

# On integrability of many-body problems with point interactions

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ABSTRACT. A study of the integrability of one-dimensional quantum mechanical many-body systems with general point interactions and boundary conditions describing the interactions which can be independent or dependent on the spin states of the particles is presented. The corresponding Bethe ansatz solutions, bound states and scattering matrices are explicitly given. Hamilton operators corresponding to special spin dependent boundary conditions are discussed.

Exactly solvable models of a single quantum particle moving in a local singular potential concentrated at one or a discrete number of points have been extensively discussed in the literature, see e.g. [1, 2, 3] and references therein. In the one dimensional case, the local singular potential (contact interactions) at, say, the origin ( $x = 0$ ) can be characterized by the boundary conditions imposed on the wave function  $\varphi$  at  $x = 0$ . There are two classes of such boundary conditions: separated and nonseparated boundary conditions, corresponding to the cases where the perturbed operator is equal to the orthogonal sum of two self-adjoint operators in  $L_2(-\infty, 0]$  and  $L_2[0, \infty)$  and when this representation is impossible, respectively. The many-body problems with pairwise interactions given by such boundary conditions are generally not exactly solvable [4, 5, 6]. In the present paper we give a systematic description for integrable models of many-body systems with pairwise interactions given by such singular potentials for the case where the boundary conditions are independent as well as for the case where they are dependent on the spin states of the particles.

We first consider the case of spin independent boundary conditions. The family of point interactions for the one dimensional Schrödinger operator  $-\frac{d^2}{dx^2}$  can be described by unitary  $2 \times 2$  matrices via von Neumann formulas for self-adjoint extensions of symmetric operators, since the second derivative operator restricted to the domain  $C_0^\infty(\mathbf{R} \setminus \{0\})$  has deficiency indices  $(2, 2)$ . The nonseparated boundary conditions describing the self-adjoint extensions have the following form

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}_{0^+} = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}_{0^-}, \quad (1)$$

where

$$ad - bc = 1, \quad \theta, a, b, c, d \in \mathbb{R}. \quad (2)$$

$\varphi(x)$  is the scalar wave function of two spinless particles with relative coordinate  $x$ . (1) also describes two particles with spin  $s$  but without any spin coupling between

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the particles when they meet (i.e. for  $x = 0$ ), in this case  $\varphi$  represents any one of the components of the wave function. The values  $\theta = b = 0$ ,  $a = d = 1$  in (1) correspond to the case of a positive (resp. negative)  $\delta$ -function potential for  $c > 0$  (resp.  $c < 0$ ). For general  $a, b, c$  and  $d$ , the properties of the corresponding Hamiltonian systems have been studied in detail, see e.g. [7, 8, 9].

The separated boundary conditions are described by

$$\varphi'(0_+) = q^+ \varphi(0_+), \quad \varphi'(0_-) = q^- \varphi(0_-), \quad (3)$$

where  $q^\pm \in \mathbb{R} \cup \{\infty\}$ .  $q^+ = \infty$  or  $q^- = \infty$  correspond to Dirichlet boundary conditions and  $q^+ = 0$  or  $q^- = 0$  correspond to Neumann boundary conditions.

To study the integrability of one dimensional systems of  $N$ -identical particles with general contact interactions described by the boundary conditions (1) or (3) that are imposed on the relative coordinates of the particles, we first consider the case of two particles ( $N = 2$ ) with coordinates  $x_1, x_2$  and momenta  $k_1, k_2$  respectively. Each particle has  $n$ -‘spin’ states designated by  $s_1$  and  $s_2$ ,  $1 \leq s_i \leq n$ . For  $x_1 \neq x_2$ , these two particles are free. The wave functions  $\varphi$  are symmetric (resp. antisymmetric) with respect to the interchange  $(x_1, s_1) \leftrightarrow (x_2, s_2)$  for bosons (resp. fermions). In the region  $x_1 < x_2$ , from the Bethe ansatz the wave function is of the form,

$$\varphi = u_{12} e^{i(k_1 x_1 + k_2 x_2)} + u_{21} e^{i(k_2 x_1 + k_1 x_2)}, \quad (4)$$

where  $u_{12}$  and  $u_{21}$  are  $n^2 \times 1$  column matrices. In the region  $x_1 > x_2$ , the wave function has the form

$$\varphi = (P^{12} u_{12}) e^{i(k_1 x_2 + k_2 x_1)} + (P^{12} u_{21}) e^{i(k_2 x_2 + k_1 x_1)}, \quad (5)$$

where according to the symmetry or antisymmetry conditions,  $P^{12} = p^{12}$  for bosons and  $P^{12} = -p^{12}$  for fermions,  $p^{12}$  being the operator on the  $n^2 \times 1$  column that interchanges  $s_1 \leftrightarrow s_2$ .

