \tilde{u}_j, \ \ x \in [0, \ell],$$

satisfying the just calculated initial conditions

$$\begin{cases} \tilde{u}_j(0) = u(x_{2j-1}), \\ \tilde{u}_j'(0) = u'(x_{2j-1}). \end{cases}$$

We introduce the notations:

$$\tilde{v}_j = \tilde{u}_j(\ell), \tilde{v}_m = 0, \ \ \partial_n \tilde{v}_j = -\tilde{u}_j'(\ell), \partial_n \tilde{v}_m = 0.$$

The vectors $\vec{\tilde{v}}^{m-1}$ span the $m-1$ dimensional subspace of $\mathbb{C}^m$ up and the relation

$$\partial_n \tilde{v}^{m-1} = P_m^\perp \tilde{M}(\lambda) P_m^\perp \vec{\tilde{v}}^{m-1}$$

determines $P_m^\perp \tilde{M}(\lambda) P_m^\perp$.

As the proof of Theorem 4.2 shows the principal block $P_m^\perp \tilde{M}(\lambda) P_m^\perp$ determines the matching conditions at the central vertex (up to the similarity transformation) and the TW-function $M_m(\lambda)$ associated with the Schrödinger operator in $L_2(\Delta_m)$.

Summing up we were able to reconstruct the real potential q and the matching conditions (5.1) up to the similarity transformation (2.14). □

Finally we would like to underline that our approach based on the Boundary Control Method allows one not only to prove the uniqueness theorems, but to develop an effective algorithm enabling one to reconstruct the tree, potential and matching conditions. This algorithm has a local character. The method can also be used to recover the geometric graph, *i.e.* the lengths of all adges, but under certain additional assumptions on the matrix S^1.

The results proven in the current article will be used to solve the inverse problem for quantum trees in its full generality, *i.e.* it will be shown how to reconstruct the tree, the potential and the matching conditions at all internal vertices. This programme will be carried out in one of our future publications.

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