Journal of Nonlinear Mathematical Physics Volume 10, Supplement 1 (2003), 93–106 Öresund Symposium

Essential Spectrum Due to Singularity

Pavel KURASOV † *and Serguei NABOKO* ‡

† *Department of Mathematics, Lund Institute of Technology,*
Box 118, 221 00 Lund, Sweden
E-mail: kurasov@maths.lth.se

‡ *Department of Mathematical Physics, St. Petersburg University,*
198904 St. Petersburg, Russia
E-mail: naboko@snoopy.phys.spbu.ru

This paper is part of the
Proceedings of the **Öresund Symposium on Partial Differential Equations***;*
Lund University, May 23–25, 2002

Abstract

It is proven that the essential spectrum of any self-adjoint operator associated with the matrix differential expression

$$\begin{pmatrix} -\dfrac{d}{dx}\rho(x)\dfrac{d}{dx}+q(x) & \dfrac{d}{dx}\dfrac{\beta(x)}{x} \\ -\dfrac{\beta(x)}{x}\dfrac{d}{dx} & \dfrac{m(x)}{x^2} \end{pmatrix}.$$

consists of two branches. One of these branches (called regularity spectrum) can be obtained by approximating the operator by regular operators (with coefficients which are bounded near the origin), the second branch (called singularity spectrum) appears due to singularity of the coefficients.

1 Introduction

The aim of this short note is to explain important spectral phenomenon which can be observed for differential operators appearing in problems of magnetic hydrodynamics. This phenomenon are due to the nonperturbative behavior of the essential spectrum. It has been observed that the essential spectrum of a matrix differential operator on a finite interval can be different from the limit of the essential spectra of the differential operators determined by the same operator matrix on an increasing sequence of smaller intervals tending to the original interval. Trivial counterpart of this phenomena is well known for infinite intervals, since for example the essential spectrum of $-\frac{d^2}{dx^2}$ on a finite interval $[-a_n, b_n]$ is empty and therefore does not give the essential spectrum of $-\frac{d^2}{dx^2}$ on the whole line when $a_n, b_n \to \infty$. The phenomenon described in the current note is more sophisticated and is due to a rather complicated interplay between the components of the matrix differential operator allowing singularities at the boundary points of a very special form only. The aim of the current paper is to present an example of a matrix differential operator on a finite interval with nonperturbative behavior of the essential spectrum with respect to change of the interval. We develop algebraic technique which can be used further to analyze similar phenomena for matrix partial differential operators. The essential spectrum of this operator consists of two branches. One branch can be described as the limit of the essential spectra of the operators restricted to the sets of functions with the support on smaller intervals. The second branch is present for the final interval only and is described by the singularities of the coefficients at the boundary points of the interval. We call this branch the **singularity spectrum**, or the **essential spectrum due to singularity** in order to separate this part of the essential spectrum from the spectrum described by operator coefficients inside the interval – the **regularity spectrum**.

Similar phenomena in connection with problems of magnetohydrodynamics have been studied by V Adamyan, J Descloux, G Geymonat, G Grubb, T Kako, H Langer, A E Lifchitz, R Mennicken, M Möller, A Shkalikov, and others [1, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 21, 22, 24]. Problems of magnetohydrodynamics are described by 3×3 matrix differential operators [14, 12]. The existence of the new branch of the essential spectrum was predicted by J Descloux and G Geymonat [4] and proven rigorously for the first time by V Hardt, R Mennicken, and S Naboko [12]. In [7] another approach to the same problem has been suggested using quasiregularity conditions, which are used intensively in our approach. Connections between the new spectral branch and the zero set for the symbol of the asymptotic Hain–Lüst operator were observed in [23]. The extension theory for matrix non-singular differential operators was studied by many scientists, in particular by F S Rofe-Beketov, L B Zelenko and H de Snoo [27, 30, 29]. The main teem of this article is to study the corresponding problems with singularities and we concentrate our attention to the phenomenon described above.

The operator under investigation in the current paper is determined by a 2×2 operator matrix and represents the simplest differential operator possessing the described phenomenon. It is defined on the interval $[0, 1]$ and the matrix coefficients have singularities at the origin. The order of the singularities as well as the differential order of the matrix coefficients are chosen in such a way that they can cancel each other in the formal determinant of

$$L := \begin{pmatrix} -\frac{d}{dx}\rho(x)\frac{d}{dx} + q(x) & \frac{d}{dx}\frac{\beta(x)}{x} \\ -\frac{\beta(x)}{x}\frac{d}{dx} & \frac{m(x)}{x^2} \end{pmatrix}. \tag{1.1}$$