Let  $k_{12} = (k_1 - k_2)/2$ . In the center of mass coordinate  $X = (x_1 + x_2)/2$  and the relative coordinate  $x = x_2 - x_1$ , we get, by substituting (4) and (5) into the boundary conditions at  $x = 0$ ,

$$u_{21} = Y_{21}^{12} u_{12}, \quad (6)$$

$$Y_{21}^{12} = \frac{2ie^{i\theta} k_{12} P^{12} + ik_{12}(a-d) + (k_{12})^2 b + c}{ik_{12}(a+d) + (k_{12})^2 b - c} \quad (7)$$

for boundary condition (1) and

$$Y_{21}^{12} = \frac{ik_{12} + q}{ik_{12} - q} \quad (8)$$

for boundary condition (3), where  $q \equiv q_+ = -q_- \in \mathbb{R} \cup \{\infty\}$ .

For  $N \geq 3$  and  $x_1 < x_2 < \dots < x_N$ , the wave function is given by

$$\begin{aligned} \psi &= u_{12\dots N} e^{i(k_1 x_1 + k_2 x_2 + \dots + k_N x_N)} + u_{21\dots N} e^{i(k_2 x_1 + k_1 x_2 + \dots + k_N x_N)} \\ &+ (N! - 2) \text{ other terms.} \end{aligned} \quad (9)$$

The columns  $u$  have  $n^N \times 1$  dimensions. The wave functions in the other regions are determined from (9) by the requirement of symmetry (for bosons) or antisymmetry (for fermions). Along any plane  $x_i = x_{i+1}$ ,  $i \in 1, 2, \dots, N-1$ , from similar considerations as above we have

$$\alpha_{\alpha_1 \alpha_2 \dots \alpha_i \alpha_{i+1} \dots \alpha_N} = Y_{\alpha_{i+1} \alpha_i}^{ii+1} u_{\alpha_1 \alpha_2 \dots \alpha_{i+1} \alpha_i \dots \alpha_N}, \quad (10)$$

where

$$Y_{\alpha_{i+1} \alpha_i}^{ii+1} = \frac{2ie^{i\theta} k_{\alpha_i \alpha_{i+1}} P^{ii+1} + ik_{\alpha_i \alpha_{i+1}}(a-d) + (k_{\alpha_i \alpha_{i+1}})^2 b + c}{ik_{\alpha_i \alpha_{i+1}}(a+d) + (k_{\alpha_i \alpha_{i+1}})^2 b - c} \quad (11)$$

for nonseparated boundary condition and

$$Y_{\alpha_{i+1} \alpha_i}^{ii+1} = \frac{ik_{\alpha_i \alpha_{i+1}} + q}{ik_{\alpha_i \alpha_{i+1}} - q} \quad (12)$$

for separated boundary condition. Here  $k_{\alpha_i \alpha_{i+1}} = (k_{\alpha_i} - k_{\alpha_{i+1}})/2$  play the role of spectral parameters.  $P^{ii+1} = p^{ii+1}$  for bosons and  $P^{ii+1} = -p^{ii+1}$  for fermions, with  $p^{ii+1}$  the operator on the  $n^N \times 1$  column that interchanges  $s_i \leftrightarrow s_{i+1}$ .

