Ronkin's and Crofton's formulae as I learnt from Mikael

Alain Yger, IMB, University of Bordeaux

Mikael Passare's Day
Stockholms universitet, September 15th, 2021

Ronkin's function and its companions

R_F and ρ_F : $F \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ (archimedean setting)

$$R_F(x) = \int_{\mathbb{T}^n_{\mathbb{R}}} \log |F(e^{x+i\theta})| \, d\nu_{\mathrm{haar}}(\theta), \quad \rho_F(x) = -R_F(-x) \qquad (x \in \mathbb{R}^n)$$

$R_{F,p}$ and $\rho_{F,p}$: $F \in \mathbb{Q}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ ($| \ |_p$, p prime, non-archimedean setting)

$$R_{F,p}(x) = \max_{\alpha \in \mathrm{Supp}(F)} (\log |c_\alpha|_p + \langle \alpha, x \rangle)$$

$$\rho_{F,p}(x) = \min_{\alpha \in \mathrm{Supp}(F)} (-\log |c_\alpha|_p + \langle \alpha, x \rangle) \qquad (x \in \mathbb{R}^n)$$

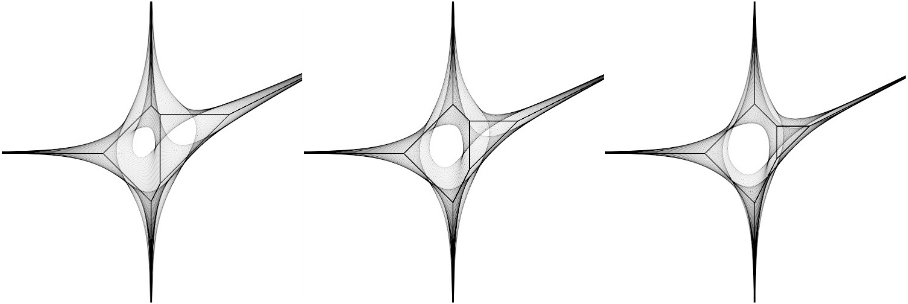
Skeletons of R_F (archimedean setting) : respective corner loci of

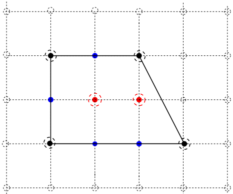
$$x \longmapsto \begin{cases} \Sigma_F(x) = \max_{\alpha \in \mathbb{Z}^n \cap \Delta(F)} (\langle \alpha, x \rangle - \check{R}_F(\alpha)) \\ S_F(x) = \max_{\{\alpha \in \mathbb{Z}^n \cap \Delta(F) \, ; \, \mathrm{int}((\langle \alpha, \cdot \rangle - \check{R}_F(\alpha))^{-1}(\{0\})) \neq \emptyset\}} (\langle \alpha \, , \, x \rangle - \check{R}_F(\alpha)) \end{cases}$$

$$(S_F \leq \Sigma_F \leq R_F)$$

Example: $F = a - X_1^3 + X_2^2 - X_1^2 X_2^2 + b\, X_1 X_2 + c\, X_1^2 X_2$

$x \in \mathbb{R}^2 \longmapsto R_F(x)$ (with corner locus of Σ_F) $(a = 1, b = 5, c = 3, 2, 1)$





Mixed integrals of concave functions (Philippon, Sombra)

$$\varphi : \mathbb{R}^n \to \{-\infty\} \cup \mathbb{R} \quad \text{concave}$$

Integral in $\mathbb{R}^n$ and its polarisation

$$\mathrm{MI}_{\mathbb{R}^n}\overbrace{(\varphi, \ldots, \varphi)}^{n+1 \text{ times}} = (n+1)! \int_{\varphi^{-1}(\mathbb{R})} \varphi(x)\, dx = (n+1)! \int_{\mathrm{Dom}(\varphi)} \varphi(x)\, dx$$

$$\mathrm{MI}_{\mathbb{R}^n}(\varphi_0, \ldots, \varphi_n) =$$

$$\sum_{k=0}^{n} (-1)^{n-k} \sum_{0 \le j_{i_0} < \cdots < j_k \le n} \int_{\mathrm{Dom}(\varphi_{j_0}) + \cdots + \mathrm{Dom}(\varphi_{j_k})} (\varphi_{j_0} \boxplus \cdots \boxplus \varphi_{j_k})(x)\, dx$$

$$(\varphi_1 \boxplus \varphi_2)(x) = \sup_{x_1 + x_2 = x} (\varphi_1(x_1) + \varphi_2(x_2)), \quad \varphi \cdot_r \lambda := \lambda \varphi(\cdot / \lambda)$$