This operator resembles matrix differential operators appearing in the problems of magnetohydrodynamics [14, 12]. We describe first all self-adjoint operators associated with this formal expression in the space $\mathbf{H} = L_2[0, 1] \oplus L_2[0, 1]$. In this description the quasiregularity conditions (2.2) play a very important role. Under this condition the self-adjoint operators associated with the matrix (1.1) are described by boundary condition at the boundary point $x = 1$. Similar problem for block operators leading to Friedrichs extensions of so-called minimal operator was analyzed by A Konstantinov and R Mennicken [19]. The differential operator is in the limit point case near the boundary point $x = 0$. In Section 3 we present the unitary transformation mapping the interval $[0,1]$ onto $[0,\infty)$ which facilitates our studies of the singular point $x=0$, being in the limit point case under quasiregularity conditions. To establish the essential spectrum of the new operator (which coincides with the essential spectrum of the original operator) we calculate its resolvent on a certain subspace of finite co-dimension. The Hain–Lüst operator[1] is introduced for this purpose in Section 4. Since the essential spectrum is invariant under compact perturbations of the resolvent we replace the original resolvent by another operator of the same type so that the difference is a compact operator. We call this procedure **cleaning of the resolvent**. To carry out this program we use the asymptotic Hain–Lüst operator. In order to use the calculus of pseudo-differential operators, the operator is extended to the whole axis without changing the essential spectrum. The spectrum is calculated in Section 6. The two branches of the essential spectrum are presented explicitly and the relations with the zeroes of the symbol of the asymptotic Hain–Lüst operator are discussed.

The aim of the current note is to explain the spectral phenomenon just described without giving detailed proofs of all propositions to be found in our article [20].

2 The differential operator and quasiregularity conditions

Consider the minimal differential operator associated with the following 2×2 operator matrix (1.1), where the real valued functions $\rho(x)$, $q(x)$, $\beta(x)$, and $m(x)$ are continuously differentiable and are not equal to zero in the closed interval $[0,1]$

$$\rho,q,\beta,m\in C^2[0,1] \qquad \text{and} \qquad |\rho|,|\beta|,|m|\geq c>0. \tag{2.1}$$

These conditions on the coefficients are far from being optimal, but we are going to use these restrictions in order to make the presentation more explicit. Condition (2.1) imply in particular that matrix elements in (1.1) have non-removable singularities at the origin.

In addition we are going to require that the coefficients satisfy so-called quasiregularity conditions at the origin

$$\begin{aligned} &\rho m-\beta^2\big|_{x=0}=0,\\ &\frac{d}{dx}\left(\rho m-\beta^2\right)\big|_{x=0}=0. \end{aligned} \tag{2.2}$$

These conditions are necessary for the essential spectrum of any self-adjoint operator corresponding to (1.1) to be bounded. This will follow from formula (6.7) below.

By $\mathbf{L}_{\min}$ we denote the minimal operator determined by (1.1) in the Hilbert space $\mathbf{H}=L_2[0,1]\oplus L_2[0,1]$ with the domain $C_0^\infty(0,1)\oplus C_0^\infty(0,1)$. We keep the same notation for the closure of this operator. The operator $\mathbf{L}_{\min}$ is symmetric but not self-adjoint in $\mathbf{H}$.

The adjoint operator is determined by the same operator matrix (1.1) on the domain of functions satisfying the following five conditions

- $$U=(u_1,u_2)\in L_2[0,1]\oplus L_2[0,1]. \tag{2.3}$$

[1] This operator was introduced for the first time by K Hain and R Lüst in application to problems of magnetohydrodynamics [11].

- $$u_1 \in W_2^1(\epsilon, 1) \text{ for any } 0 < \epsilon < 1. \tag{2.4}$$

- The function
$$\omega_U(x) := -\rho(x)u_1'(x) + \frac{\beta(x)}{x}u_2(x) \tag{2.5}$$
is absolutely continuous on $[0, 1]$.

- $$\frac{d}{dx}\omega_U(x) = \frac{d}{dx}\left(-\rho(x)\frac{d}{dx}u_1 + \frac{\beta(x)}{x}u_2\right) \in L_2[0, 1]. \tag{2.6}$$

- $$-\frac{\beta(x)}{x}\frac{d}{dx}u_1 + \frac{m}{x^2}u_2 \in L_2[0, 1]. \tag{2.7}$$

The function ω_U is called **transformed derivative** and is well-defined for any function $U = (u_1, u_{2)})$, $u_1 \in W^1_{2,\text{loc}}(0, 1) \cap L_2[0, 1]$, $u_2 \in L_2[0, 1]$. The transformed derivative appearing in the boundary conditions for the matrix differential operator L plays the same rôle as the usual derivative for the standard one-dimensional Schrödinger operator. The function ω_U corresponding to $U \in \text{Dom}(L^*)$ belongs to $W_2^1(0, 1)$, since it is absolutely continuous and (2.6) holds.

