Few-Body Systems 0, 1–4 (2009)

Few-Body Systems

How to model p-scattering using point interactions and related problems

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Abstract. A new type of point interactions for the Laplacian in $\mathbb{R}^3$ is constructed generalizing classical Fermi pseudopotential. This model leads to a new resolvent formula and a non-trivial scattering matrix in p-channel.

1 Introduction

It is well-known that the celebrated Fermi delta potential [2, 3] leads to non-trivial scattering in the s-channel only. We propose a new family of point interaction models which may be used to describe particles with non-trivial interaction also in the p-channel while preserving exact solvability and point character of the interaction [4]. These models are given by self-adjoint operators and their spectral and scattering properties are discussed. Similar physical models have been discussed during the conference (see contributions by J. Macek, S. Wycech and others). The developed method can be also applied to model the system of three quantum particles. One may expect that the corresponding operator is semibounded (in contrast to the Landau Hamiltonian studied by Skorniakov-Ter-Martirosyan and Minlos-Faddeev in the sixties [5]).

2 Fermi-Berezin-Faddeev point interaction

The stationary Schrödinger operator with delta potential is formally defined by

$$-\Delta + \alpha\delta \equiv -\Delta + \alpha\delta\langle\delta, \cdot\rangle. \tag{1}$$

F.A. Berezin and L.D. Faddeev [2] interpreted this operator as the Laplace operator L_θ defined on the domain of functions from the Sobolev space $W_2^2(\mathbb{R}^3 \setminus \{0\})$[1] possessing the asymptotic representation $U(x) = \frac{u_-}{4\pi|\mathbf{x}|} + u_0 + o(1)$, as $\mathbb{R}^3 \ni \mathbf{x} \to 0$

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[1]The Sobolev space W_2^s is best characterized using Fourier transform as $(1+|\mathbf{p}|)^s\hat{f}(\mathbf{p}) \in L_2(\mathbb{R}^3)$.

and the boundary conditions $u_0 = \cot\theta \; u_-$, $\theta \in [0, \pi)$. In general there is no relation between the parameters α and θ, but it may be established using homogeneity requirements [1].

The operator L_θ can also be seen as the differential Laplace operator defined on the set of functions satisfying the representation $U = U_r + u_1 \frac{e^{-\beta|x|}}{4\pi|x|}$, where $U_r \in W_2^2(\mathbb{R}^3)$, $u_1 \in \mathbb{C}$, $\beta > 0$, and the boundary condition $U_r(0) = (\cot\theta + \frac{\beta}{4\pi})u_1$. The parameters β and θ determining L_θ in this representation are not independent.

The operator L_θ is self-adjoint in $L^2(\mathbb{R}^3)$, its absolutely continuous spectrum is $[0, \infty)$ and it has a unique eigenvalue $E_0 = -(4\pi \cot\theta)^2$, provided $\cot\theta < 0$. The corresponding bound state eigenfunction is spherically symmetric and the scattering amplitude does not depend on the angle between the incoming and outgoing waves.

3 Why p-type point interactions are impossible in $L_2(\mathbb{R}^3)$?

Several attempts to define higher order point interactions lead to operators in Pontryagin spaces (with indefinite metrics), making these models not very attractive for physical applications.[2] The impossibility to define such interactions in the original Hilbert space follows from the fact that all self-adjoint extensions of the operator $-\Delta|_{C_0^\infty(\mathbb{R}^3 \setminus \{0\})}$ coincide with the family L_θ. It is expected that the operator with a high order interaction at the origin is defined on the functions possessing the representation $U_r(x) + u_1 g_1$, where g_1 is a certain solution of the Helmholtz equation having singularity at the origin. Every such solution different from $e^{-\beta|\mathbf{x}|}/4\pi|\mathbf{x}|$ has a non square integrable singularity. The corresponding boundary condition should contain derivatives of U_r at the origin, which are properly defined only if U_r belongs to the Sobolev space W_2^s with $s > 2$. For example the first derivatives are defined if U_r belongs to W_2^3, which is precisely the domain of the Laplacian considered as an operator in W_2^1 instead of L_2.[3]