Adelic collections of metrized hermitian bundles

geometric complexity (in $\mathbb{P}^n$) :

$$\deg_{\mathbb{A}^n_{\mathbb{C}}}(p^{-1}(\{0\})) = \deg_{\mathbb{P}^n_{\mathbb{C}}}(P^{-1}(\{0\}) = \int_{\mathbb{C}^n} [\mathrm{div}(P)] \wedge c_1^{n-1}(\mathcal{O}(1), |\ |)$$

arithmetic complexity (in $\mathbb{P}^n$) : $p \in \mathbb{Z}[X]$ (with GCD(coeff.) = 1)

$$\Longrightarrow h_{\overline{\mathcal{O}(1)},\dots,\overline{\mathcal{O}(1)}}^{\mathrm{canonical}}(p^{-1}(\{0\})) = h_{\overline{\mathcal{O}(1)},\dots,\overline{\mathcal{O}(1)}}^{\mathrm{canonical}}(P^{-1}(\{0\})) = \deg p \times \boxed{R_p(0)}$$

$$-dd^c \log |s|^2 + [\mathrm{div}(s)] = \boxed{c_1(L, |\ |)}$$

$\mathbb{X} = \mathbb{X}_\Sigma$ complete algebraic toric variety, attached to a complete $\mathbb{Z}^n$-rational fan in $\mathbb{R}^n$.

$$\mathbb{X} \longrightarrow X = \mathbb{X}_{\mathbb{C}} = (\mathbb{X})^{\mathrm{an}}_{v_\infty}, (\mathbb{X})^{\mathrm{an}}_{v_p} \text{ for all primes (Berkovich analytifications)}$$

L toric Cartier divisor on $\mathbb{X}_\Sigma$

$$\boxed{\begin{array}{c} |\ |_v \text{ on } L_v^{\mathrm{an}} \\ \text{toric, semi}-\text{positive}\end{array}} \longleftrightarrow \boxed{\psi_v : \mathbb{R}^n \longmapsto \{-\infty\} \cup \mathbb{R},\ \begin{cases}\text{concave} \\ |\psi_v - \Psi| = O(1)\end{cases}}$$

Here $\Psi = \Psi_L$ being the (concave) virtual support function of $\mathcal{O}(L)$, namely the piecewise linear rational concave function such that global sections of the line bundle L correspond to Laurent polynomials in the polyhedron with vertices in $\mathbb{Z}^n$ defined by $\mathrm{Stab}(\Psi) = \{u \in \mathbb{R}^n\,;\, \langle u, \cdot\rangle \geq \Psi\}$).

R. Gualdi's formula (2018) for $F \in \mathbb{Q}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$

Let $\mathbb{X}_\Sigma$ be complete toric variety attached to a complete rational fan in $\mathbb{R}^n$ which is compatible with the Newton polyhedron of F, $L_0, \dots, L_{n-1}$ toric divisors on $\mathbb{X}_\Sigma$, with an adelic prescribed choice of semi-positive toric metrics $|\ |_{j,v}$ respectively on each $L_{j,v}$.

$$\sum_{v \text{ place}} \mathrm{MI}_{\mathbb{R}^n}(\check{\psi}_v^0, \dots, \check{\psi}_v^{n-1}, \boxed{\check{\rho}_{F,v}}) = h_{(\bar{L}_{0,v})_v, \dots, (\bar{L}_{n-1,v})_v}(F^{-1}(\{0\}).$$

Arithmetic complexity and potential theory

Let Z be a purely dimensional k cycle on $\mathbb{X}$, defined over $\mathbb{Q}$. Let $L_0, \dots, L_k$ $k+1$ toric line metrized bundles which are metrized with semi-positive toric metrics in an adelic way.

Local logarithmic height through potential theory

For any place v of $\mathbb{Q}$

$$h_{\bar{L}_{0,v},\dots,\bar{L}_{k,v}}[s_0,\dots,s_k](Z) = \int_{\mathbb{X}_v^{\rm an}} [Z]\wedge G_v[\bar{L};s],$$

$$G_v = G_v[\bar{L};s] = -\sum_{\kappa=0}^{k} \log|s_{k-\kappa}| \bigwedge_{j<k-\kappa} c_1(\bar{L}_{j,v}) \wedge \bigwedge_{j>k-\kappa} [({\rm div}(s_j))_v^{\rm an}]$$

$$\boxed{dd^c G_{v_\infty} + [Z_{\mathbb{C}}] = \bigwedge_{j=0}^{k} c_1(\bar{L}_j)} \quad \text{or} \quad \boxed{d'_p d''_p G_{v_p} + [Z_{v_p}^{\rm an}] = \bigwedge_{j=0}^{k} c_1(\bar{L}_{j,v_p})}.$$

Sheaves of forms and currents on the Berkovich space $\mathbb{X}_{v_p}$ for p prime, as well as operators d'_p, d''_p, have been introduced by A. Ducros, A. Chambert-Loir and W. Gubler. Adding for all $v = v_p$ and v_∞ eliminates dependence in s.