For consistency  $Y$  must satisfy the Yang-Baxter equation with spectral parameter [10, 13, 14, 15, 16], i.e.,

$$Y_{ij}^{m,m+1} Y_{kj}^{m+1,m+2} Y_{ki}^{m,m+1} = Y_{ki}^{m+1,m+2} Y_{kj}^{m,m+1} Y_{ij}^{m+1,m+2},$$

or

$$Y_{ij}^{mr} Y_{kj}^{rs} Y_{ki}^{mr} = Y_{ki}^{rs} Y_{kj}^{mr} Y_{ij}^{rs} \quad (13)$$

if  $m, r, s$  are all unequal, resp.

$$Y_{ij}^{mr} Y_{ji}^{mr} = 1, \quad Y_{ij}^{mr} Y_{kl}^{sq} = Y_{kl}^{sq} Y_{ij}^{mr} \quad (14)$$

if  $m, r, s, q$  are all unequal.

The operators  $Y$  given by (11) satisfy the relation (14) for all  $\theta, a, b, c, d$ . However the relations (13) are satisfied only when  $\theta = 0$ ,  $a = d$  and  $b = 0$ , that is, according to the constraint (2),  $\theta = 0$ ,  $a = d = \pm 1$ ,  $b = 0$ ,  $c$  arbitrary. The case  $a = d = 1$ ,  $\theta = b = 0$  corresponds to the usual  $\delta$ -function interactions, which has been investigated in [10, 11, 12]. The case  $a = d = -1$ ,  $\theta = b = 0$  is related to another singular interactions between any pair of particles (for  $a = d = -1$  and  $\theta = b = c = 0$  see [7, 8]), which is in fact unitarily equivalent to the  $\delta$ -interaction, under a non-smooth ‘‘kink type’’ gauge transformation  $U = \prod_{i>j} \text{sgn}(x_i - x_j)$ . Associated with the separated boundary condition, the operators  $Y$  given by (12) satisfy both the relations (13) and (14) for arbitrary  $q$ . Therefore with respect to  $N$ -particle (either boson or fermion) problems, there are two non-equivalent integrable one parameter families with contact interactions described respectively by one of the following conditions on the wave function along the plane  $x_i = x_j$  for any pair of particles with coordinates  $x_i$  and  $x_j$ ,

$$\varphi(0_+) = +\varphi(0_-), \quad \varphi'(0_+) = c\varphi(0_-) + \varphi'(0_-), \quad c \in \mathbb{R}; \quad (15)$$

$$\varphi'(0_+) = q\varphi(0_+), \quad \varphi'(0_-) = -q\varphi(0_-), \quad q \in \mathbb{R} \cup \{\infty\}. \quad (16)$$

The wave functions are given by (9) with the  $u$ 's determined by (10) and initial conditions. The operators  $Y$  in (10) are given respectively by

$$Y_{\alpha_{i+1}\alpha_i}^{ii+1} = \frac{i(k_{\alpha_i} - k_{\alpha_{i+1}})P^{ii+1} + c}{i(k_{\alpha_i} - k_{\alpha_{i+1}}) - c}; \quad (17)$$

and

$$Y_{\alpha_{i+1}\alpha_i}^{ii+1} = \frac{i(k_{\alpha_i} - k_{\alpha_{i+1}}) + 2q}{i(k_{\alpha_i} - k_{\alpha_{i+1}}) - 2q}. \quad (18)$$

When  $q < 0$ , there exist  $2^{N(N-1)/2}$  bound states for the case (18) of separated boundary conditions, with wavefunction

$$\psi_{N,\underline{\epsilon}} = u_{\underline{\epsilon}} \prod_{k>l} (\theta(x_k - x_l) + \epsilon_{kl}\theta(x_l - x_k)) e^{q \sum_{i>j} |x_i - x_j|} \quad (19)$$

and eigenvalue  $E = -q^2 N(N^2 - 1)/3$ , where  $u_{\underline{\epsilon}}$  is the spin wave function and  $\underline{\epsilon} \equiv \{\epsilon_{kl} : k > l\}$ ;  $\epsilon_{kl} = \pm 1$ , labels the  $2^{N(N-1)/2}$ -fold degeneracy.

We consider now the case of spin dependent boundary conditions. For a particle with spin  $s$ , the wave function has  $n = 2s + 1$  components. Therefore two particles with contact interactions have a general boundary condition described in the center of mass coordinate system by:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0+} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0-}, \quad (20)$$

where  $\psi$  and  $\psi'$  are  $n^2$ -dimensional column vectors,  $A, B, C$  and  $D$  are  $n^2 \times n^2$  matrices. The boundary condition (20) can include not only the usual contact interaction between the particles, but also a spin coupling of the two particles if the matrices  $A, B, C, D$  are not diagonal.