Integrating by parts one obtains the following sesquilinear boundary form of the adjoint operator for arbitrary $U, V \in \text{Dom}\,(\mathbf{L}_{\min})$:

$$\begin{aligned} &\langle \mathbf{L}^*_{\min}U, V\rangle - \langle U, \mathbf{L}^*_{\min}V\rangle \\ &\quad = \lim_{\epsilon\searrow 0, \tau\nearrow 1} \left\{\omega_U(x)\overline{v}_1(x)|_{x=\epsilon}^{\tau} - u_1(x)\overline{\omega_V}(x)|_{x=\epsilon}^{\tau}\right\}. \end{aligned} \tag{2.8}$$

Using this boundary form we are able to calculate the deficiency indices of the minimal operator and describe its self-adjoint extensions.

Theorem 2.1. *Suppose that the quasiregularity conditions (2.2) are satisfied and the functions ρ, β, m do not vanish (i.e. the operator matrix is not regular). Then the operator $\mathbf{L}_{\min}$ is a symmetric operator in the Hilbert space $\mathcal{H}$ with the deficiency indices* $(1, 1)$ *and all self-adjoint extensions of $\mathbf{L}_{\min}$ are described by the standard boundary condition at point $x = 1$*

$$\omega_U(1) = h_1u_1(1), \qquad h_1 \in \mathbf{R} \cup \{\infty\}. \tag{2.9}$$

In the case $h = \infty$, the corresponding boundary condition should be written as $u_1(1) = 0$.

Proof. The point $x = 1$ is a regular boundary point, since the functions ρ^{-1}, $\frac{\beta}{x}$, $\frac{m}{x^2}$ are infinitely differentiable in a neighborhood of this point. The symmetric boundary condition at the point $x = 1$ can be written in the form

$$\omega_U(1) = h_1u_1(1), \tag{2.10}$$

where $h_1 \in \mathbf{R} \cup \infty$ is a real constant parametrizing all symmetric conditions. The extension of the operator $\mathbf{L}_{\min}$ to the set of infinitely differentiable functions with support separated from the origin and satisfying condition (2.10) at the point $x = 1$ will be denoted by $\mathbf{L}_{h_1}$.

Let us study the deficiency indices of the operator $\mathbf{L}_{h_1}$. The operator adjoint to $\mathbf{L}_{h_1}$ is the restriction of $\mathbf{L}^*_{\min}$ to the set of functions satisfying (2.10). This operator is defined by the operator matrix with real coefficients, therefore the deficiency indices of $\mathbf{L}_{h_1}$ are equal. Moreover the differential equation on the deficiency element g^λ for any $\lambda \notin \mathbf{R}$ [2] is given by

$$\begin{aligned} &\frac{d}{dx}\left(-\rho(x)\frac{d}{dx}g_1^\lambda + \frac{\beta(x)}{x}g_2^\lambda\right) + q(x)g_1^\lambda = \lambda g_1^\lambda, \\ &-\frac{\beta(x)}{x}\frac{d}{dx}g_1^\lambda + \frac{m(x)}{x^2}g_2^\lambda = \lambda g_2^\lambda; \end{aligned} \qquad (2.11)$$

and it can be reduced to the following scalar differential equation for the first component

$$-\frac{d}{dx}\left(\rho(x) + \frac{\beta(x)}{x}\frac{1}{\lambda - m(x)/x^2}\frac{\beta(x)}{x}\right)\frac{d}{dx}g_1^\lambda + q(x)g_1^\lambda = \lambda g_1^\lambda. \qquad (2.12)$$

The component g_2^λ can be calculated from g_1^λ using the formula

$$g_2^\lambda = -\frac{1}{\lambda - m(x)/x^2}\frac{\beta(x)}{x}\frac{d}{dx}g_1^\lambda.$$

Equation (2.12) is a second order ordinary differential equation with continuously differentiable coefficients. Since the principle coefficient in this equation for nonreal λ is separated from zero on the interval $(\epsilon, 1]$, the solutions are two times continuously differentiable functions [13].

Boundary condition (2.10) implies that the first component satisfies the boundary condition at point $x = 1$

$$-\left(\rho(1) + \frac{\beta^2(1)}{\lambda - m(1)}\right)\frac{d}{dx}g_1^\lambda(1) = h_1 g_1^\lambda(1). \qquad (2.13)$$

This condition is nondegenerate, since λ is nonreal. Therefore the subspace of solutions to equation (2.11) satisfying condition (2.10) has dimension 1. But these solutions do not necessarily belong to the Hilbert space $\mathcal{H} = L_2[0,1] \oplus L_2[0,1]$. If the nontrivial solution is from the Hilbert space, $g^\lambda \in \mathcal{H}$, then the operator $\mathbf{L}_{h_1}$ is symmetric with deficiency indices $(1,1)$. Otherwise the operator $\mathbf{L}_{h_1}$ is essentially self-adjoint [28]. If the principal coefficient of equation (2.12) is bounded and separated from zero on the interval $[0,1]$, then $g^\lambda \in \mathcal{H}$ and the operator $\mathbf{L}_{h_1}$ has deficiency indices $(1,1)$. The last condition is satisfied if for example $m(0) \neq 0$ and $\rho(0)m(0) - \beta^2(0) \neq 0$, since $\Im\lambda \neq 0$. Complete analysis of equation (2.12) can be carried out using WKB method [26]. We are going instead to analyze the boundary form.