Crofton's formula

Primitive version

Let $f_0, \dots, f_m$ be analytic functions with no common zero on a complex analytic space X. Then

$$[dd^c \log \|f\|^2] = \int_{\mathbb{P}^m_{\mathbb{C}}} [\mathrm{div}(\langle t, f \rangle)]\, c^m_{\mathrm{FS}}(t) = \int_{\mathbb{P}^m_{\mathbb{C}}} [\mathrm{div}(\langle t, f \rangle)] (dd^c \log \|t\|^2)^m$$

The projective setting

The concave function

$$x \longmapsto \frac{1}{2} \log \Big(1 + \sum_{k=1}^{n} e^{-2x_k} \Big)$$

is reproduced through the convolution formula

$$\psi(x) = \psi_{\mathrm{Croft}}(x) = n! \int_{\mathbb{R}^n} \rho_{1+x_1+\cdots+x_n}(x+y)\, d\mathrm{MA}_{\mathbb{R}}[\psi, \dots, \psi](y).$$

This leads naturally to the concept of croftonization of semi-positive metrics on toric divisors.

Crofton's formula and Stückrad-Vogel generalized cycles

Stückrad-Vogel cycles as direct images (after [ASWY] and [ASEWY])

Let $s_0, \ldots, s_m$ be global sections of an holomorphic line bundle L over a complex n - dimensional analytic space X. For any $r = 1, \ldots, n$, one has

$$1_{s^{-1}(\{0\})}\, (dd^c \log \|s\|_\circ^2)^r =$$

$$(\Pi_r)_* \Big(\boxed{1_{\Pi_r^{-1}(s^{-1}(\{0\})} \cdot \mathrm{div}(\langle \mathrm{t}_1, s(z)\rangle)\, \dot{\cap} \cdots \dot{\cap}\, \mathrm{div}(\langle \mathrm{t}_r, s(z)\rangle)} \wedge \bigwedge_{\rho=1}^{r} c_{\mathrm{FS}}^m(\mathrm{t}_\rho) \Big)$$

where

$$\Pi_r : (z, \mathrm{t}_1, \cdots, t_r) \in X \times (\mathbb{P}^m_{\mathbb{C}})^r \longrightarrow z \in X.$$

$$Y \,\dot{\cap} \Big(\sum_\iota \nu_\iota Z_\iota \Big) := \sum_{\{\iota\,;\, Z_\iota \not\subset Y\}} \nu_\iota \,(Y \cap_{\mathrm{proper}} Z_\iota)$$

If $\mathbb{X}$ is an algebraic variety defined over $\mathbb{Q}$, such is $\mathbb{X} \times (\mathbb{P}^m)^r$ and the intersection cycle involved within the box remains defined over $\mathbb{Q}$ when the line bundle and its sections s_j are.

From Jacobi-Kronecker-Lagrange to a parametric Hilbert's Nullstellensatz (revisiting C. Berenstein - A.Y (1991))

Setting the polynomial entries in normal position

Let $\mathbb{K} \subset \mathbb{C}$ and $\mathrm{p} = (p_0, ..., p_n) \in (\mathbb{K}[X_1, ..., X_n])^{n+1}$ with $\deg p_j \leq d$ for any j such that the algebraic hypersurfaces $p_j^{-1}(\{0\})$ intersect properly in $\mathbb{C}^n$ (which implies $\mathrm{p}^{-1}(\{(0, ..., 0)\}) = \emptyset$). Let $p_j(Y) - p_j(X) = \sum_1^n q_{j,k}(Y, X)(Y_k - X_k)$ with $\max \deg q_{j,k} \leq \deg p_j - 1$.

Introducing $n + 1$ auxiliary blocks of parameters $\mathrm{t}_j = (t_{j,1}, ..., t_{j,n})$

Let $\mathrm{D} = (n + 1)d^n + n$ and for $j = 1, ..., n$,

$$\phi_j(\mathrm{t}, Y) = \boxed{(1 + \langle \mathrm{t_j}, Y\rangle)^{\mathrm{D}+1}} \, p_j(Y)$$

$$\mathbb{G}_j(\mathrm{t}, Y, X) = (1 + \langle \mathrm{t}_j, X\rangle)^{\mathrm{D}+1} \Big(\sum_{k=1}^{n} q_{j,k}(Y, X) \, dY_k \Big) + p_j(Y) \, U_j(\mathrm{t}, Y, X) \langle \mathrm{t}_j, dY\rangle,$$

where $U_j(\mathrm{t}, Y, X) = \sum_{\nu=0}^{\mathrm{D}} (1 + \langle \mathrm{t}_j, Y\rangle)^{\nu} (1 + \langle \mathrm{t}_j, X\rangle)^{\mathrm{D}-\nu}$.