The matrices  $A, B, C$ , and  $D$  are subject to restrictions due to the required symmetry condition of the Schrödinger operator. For any  $u, v \in C^\infty(\mathbb{R} \setminus \{0\})$ ,  $\langle -\frac{d^2}{dx^2} u, v \rangle_{L_2(\mathbb{R}, \mathbb{C}^n)} - \langle u, -\frac{d^2}{dx^2} v \rangle_{L_2(\mathbb{R}, \mathbb{C}^n)} = 0$ , which, together with (20) imply

$$A^\dagger D - C^\dagger B = 1, \quad B^\dagger D = D^\dagger B, \quad A^\dagger C = C^\dagger A, \quad (21)$$

where  $\dagger$  stands for the conjugate and transpose. Obviously (1) is the special case of (20) for  $s = 0$ .

In the following we study quantum systems with contact interactions described by the boundary condition (20), in particular,  $N$ -body systems with  $\delta$ -interactions. We first consider two spin- $s$  particles with  $\delta$ -interactions. The Hamiltonian is then of the form

$$H = \left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) \mathbf{I}_2 + 2h\delta(x_1 - x_2), \quad (22)$$

where  $\mathbf{I}_2$  is the  $n^2 \times n^2$  identity matrix,  $h$  is an  $n^2 \times n^2$  Hermitian matrix. If the matrix  $h$  is proportional to the unit matrix  $\mathbf{I}_2$ , then  $H$  is reduced to the usual two-particle Hamiltonian with contact interactions but no spin coupling.

Let  $e_\alpha$ ,  $\alpha = 1, \dots, n$ , be the basis (column) vector with the  $\alpha$ -th component as 1 and the rest components 0. The wave function of the system (22) is of the form

$$\psi = \sum_{\alpha, \beta=1}^n \phi_{\alpha\beta}(x_1, x_2) e_\alpha \otimes e_\beta. \quad (23)$$

In the center of mass coordinate system, the operator (22) has the form

$$H = - \left( \frac{1}{2} \frac{\partial^2}{\partial X^2} + 2 \frac{\partial^2}{\partial x^2} \right) \mathbf{I}_2 + 2h\delta(x). \quad (24)$$

The functions  $\phi = \phi(x, X)$  from the domain of this operator satisfy the following boundary condition at  $x = 0$ ,

$$\phi'_{\alpha\beta}(0^+, X) - \phi'_{\alpha\beta}(0^-, X) = \sum_{\alpha, \beta=1}^n h_{\gamma\lambda, \alpha\beta} \phi_{\gamma\lambda}(0, X), \quad \phi_{\alpha\beta}(0^+, X) = \phi_{\alpha\beta}(0^-, X), \quad (25)$$

$\alpha, \beta = 1, \dots, n$ , where the indices of the matrix  $h$  are arranged as  $11, 12, \dots, 1n; 21, 22, \dots, 2n; \dots; n1, n2, \dots, nn$ . (25) is a special case of (20) for  $A = D = \mathbf{I}_2$ ,  $B = 0$  and  $C = h$ .  $h$  acts on the basis vector of particles 1 and 2 by  $he_\alpha \otimes e_\beta = \sum_{\gamma, \lambda=1}^n h_{\alpha\beta, \gamma\lambda} e_\gamma \otimes e_\lambda$ .