4 Cascade model for p-scattering

A mathematically rigorous interpretation for the formal operator

$$-\Delta + \sum_{i=1}^{3} \alpha \partial_{x_i} \delta \langle \partial_{x_i} \delta, \cdot \rangle, \qquad \alpha \in \mathbb{R} \tag{2}$$

can be given in the following way.[4] Consider the following three singular solutions to the Helmholtz equation $(-\Delta + \beta_1^2) g_j = 0$, $\mathbf{x} \neq 0$, where $\beta_1 > 0$

$$g_j = \frac{\partial}{\partial x_j} \frac{e^{-\beta_1|\mathbf{x}|}}{4\pi|\mathbf{x}|} = -\frac{\beta_1|\mathbf{x}| + 1}{4\pi|\mathbf{x}|^3} e^{-\beta_1|\mathbf{x}|} x_j \notin L_2(\mathbb{R}^3), \quad g_j = (-\Delta + \beta_1^2)^{-1} \partial_{x_j} \delta$$

[2] See in particular papers by Yu. Shondin, A. Tip, J.F. van Diejen, A. Dijksma, H. Langer and C.G. Zeinstra.

[3] This fact is usually known as *Sobolev embedding theorem.*

[4] The first model of this type was developed using abstract mathematical language by K. Watanabe and the author when they tried to analyse an article by I. Andronov. It was realized later that this model leads to a natural generalization of von Neumann extension theory [4].

and the Hilbert space

$$\mathbb{H} = W_2^1(\mathbb{R}^3) \dot{+} \mathcal{L}\{g_1, g_2, g_3\} \ni \mathbb{U} = U + \sum_{i=1}^{3} u_1^i g_i = U - \frac{\beta_1|\mathbf{x}| + 1}{4\pi|\mathbf{x}|^3} e^{-\beta_1|\mathbf{x}|} \mathbf{x}^t \cdot \mathbf{u}_1,$$

with the norm $||\mathbb{U}||_{\mathbb{H}}^2 = ||\sqrt{-\Delta + \beta_1^2} U||_{L_2}^2 + \gamma ||\mathbf{u}_1||^2$, where $\gamma > 0$. The operator associated with (2) in $\mathbb{H}$ is defined as a restriction of the operator $\mathbb{A}$ acting as the differential Laplace operator outside the origin $\mathbb{A}\mathbb{U} = -(\mathbb{U}_{x_1x_1} + \mathbb{U}_{x_2x_2} + \mathbb{U}_{x_3x_3}), \ \ \mathbf{x} \neq 0$. Consider another positive parameter $\beta \neq \beta_1$ and introduce

$$G_i = \frac{1}{\beta_1^2 - \beta^2} (g_i(-\beta^2) - g_i(-\beta_1^2)), \quad G_i = (-\Delta + \beta^2)^{-1} g_i.$$

Then the family of self-adjoint in $\mathbb{H}$ operator $\mathbb{A}_\theta$, $\theta \in [0, \pi)$ is defined on the functions possessing the representation

$$\mathbb{U} = U_r + \sum_{i=1}^{3} u^i G_i + \sum_{i=1}^{3} u_1^i g_i, \quad U_r \in W_2^3(\mathbb{R}^3), \mathbf{u}, \mathbf{u}_1 \in \mathbb{C}^3,$$

and the boundary conditions $\sin\theta \, (\nabla U_r(0) + \gamma \mathbf{u}_1) = \cos\theta \, \mathbf{u}$, by the formula

$$\mathbb{A}_\theta \left(U_r + \sum_{i=1}^{3} u^i G_i + \sum_{i=1}^{3} u_1^i g_i \right) = -\Delta U_r - \beta^2 \sum_{i=1}^{3} u^i G_i + \sum_{i=1}^{3} (u^i - \beta_1^2 u_1^i) g_i.$$

The self-adjoint operator $\mathbb{A}_\theta$ - rigorous interpretation for the formal operator (2), - is defined on the functions forming cascade of less and less singular elements. It is described by four real parameters $\beta, \beta_1, \gamma, \theta$ (not all independent).