Consider the vector function

$$E = \begin{pmatrix} -\int_x^1 \frac{\beta(t)}{t\rho(t)}dt \\ 1 \end{pmatrix},$$

which belongs to the domain of the adjoint operator $\mathbf{L}^*_{\min}$ due to quasiregularity conditions. Therefore

$$\varphi E \in \mathrm{Dom}\,(\mathbf{L}^*_{h_1}).$$

Then for any function $U \in \mathrm{Dom}\,(\mathbf{L}^*_{h_1})$ the boundary form is given by

$$\langle \mathbf{L}^*_{h_1} U, \varphi E \rangle - \langle U, \mathbf{L}^*_{h_1} \varphi E \rangle = - \lim_{\epsilon \searrow 0} \omega_U(\epsilon) \overline{e_1(\epsilon)},$$

since $\omega_E(\epsilon) \equiv 0$. Note that e_1 diverges to infinity due to our assumption $\beta(0) \neq 0$

$$v_1(\epsilon) \sim_{\epsilon \searrow 0} \frac{\beta(0)}{\rho(0)} \ln \epsilon \to \infty.$$

Since the limit $\lim_{\epsilon \searrow 0} \omega_U(\epsilon)$ exists it should be equal to zero

$$\omega_U(0) = 0. \tag{2.14}$$

Hence taking into account that $\omega_U \in W_2^1[0,1]$ one concludes that

$$\omega_U(\epsilon) = o\left(\sqrt{\epsilon}\right). \tag{2.15}$$

On the other hand condition (2.6) implies that

$$x \frac{d}{dx} u_1 = \frac{\beta}{\rho} u_2 - \frac{x}{\rho} \omega_U \in L_2[0,1]. \tag{2.16}$$

It follows from Cauchy inequality that

$$u_1(\epsilon) = O\left(\frac{1}{\sqrt{\epsilon}}\right). \tag{2.17}$$

Formulas (2.15) and (2.17) imply that the boundary form is identically equal to zero. Therefore the operator $L(h_1)$ is essentially self-adjoint in this case. ∎

Complete analysis of self-adjoint extensions of the operator $\mathbf{L}_{\min}$ including the case where the quasiregularity conditions are not satisfied can be found in [20].

3 Transformation of the operator

The operator $\mathbf{L}_{h_1}$ is essentially self-adjoint under the quasiregularity conditions, in other words the operator matrix is in the limit point case at the origin. Therefore in order to calculate the essential spectrum of this operator it is convenient to transform the finite interval $[0,1]$ onto the half-infinite interval $[0,\infty)$ for example using the following change of variables

$$\begin{aligned} & x = e^{-y}, \\ & dx = -e^{-y} dy = -x dy, \end{aligned} \tag{3.1}$$

and the corresponding unitary transformation between the spaces $L_2[0,1]$ and $L_2[0,\infty)$

$$\Phi : \psi(x) \mapsto \tilde{\psi}(y) = \psi(e^{-y})e^{-y/2}. \tag{3.2}$$

The transformed operator matrix is given by

$$\begin{aligned} K &= \begin{pmatrix} -\frac{d}{dy}\frac{\rho}{x^2}\frac{d}{dy} + \left(q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2}\right) & -\frac{d}{dy}\frac{\beta}{x^2} + \frac{\beta}{2x^2} \\ \frac{\beta}{x^2}\frac{d}{dy} + \frac{1}{2}\frac{\beta}{x^2} & \frac{m}{x^2} \end{pmatrix} \\ &:= \begin{pmatrix} A & C^* \\ C & D \end{pmatrix}. \end{aligned} \tag{3.3}$$

The unitary transformation maps $C_0^\infty(0,1)$ onto $C_0^\infty(0,\infty)$ and therefore the deficiency indices of the minimal operator $\mathbf{K}_{\min}$ defined by (3.3) on the domain $C_0^\infty(0,\infty) \oplus C_0^\infty(0,\infty)$ are equal to $(1,1)$ and all its self-adjoint extensions are described by one boundary condition at the origin. Let us denote by $\mathbf{K}$ any of these operators self-adjoint in the space $\mathbf{H} = L_2(0,\infty) \oplus L_2(0,\infty)$.