The wave functions are still of the forms (4) (resp. 5) in the region  $x_1 < x_2$  (resp.  $x_1 > x_2$ ). Substituting them into the boundary conditions (25), we get

$$\begin{cases} u_{12} + u_{21} = P^{12}(u_{12} + u_{21}), \\ ik_{12}(u_{21} - u_{12}) = hP^{12}(u_{12} + u_{21}) + ik_{12}P^{12}(u_{12} - u_{21}). \end{cases} \quad (26)$$

Eliminating the term  $P^{12}u_{12}$  from (26) we obtain the same relation as (6),  $u_{21} = Y_{21}^{12}u_{12}$ . Nevertheless the  $Y$  operator is given by

$$Y_{21}^{12} = [2ik_{12} - h]^{-1}[2ik_{12}P^{12} + h]. \quad (27)$$

For a system of  $N$  identical particles with  $\delta$ -interactions, the Hamiltonian is given by

$$H = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} \mathbf{I}_N + \sum_{i < j} h_{ij} \delta(x_i - x_j), \quad (28)$$

where  $\mathbf{I}_N$  is the  $n^N \times n^N$  identity matrix,  $h_{ij}$  is an operator acting on the  $i$ -th and  $j$ -th bases as  $h$  and the rest as identity, e.g.,  $h_{12} = h \otimes \mathbf{1}_3 \otimes \dots \mathbf{1}_N$ , with  $\mathbf{1}_i$  the  $n \times n$  identity matrix acting on the  $i$ -th basis. The wave function in a given region, say  $x_1 < x_2 < \dots < x_N$ , is of the form (9), with

$$u_{\alpha_1 \alpha_2 \dots \alpha_j \alpha_{j+1} \dots \alpha_N} = Y_{\alpha_{j+1} \alpha_j}^{jj+1} u_{\alpha_1 \alpha_2 \dots \alpha_{j+1} \alpha_j \dots \alpha_N} \quad (29)$$

and

$$Y_{\alpha_{j+1} \alpha_j}^{jj+1} = [2ik_{\alpha_j \alpha_{j+1}} - h_{jj+1}]^{-1} [2ik_{\alpha_j \alpha_{j+1}} P^{jj+1} + h_{jj+1}]. \quad (30)$$

From the Yang-Baxter equations it is straightforward to show that the operator  $Y$  given by (30) satisfies all the Yang-Baxter relations if

$$[h_{ij}, P^{ij}] = 0. \quad (31)$$

Therefore if the Hamiltonian operators for the spin coupling commute with the spin permutation operator, the  $N$ -body quantum system (28) can be exactly solved. The

wave function is then given by (9) and (29) with the energy  $E = \sum_{i=1}^N k_i^2$ .

For the case of spin- $\frac{1}{2}$ , a Hermitian matrix satisfying (31) is generally of the form

$$h^{\frac{1}{2}} = \begin{pmatrix} a & e_1 & e_1 & c \\ e_1^* & f & g & e_2 \\ e_1^* & g & f & e_2 \\ c^* & e_2^* & e_2^* & b \end{pmatrix}, \quad (32)$$

where  $a, b, c, f, e_1, e_2 \in \mathbb{C}$ ,  $g \in \mathbb{R}$ . We recall that for a complex vector space  $V$ , a matrix  $R$  taking values in  $\text{End}_c(V \otimes V)$  is called a solution of the Yang-Baxter equation without spectral parameters, if it satisfies  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ , where  $R_{ij}$  denotes the matrix on the complex vector space  $V \otimes V \otimes V$ , acting as  $R$  on the  $i$ -th and the  $j$ -th components and as identity on the other components. When  $V$  is a two dimensional complex space, the solutions of the Yang-Baxter equation include the ones such as  $R_q$  which gives rise to the quantum algebra  $SU_q(2)$  and the integrable Heisenberg spin- $\frac{1}{2}$  chain models such as the XXZ model ( $R$  corresponds to the spin coupling operator between the nearest neighbor spins in Heisenberg spin chain models)[**13**, **14**, **15**, **16**]. Nevertheless in general  $h^{\frac{1}{2}}$  does not satisfy the Yang-Baxter equation without spectral parameters:  $h_{12}^{\frac{1}{2}}h_{13}^{\frac{1}{2}}h_{23}^{\frac{1}{2}} \neq h_{23}^{\frac{1}{2}}h_{13}^{\frac{1}{2}}h_{12}^{\frac{1}{2}}$ . But (32) includes the Yang-Baxter solutions, such as  $R_q$ , that gives integrable spin chain models (for an extensive investigation of the Yang-Baxter solutions see [**17**, **18**]). Therefore for an  $N$ -body system to be integrable, the spin coupling in the contact interaction (28) is allowed to be more general than the spin coupling in a Heisenberg spin chain model with nearest neighbors interactions.