5 Properties of the cascade model

The operator $\mathbb{A}_\theta$ is self-adjoint in the Hilbert space $\mathbb{H}$, and describes a certain point interaction, which is not of s-type as Berezin-Faddeev one. The operator commutes with the rotations around the origin and reflections in planes passing through the origin, *i.e.* this constructed point interaction is spherically symmetric. The spectrum has an absolutely continuous branch $[0, \infty)$ and negative eigenvalues having multiplicity three. The spectral properties of the operator are encoded in the following rational Nevanlinna function

$$Q(\lambda) = \frac{1}{12\pi} \left\{ ik + \frac{\beta_1^2}{ik - \beta_1} + \beta + \frac{\beta_1^2}{\beta + \beta_1} \right\} + \frac{\gamma}{-\beta_1^2 - k^2}, \quad k\sqrt{\lambda}.$$

Continuous spectrum (generalized) eigenfunctions are

$$\mathbb{V}(\lambda, \mathbf{k}/k, \mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{i}{(k^2 + \beta_1^2)(Q(k^2) + \cot\theta)} \frac{ik|\mathbf{x}| - 1}{4\pi|\mathbf{x}|^3} e^{ik|\mathbf{x}|} \mathbf{x}^t \cdot \mathbf{k},$$

and we see that the scattering amplitude $\frac{-\lambda \cos(\widehat{\mathbf{x}, \mathbf{k}})}{4\pi(\lambda + \beta_1^2)(Q(\lambda) + \cot\theta)}$ depends on the angle between the incoming and outgoing waves. Hence the scattering matrix is non-trivial in the p-channel.

The bound state eigenfunctions are

$$\mathbb{V}_{\lambda_0} = -\frac{\chi|\mathbf{x}|+1}{4\pi|\mathbf{x}|^3} e^{-\chi|\mathbf{x}|}\mathbf{x}\cdot\mathbf{a}, \quad \lambda_0 = -\chi^2,$$

where $\chi > 0$ is a solution to the equation $Q(-\chi^2) + \cot\theta = 0$.

The function Q appears also in the denominator of the resolvent $(\mathbb{A}-\lambda)^{-1}$ in $\mathbb{H}$. The restriction of the resolvent to the infinite dimensional subspace $W_2^1(\mathbb{R}^3) \subset \mathbb{H}$ has the form which reminds of classical Krein's resolvent formula[5]

$$\begin{aligned}(\mathbb{A}_\theta-\lambda)^{-1}U &= \frac{1}{-\Delta-\lambda}U \\ &-\frac{1}{(\lambda+\beta_1^2)(Q(\lambda)+\cot\theta)}\left(\int_{\mathbb{R}^3}\frac{(ik|\mathbf{y}|-1)e^{ik|\mathbf{y}|}}{4\pi|\mathbf{y}|^3}\mathbf{y}^t U(\mathbf{y})d^3\mathbf{y}\right)\frac{(ik|\mathbf{x}|-1)e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|^3}\mathbf{x}.\end{aligned} \tag{3}$$

Note that the function appearing in the denominator is not any longer a Nevanlinna function as it is growing like $\lambda^{3/2}$, $\lambda\to\infty$.

6 Perspectives

The suggested model can be generalized to include even higher order point interactions. Analytic properties of these operators and new families of eigenfunction expansions based on the resolvent formula (3) were discussed by A. Luger and the author.

Acknowledgement. The author would like to thank the organizers for putting together an extremely stimulating conference, which allowed to develop new ideas and find listeners to the old ones.

References

1. S. Albeverio and P. Kurasov, Singular Perturbations of Differentail Operators, Cambridge Univ. Press, 2001.

2. F.A.Berezin, L.D.Faddeev, Remark on the Schrödinger equation with singular potential. (Russian) *Dokl. Akad. Nauk SSSR*, **137** (1961), 1011–1014.

3. E. Fermi, Sul moto dei neutroni nelle sostanze idrogenate, *Ricerca Scientifica*, **7** (1936), 13–52 (In Italian.), English translation in E.Fermi, Collected papers, vol. I, Italy 1921–1938, Univ. of Chicago Press, Chicago, 1962, pp. 980–1016.

4. P. Kurasov, Triplet extensions I: semibounded operators in the scale of Hilbert spaces, accepted for publication in *J. d'Analyse Mathématique.*

5. R.A. Minlos and L.D. Faddeev, Comment on the problem of three particles with point interactions, *Soviet Physics JETP*, **14** (1962), 1315–1316.

[5]Note that we consider here not the bordered resolvent, but just the restriction of $(\mathbb{A}-\lambda)^{-1}$ to the subspace $W_2^1 \subset \mathbb{H}$.