4 Resolvent equation and Hain–Lüst operator

It is sufficient to study the resolvent equation for the operator $\mathbf{K}_{\min}$, since the difference between any two self-adjoint extensions of this operator is a rank one operator and therefore the essential spectra of these two operators coincide. The resolvent equation for μ, $\Re\mu \neq 0$

$$(\mathbf{K}_{\min} - \mu)^{-1} F = U$$

can be written as follows

$$\begin{aligned} f_1 &= (A-\mu)u_1 + C^* u_2, \\ f_2 &= Cu_1 + (D-\mu)u_2. \end{aligned}$$

Using the fact that the operator $(D-\mu)$ is invertible for nonreal μ one can calculate u_2 from the second equation

$$u_2 = (D-\mu)^{-1} f_2 - (D-\mu)^{-1} C u_1$$

and substitute it into the first equation to get

$$f_1 = \left((A-\mu) - C^*(D-\mu)^{-1}C\right) u_1 + C^*(D-\mu)^{-1} f_2.$$

The last equation can easily be resolved using **Hain–Lüst operator**, which is analogous to the regularized determinant of the matrix K

$$\begin{aligned} T(\mu) &= (A-\mu I) - C^*(D-\mu I)^{-1}C \\ &= -\frac{d}{dy}\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}\right)\frac{d}{dy} - \mu \\ &\quad + \left\{ q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2} - \frac{\beta^2}{4x^2(m-\mu x^2)} - x\frac{d}{dx}\left(\frac{\beta^2}{2x^2(m-\mu x^2)}\right)\right\}. \end{aligned} \tag{4.1}$$

One can prove the following lemma.

Lemma 4.1. *Let $\mu \notin \mathrm{Range}_{x\in[0,1]}\left(\frac{m(x)}{x^2}\right)$, then the coefficients of the Hain–Lüst operator (4.1)*

$$f(x) = \frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)};$$

and

$$g(x) = q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2} - \frac{\beta^2}{4x^2(m-\mu x^2)} - x\frac{d}{dx}\left(\frac{\beta^2}{2x^2(m-\mu x^2)}\right) - \mu,$$

are uniformly bounded functions if and only if the quasiregularity conditions (2.2) hold.

The resolvent matrix can be presented by

$$\mathbf{M}(\mu) \equiv (\mathbf{K}_{\min} - \mu)^{-1} \tag{4.2}$$
$$= \begin{pmatrix} T^{-1}(\mu) & -T^{-1}(\mu)\left[C^*(D-\mu I)^{-1}\right] \\ -\left[(D-\mu I)^{-1}C\right]T^{-1}(\mu) & (D-\mu I)^{-1} + \left[(D-\mu I)^{-1}C\right]T^{-1}(\mu)\left[C^*(D-\mu I)^{-1}\right] \end{pmatrix}.$$

The last expression determines the resolvent of any self-adjoint extension $\mathbf{K}$ of the minimal operator $\mathbf{K}_{\min}$ on the subspace $\mathcal{R}(\mathbf{K}_{\min})$ which has finite codimension. Therefore this resolvent matrix determines the essential spectrum of any self-adjoint extension $\mathbf{K}$. In order to calculate the essential spectrum we are going to consider perturbations of the calculated resolvent by compact operators. This is discussed in the following section.

5 Asymptotic Hain–Lüst operator and cleaning of the resolvent

Let us introduce the **asymptotic Hain–Lüst operator** for the generic case $m(0) \neq 0$

$$T_{\mathrm{as}}(\mu) = a(\mu)\left(-\frac{d^2}{dy^2} + c(\mu)\right) \equiv a(\mu)\left(p^2 + c(\mu)\right), \tag{5.1}$$

where

$$\begin{aligned} a(\mu) &= \lim_{x\to 0}\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}\right) = l_0 - \mu\frac{\rho(0)}{m(0)}, \\ l_0 &= \lim_{x\to 0}\left(\frac{\rho - \frac{\beta^2}{m}}{x^2}\right), \\ c(\mu) &= \frac{1}{4} - \frac{\mu}{a(\mu)}. \end{aligned} \tag{5.2}$$

The domain of the asymptotic Hain–Lüst coincides with the set of functions from the Sobolev space W_2^2 satisfying the Dirichlet boundary condition at the origin:

$$\{\psi \in W_2^2([0,\infty)), \psi(0) = 0\}.$$

We obtain the asymptotic Hain–Lüst operator by substituting the coefficients of the second order differential Hain–Lüst operator by their limit values at the singular point. It will be shown that the additional branch of essential spectrum of $\mathbf{K}$ (and any $\mathbf{L}_{h_1}$) is determined exactly by the symbol of asymptotic Hain–Lüst operator.

One can prove using methods of [20] that the difference between the two operator valued Herglotz functions $-T^{-1}(\mu)$ and $-T_{\rm as}^{-1}(\mu)$ is a compact operator for sufficiently large $|\mu|$. It can be proven then that the difference between the resolvent matrix $\mathbf{M}(\mu)$ and the asymptotic matrix

$$\mathbf{M}_{\rm as}(\mu) = \tag{5.3}$$
$$\doteq \begin{pmatrix} \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} & -\frac{b(0,\mu)}{a(\mu)}\frac{ip+1/2}{p^2+c(\mu)} \\ -\frac{b(0,\mu)}{a(\mu)}\frac{-ip+1/2}{p^2+c(\mu)} & \frac{x^2}{m-\mu x^2}+\frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)}+\frac{b^2(x,\mu)}{\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}} \end{pmatrix},$$

is a compact operator. Therefore to calculate the essential spectrum of $\mathbf{K}$ it is enough to study the last resolvent matrix.