For  $N = 2$ , from (26) the bound states have the form,

$$\psi_\alpha^2 = u_\alpha e^{\frac{c+a\Lambda_\alpha}{2}|x_2-x_1|}, \quad \alpha = 1, \dots, n^2, \quad (33)$$

where  $u_\alpha$  is the common  $\alpha$ -th eigenvector of  $h$  and  $P^{12}$ , with eigenvalue  $\Lambda_\alpha$ , s.t.  $hu_\alpha = \Lambda_\alpha u_\alpha$  and  $c + a\Lambda_\alpha < 0$ ,  $P^{12}u_\alpha = u_\alpha$ . The eigenvalue of the Hamiltonian  $H$  corresponding to the bound state (33) is  $-(c + a\Lambda_\alpha)^2/2$ . We remark that, whereas for the case of the boundary condition (1), for a  $\delta$  interaction one has a unique bound state, here we have  $n^2$  bound states. By generalization we get the bound state for the  $N$ -particle system,

$$\psi_\alpha^N = v_\alpha e^{-\frac{c+a\Lambda_\alpha}{2} \sum_{i>j} |x_i-x_j|}, \quad \alpha = 1, \dots, n^2, \quad (34)$$

where  $v_\alpha$  is the wave function of the spin part satisfying  $P^{ij}v_\alpha = v_\alpha$  and  $h_{ij}v_\alpha = \Lambda_\alpha v_\alpha$ , for any  $i \neq j$ .

It is worth mentioning that  $\psi_\alpha^N$  is of the form (9) in each of the above regions. For instance comparing  $\psi_\alpha^N$  with (9) in the region  $x_1 < x_2 \dots < x_N$  we get

$$k_1 = -i\frac{c+a\Lambda_\alpha}{2}(N-1), \quad k_2 = k_1 + ic, \quad k_3 = k_2 + ic, \dots, k_N = -k_1, \quad (35)$$

for  $\alpha = 1, \dots, n^2$ . The energy of the bound state  $\psi_\alpha^N$  is

$$E_\alpha = -\frac{(c + a\Lambda_\alpha)^2}{12}N(N^2 - 1). \quad (36)$$

Now we pass to the scattering matrix. For real  $k_1 < k_2 < \dots < k_N$ , in each coordinate region such as  $x_1 < x_2 < \dots < x_N$ , the following term in (9) describes an outgoing wave  $\psi_{out} = u_{12\dots N}e^{i(k_1x_1+\dots+k_Nx_N)}$ . An incoming wave with the same exponential as  $\psi_{out}$  is given by  $\psi_{in} = [P^{1N}P^{2(N-1)}\dots]u_{N(N-1)\dots 1}e^{i(k_Nx_N+\dots+k_1x_1)}$  in the region  $x_N < x_{N-1} < \dots < x_1$ . From (29) the scattering matrix  $S$  defined by  $\psi_{out} = S\psi_{in}$  is given by  $S = [X_{21}X_{31}\dots X_{N1}][X_{32}X_{42}\dots X_{N2}]\dots[X_{N(N-1)}]$ , where  $X_{ij} = Y_{ij}^{ij}P^{ij}$ .

The scattering matrix  $S$  is unitary and symmetric due to the time reversal invariance of the interactions.  $\langle s'_1s'_2\dots s'_N|S|s_1s_2\dots s_N \rangle$  stands for the  $S$  matrix element of the process from the state  $(k_1s_1, k_2s_2, \dots, k_Ns_N)$  to the state  $(k_1s'_1, k_2s'_2, \dots, k_Ns'_N)$ .

The scattering of clusters (bound states) can be discussed in a similar way as in [12]. For instance for the scattering of a bound state of two particles ( $x_1 < x_2$ ) on a bound state of three particles ( $x_3 < x_4 < x_5$ ), the scattering matrix is  $S = [X_{32}X_{42}X_{52}][X_{31}X_{41}X_{51}]$ .