In Section 4 to handle pseudodifferential operators we discussed the idea to "extend" the operator $\mathbf{K}$ to a certain operator $\mathbb{K}$ acting in the Hilbert space $\mathbb{H} = L_2(\mathbf{R}) \oplus L_2(\mathbf{R}) \supset L_2[0,\infty) \oplus L_2[0,\infty)$, preserving the essential spectrum (if one does not count the multiplicity). This procedure can easily be carried out for the cleaned resolvent. Let us continue all involved functions $b(x(y),\mu)$, $\rho(x(y))$ and $m(x(y))$ to the whole real line as even functions of y. Consider the operator generated by the continuous matrix symbol

$$X(x(y)) + P(p).$$

This operator is bounded operator defined on the whole Hilbert space $\mathbb{H}$. The essential spectrum of the new operator coincides (without counting multiplicity) with the essential spectrum of the original operator $\mathbf{M}(\mu)$. Really Glasman's splitting procedure [2] and Weyl theorem on compact perturbations [18] imply that the essential spectrum of the new operator coincides with the union of the essential spectra of the two operators generated by the operator matrix on the two half-axes:

$$\frac{1}{p^2+c(\mu)}\Big|_{L_2(\mathbf{R})} \doteq \frac{1}{p^2+c(\mu)}\Big|_{L_2(-\infty,0]} \oplus \frac{1}{p^2+c(\mu)}\Big|_{L_2[0,\infty)},$$

where $\frac{1}{p^2+c(\mu)}|_{L_2(-\infty,0]}$ and $\frac{1}{p^2+c(\mu)}|_{L_2[0,\infty)}$ denote the resolvents of the Laplace operator p^2 on the corresponding semiaxis with the Dirichlet boundary condition at the origin. In the last formula p denotes the momentum operator in the left hand side and the differential expression in the right one.

One can easily prove that the unitary transformation

$$\begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} \mapsto \begin{pmatrix} f_1(-y) \\ -f_2(-y) \end{pmatrix}$$

relates the matrix operators generated in the orthogonal decomposition of the Hilbert space

$$\mathbb{H} = (L_2(-\infty,0] \oplus L_2(-\infty,0]) \oplus (L_2[0,\infty) \oplus L_2[0,\infty)).$$

Hence the two operators appearing in this orthogonal decomposition are unitary equivalent and therefore have the same essential spectrum.

6 The essential spectrum

To calculate the essential spectrum of the operator $\mathbb{K}$ we are going to use the following proposition from [20, Proposition 10.1]

Proposition 6.1. *Let* $\mathbf{M}$ *be any* $n \times n$ *matrix separable pseudodifferential operator generated in the Hilbert space* $L_2(\mathbf{R}, \mathbf{C}^n)$ *by the symbol*

$$M(y,p) = Q + Y(y) + P(p), \qquad p = \frac{1}{i}\frac{d}{dy}, \tag{6.1}$$

where Q *is a constant diagonalizable matrix with simple spectrum, and the matrix functions* $Y(y)$ *and* $P(p)$ *are essentially bounded and satisfy the following two asymptotic conditions*

$$\lim_{x\to\infty} Y(y) = 0, \qquad \lim_{p\to\infty} P(p) = 0. \tag{6.2}$$

Then the essential spectrum of the operator $\mathbf{M}$ *is given by*

$$\sigma_{\mathrm{ess}}(\mathbf{M}) = \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{Y}). \tag{6.3}$$

The matrix function $M_{\mathrm{as}}(\mu)$ can be written as a sum of the following three matrices

$$M_{\mathrm{as}}(\mu) = Q + Y(y) + P(p),$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\rho(0)}{m(0)a(\mu)} \end{pmatrix},$$

$$Y(y) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{x^2}{m-\mu x^2} + \dfrac{b^2(x,\mu)}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}} - \dfrac{\rho(0)}{m(0)a(\mu)} \end{pmatrix},$$

$$P(p) = \begin{pmatrix} \dfrac{1}{a(\mu)}\dfrac{1}{p^2+c(\mu)} & -\dfrac{b(0,\mu)}{a(\mu)}\dfrac{ip+1/2}{p^2+c(\mu)} \\ -\dfrac{b(0,\mu)}{a(\mu)}\dfrac{-ip+1/2}{p^2+c(\mu)} & \dfrac{b^2(0,\mu)}{a(\mu)}\dfrac{1/4-c(\mu)}{p^2+c(\mu)} \end{pmatrix}.$$

The matrices Q, $Y(y)$, $P(p)$ satisfy all necessary conditions of Proposition 6.1. Thus the essential spectrum is given by

$$\sigma_{\mathrm{ess}}(\mathbf{M}(\mu)) = \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{Y}).$$

The determinants of the two matrices are equal to zero and therefore one of their eigenvalues is always 0. Thus the essential spectra coincide with the range of the second (nontrivial) eiqenvalues when y resp. p runs over the whole real axis and can be calculated using the traces of $Q+Y$ and $Q+P$. The nontrivial eigenvalues coincide with the traces of the corresponding 2×2 matrices $Q+P(p)$ and $Q+Y(y)$. The trace of the matrix $M(\mu)$ is given by

$$\begin{aligned}\mathrm{Tr}\,(M(\mu)) &= \mathrm{Tr}\,(Y(y)) + \mathrm{Tr}\,(P(p)) - \mathrm{Tr}\,(Q)\\ &= \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} + \frac{x^2}{m-\mu x^2} + \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)} + \frac{b^2(x,\mu)}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}}.\end{aligned}$$

The last expression can be factorized into the sum of three factors

$$\begin{aligned}&\mathrm{Tr}\,(M(\mu)) = \varphi(x(y)) + \psi(p) - \mathrm{Tr}\,(Q),\\ &\mathrm{Tr}\,Q = \frac{\rho(0)}{m(0)a(\mu)},\end{aligned}$$

where the functions $\varphi(x(y))$ and $\psi(p)$ tend to zero as y resp. p tend to ∞. The factorization is unique and obvious

$$\begin{aligned}\varphi(x) &= \frac{x^2}{m-\mu x^2} + \frac{b^2(x,\mu)}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}},\\ \psi(p) &= \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} + \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)} + \frac{\rho(0)}{m(0)a(\mu)}.\end{aligned} \tag{6.4}$$

Proposition 6.1 implies that the essential spectrum of the resolvent operator is given by

$$\sigma_{\mathrm{ess}}(\mathbf{M}(\mu)) = \left(\mathrm{Range}(\varphi(x)) \cup \mathrm{Range}(\psi(x)) + \varphi(0)\right). \tag{6.5}$$

Straightforward calculations imply

$$\sigma_{\mathrm{ess}}(\mathbf{L}) = \mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\} \cup \left[\frac{l_0}{4+\frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right], \tag{6.6}$$

where l_0 is given by (5.2). The parameter μ disappears eventually as one can expect. This parameter is pure axillary.

We conclude that the essential spectrum of $\mathbf{L}$ consists of two parts having different origin. The so-called regularity spectrum [23]

$$\mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\} \tag{6.7}$$

is determined by all coefficients of the operator matrix on the whole interval $[0,1]$. This part of the spectrum coincides with the limit of the essential spectra of the truncated operators $\mathbf{L}(\epsilon)$

$$\mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\} = \overline{\cup_{\epsilon>0}\sigma_{\mathrm{ess}}(\mathbf{L}(\epsilon))}.$$

On the contrary the singularity spectrum

$$\left[\frac{l_0}{4+\frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right]$$

is due to the singularity of the operator matrix at the origin is depends on the behavior of the matrix coefficients at the origin only. This part of the essential spectrum is absent for all truncated operators $\mathbf{L}(\epsilon)$ and cannot be obtained by the limit procedure $\epsilon \to 0$. This fact explains the name singularity spectrum given in [23]. The appearance of this interval of the essential spectrum generated by the singularity was predicted by J Descloux and G Geymonat. Note that the end point $\frac{l_0}{\frac{\rho(0)}{m(0)}}$ of the singularity spectrum always belongs to the interval of regularity spectrum, since

$$\lim_{x\to 0} \frac{m - \frac{\beta^2}{\rho}}{x^2} = \frac{l_0}{\frac{\rho(0)}{m(0)}}.$$

Remark. Let us remind that the essential spectrum has been calculated provided $m(0) \neq 0$ and the quasiregularity conditions are satisfied. If $m(0) = 0$, the quasiregularity conditions imply that $\beta(0) = 0$ and hence $m'(0) = 0$. No singularity appears in the coefficients of the matrix L given by (1.1). Therefore the operator is regular and its essential spectrum equals to $\text{Range}_{x\in[0,1]} \left\{ \frac{m - \frac{\beta^2}{\rho}}{x^2} \right\}$ [3]. No singularity spectrum appears in this case.

References

[1] Adamyan V, Langer H, Mennicken R and Saurer J, Spectral Components of Self-Adjoint Block Operator Matrices with Unbounded Entries, *Math. Nachr.* **178** (1996), 43–80.

[2] Akhiezer N I and Glazman I M, Theory of Linear Operators in Hilbert Space, Pitman, Boston, 1981.

[3] Atkinson F V, Langer H, Mennicken R and Shkalikov A, The Essential Spectrum of Some Matrix Operators, *Math. Nachr.* **167** (1994), 5–20.