The integrability of many particles systems with contact spin coupling interactions governed by separated boundary conditions can also be studied. Instead of (20) we need to deal with the case

$$\phi'(0_+) = G^+\phi(0_+), \quad \phi'(0_-) = G^-\phi(0_-), \quad (37)$$

where  $G^\pm$  are Hermitian matrices. For  $G^+ = G^- \equiv G$ ,  $G^\dagger = G$ , there is a Bethe Ansatz solution to (9) with  $Y_{\alpha_{i+1}\alpha_i}^{ii+1}$  in (29) given by

$$Y_{\alpha_{i+1}\alpha_i}^{ii+1} = \frac{ik_{\alpha_i\alpha_{i+1}} + G}{ik_{\alpha_i\alpha_{i+1}} - G}. \quad (38)$$

Let  $\Gamma$  be the set of  $n^2$  eigenvalues of  $G$ . For any  $\lambda_\alpha \in \Gamma$  such that  $\lambda_\alpha < 0$ , there are  $2^{N(N-1)/2}$  bound states for the  $N$ -particle system,

$$\psi_{\alpha_\epsilon}^N = v_{\alpha_\epsilon} \prod_{k>l} (\theta(x_k - x_l) + \epsilon_{kl}\theta(x_l - x_k)) e^{\lambda_\alpha \sum_{i>j} |x_i - x_j|}, \quad (39)$$

where  $v_{\alpha_\epsilon}$  is the spin wave function and  $\epsilon \equiv \{\epsilon_{kl} : k > l\}$ ;  $\epsilon_{kl} = \pm$ , labels the  $2^{N(N-1)/2}$ -fold degeneracy. The spin wave function  $v$  here satisfies  $P^{ij}v_{\alpha_\epsilon} = \epsilon_{ij}v_{\alpha_\epsilon}$  for any  $i \neq j$ , that is,  $p^{ij}v_{\alpha_\epsilon} = \epsilon_{ij}v_{\alpha_\epsilon}$  for bosons and  $p^{ij}v_{\alpha_\epsilon} = -\epsilon_{ij}v_{\alpha_\epsilon}$  for fermions.

Again  $\psi_{\alpha_\epsilon}^N$  is of the form (9) in each of the regions  $x_{i_1} < x_{i_2} < \dots < x_{i_N}$ . For instance comparing  $\psi_{\alpha_\epsilon}^N$  with (9) in the region  $x_1 < x_2 \dots < x_N$  we get  $k_1 = i\lambda_\alpha(N-1)$ ,  $k_2 = k_1 - 2i\lambda_\alpha$ ,  $k_3 = k_2 - 2i\lambda_\alpha, \dots, k_N = -k_1$ . The energy of the bound state  $\psi_{\alpha_\epsilon}^N$  is  $E_\alpha = -\lambda_\alpha^2 N(N^2 - 1)/3$ .

We have investigated the integrable models of  $N$ -body systems with contact spin coupling interactions. Without taking into account the spin coupling, the boundary condition (1) is characterized by four parameters (separated boundary conditions are a special limiting case of these). Obviously the general boundary condition (20) we considered in this article has much more parameters. A complete classification of the dynamic operators associated with different parameter regions remains to be done. As we have seen, the case  $A = D = \mathbf{I}_2$ ,  $B = 0$ ,  $C = h$  corresponds to a

Hamiltonian with  $\delta$ -interactions of the form (22) (for  $N = 2$ ). It can be further shown that (for  $N = 2$ ) the following boundary condition

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0+} = \begin{pmatrix} \mathbf{I} & B \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0-} \quad (40)$$

corresponds to a Hamiltonian  $H$  of the form:  $H = -D_x^2(1+B\delta) - BD_x\delta'$ , where  $B$  is an  $n^2 \times n^2$  Hermitian matrix,  $D_x$  is defined by  $(D_x f)(\varphi) = -f(\frac{d}{dx}\varphi)$ , for  $f \in C_0^\infty(\mathbb{R}/\{0\})$  and  $\varphi$  a test function with a possible discontinuity at the origin. The boundary condition

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0+} = \begin{pmatrix} \frac{2+iB}{2-iB} & 0 \\ 0 & \frac{2-iB}{2+iB} \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0-} \quad (41)$$

describes the Hamiltonian  $H = -D_x^2 + iB(2D_x\delta - \delta')$ .

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