[4] Descloux J and Geymonat G, Sur le spectre essentiel d'un operateur relatif á la stabilité d'un plasma en géometrie toroidal, *C.R. Acad. Sci. Paris* **290** (1980), 795–797.

[5] Faierman M, Lifschitz A, Mennicken R and Möller M, On the Essential Spectrum of a Differentially Rotating Star, *ZAMM Z. Angew. Math. Mech.* **79** (1999), 739–755.

[6] Faierman M, Mennicken R and Möller M, The Essential Spectrum of a System of Singular Ordinary Differential Operators of Mixed Order. I. The General Problem and an Almost Regular Case, *Math. Nachr.* **208** (1999), 101–115.

[7] Faierman M, Mennicken R and Möller M, The Essential Spectrum of a System of Singular Ordinary Differential Operators of Mixed Order. II. The Generalization of Kako's Problem, *Math. Nachr.* **209** (2000), 55–81.

[8] Faierman M and Möller M, On the Essential Spectrum of a Differentially Rotating Star in the Axisymmetric Case, *Proc. Roy. Soc. Edinburgh Sect. A*, **130** (2000), 1–23.

[9] Goedbloed J P, Lecture Notes on Ideal Magnetohydrodynamics, Rijnhiuzen Report, Form-Instituut voor Plasmafysica, Nieuwegein, 1983, 83–145.

[10] Grubb G and Geymonat G, The Essential Spectrum of Elliptic Systems of Mixed Order, *Math. Anal.* **227** (1977), 247–276.

[11] Hain K and Lüst R, Zur Stabilität zylindersymmetrischer PLasmakonfigurationen mit Volumenströmmen, *Z. Naturforsch.* **A13** (1958), 936–940.

[12] Hardt V, Mennicken R and Naboko S, System of Singular Differential Operators of Mixed Order and Applications to 1-dimensional MHD Problems, *Math. Nachr.* **205** (1999), 19–68.

[13] Hartman P, Ordinary Differential Equations, John Wiley, New York, 1964.

[14] Kako T, On the Essential Spectrum of MHD Plasma in Toroidal Region, *Proc. Japan Acad. Ser. A Math. Sci.* **60** (1984), 53–56.

[15] Kako T, On the Absolutely Continuous Spectrum of MHD Plasma Confined in the Flat Torus, *Math. Methods Appl. Sci.*, **7** (1985), 432–442.

[16] Kako T, Essential Spectrum of Linearized Operator for MHD Plasma in Cylindrical Region, *Z. Angew. Math. Phys.* **38** (1987), 433–449.

[17] Kako T and Descloux J, Spectral Approximation for the Linearized MHD Operator in Cylindrical Region, *Japan J. Indust. Appl. Math.* **8** (1991), 221–244.

[18] Kato T, Perturbation Theory for Linear Operators, Second Edition, Springer, Berlin, 1976.

[19] Konstantinov A and Mennicken R, On the Friedrichs Extension of Some Block Operator Matrices, *Integral Equations Operator Theory* **42** (2002), 472–481.

[20] Kurasov P and Naboko S, On the Essential Spectrum of a Class of Singular Matrix Differential Operators. I. Quasiregularity Conditions and Essential Self-Adjointness, *Math. Phys. Anal. Geom.* **5** (2002), 243–286.

[21] Langer H and Möller M, The Essential Spectrum of a Non-Elliptic Boundary Value Problem, *Math. Nachr.* **178** (1996), 233–248.

[22] Lifchitz A E, Magnetohydrodynamics and Spectral Theory, Kluwer Acad. Publishers, Dordrecht, 1989.

[23] Mennicken R, Naboko S and Tretter Ch, Essential Spectrum of a System of Singular Differential Operators, *Proc. AMS*, to appear.

[24] Möller M, On the Essential Spectrum of a Class of Operators in Hilbert Space, *Math. Nachr.* **194** (1998), 185–196.

[25] Naimark M A, Linear Differential Operators, Ungar, New York, 1968.

[26] Olver F W, Introduction to Asymptotic and Special Functions, Academic Press, New York, 1974.

[27] Rofe-Beketov F S, Selfadjoint Extensions of Differential Operators in a Space of Vector-Valued Functions, *Dokl. Akad. Nauk SSSR* **184** (1969), 1034–1037 (in Russian).

[28] Reed M and Simon B, Methods of Modern Mathematical Physics, Vol. I–IV, Second Edition, Academic Press, 1984.

[29] de Snoo H, Regular Sturm–Liouville Problems whose Coefficients Depend Rationally on the Eigenvalue Parameter, *Math. Nachr.* **182** (1996), 99–126.

[30] Zelenko L B and Rofe-Beketov F S, The Limit Spectrum of Systems of First Order Differential Equations with Slowly Changing Coefficients, *Differencial'nye Uravnenija* **7** (1971), 1982–1991 (in Russian).