Parametric solution for Hilbert's nullstellensatz (Bézout identity formulation)

$$
\begin{aligned}
1 &= \sum_{j=0}^{n} \mathrm{Res}_{\mathbb{K}(t)[Y]/\mathbb{K}(t)} \left[\begin{matrix} \frac{1}{\phi_0(\mathtt{t},Y)} \bigwedge_{\ell \neq j} \mathbb{G}_\ell(\mathtt{t}, X, Y) \\ \phi_1(\mathtt{t}, Y), \dots, \phi_n(\mathtt{t}, Y) \end{matrix} \right] \phi_j(\mathtt{t}, X) \\
&= \sum_{j=0}^{n} \mathrm{Res}_{\mathbb{K}(t)[Y]/\mathbb{K}(t)} \left[\begin{matrix} \frac{1}{\phi_0(\mathtt{t},Y)} \bigwedge_{\ell \neq j} \mathbb{G}_\ell(\mathtt{t}, X, Y) \\ \phi_1(\mathtt{t}, Y), \dots, \phi_n(\mathtt{t}, Y) \end{matrix} \right] \boxed{(1 + \langle \mathtt{t}_j, X \rangle)^{\mathrm{D}+1}} \, p_j(X).
\end{aligned}
$$

Is the NP-complete Nullstellensatz decision problem in the P class?

P=NP true over $\mathbb{C} \Longleftrightarrow$ P=NP true over $\overline{\mathbb{Q}}$ (Shub-Smale 1996). Moreover they also show that proving that the sequence $(k!)_{k\in\mathbb{N}}$ is *ultimately hard to compute* would be enough to disprove that P=NP over $\overline{\mathbb{Q}}$, hence P=NP over $\mathbb{C}$. It could be interesting to observe that such a sequence $(k!)_{k\in\mathbb{N}}$ connects precisely Leibniz differential calculus with multivariate residue calculus.

Where Cayley, Jacobi, Kronecker aware of such approach to Bézout identity ? A pioneer reference: E. Netto, Vorlesungen über Algebra, II, Teubner, Leipzig, 1900.

Bochner-Martinelli weights or currents, Crofton's formula

Bochner-Martinelli weight in $\mathbb{P}^n_{[z]} \times \mathbb{P}^n_{[\zeta]}$ and its Crofton's realization

$$\mathrm{w}_{\mathrm{FS}}(z,\zeta) = \Big[\frac{z\cdot\zeta}{\|\zeta\|^2}\Big] + [c_{\mathrm{FS}}(\zeta)] = \Pi_*\Big(\mathrm{PV}\Big[\frac{\langle \mathrm{t}, z\rangle}{\langle \mathrm{t},\zeta\rangle}\, c^n_{\mathrm{FS}}(\mathrm{t})\Big] + [c_{\mathrm{FS}}(\zeta)]\wedge[c^n_{\mathrm{FS}}(\mathrm{t})]\Big)$$

where $\Pi : (z,\zeta,\mathrm{t}) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \mapsto (z,\zeta)$.

Koszul $(0, r-1)$ current and its Crofton's realization ($P_0, ..., P_m$ sections of $\mathcal{O}(d)$)

$$U_{\mathrm{P},r} = \Big[\Big(\frac{\|\mathrm{P}\|^2}{\|\zeta\|^2}\Big)^{\lambda}\Big(\sum_{j=0}^{m}\frac{\bar{P}_j}{\|\mathrm{P}\|^2}\, e_j\Big) \wedge \Big(\bar\partial\Big(\sum_{j=0}^{m}\frac{\bar{P}_j}{\|\mathrm{P}\|^2}\, e_j\Big)\Big)^{\wedge^{r-1}}\Big]_{\lambda=0} =$$

$$(\pi_r)_*\Big(\Big[|\langle \boldsymbol{\tau}_r, \mathrm{P}\rangle|^{2\lambda^r}\frac{\langle \boldsymbol{\tau}_r, e^*\rangle}{2i\pi\langle \boldsymbol{\tau}_r, \mathrm{P}\rangle} \wedge \Big(\bigwedge_{\rho=1}^{r-1}\bar\partial\Big(\frac{|\langle \boldsymbol{\tau}_\rho, \mathrm{P}\rangle|^{2\lambda^\rho}}{2i\pi}\frac{\langle \boldsymbol{\tau}_\rho, e^*\rangle}{\langle \boldsymbol{\tau}_\rho, \mathrm{P}\rangle}\Big)\Big) \wedge \bigwedge_{\rho=1}^{r}(c^m_{\mathrm{FS}}(\boldsymbol{\tau}_\rho)\Big]_{\lambda=0}\Big)$$

$$\pi_r : (\zeta, \boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_r) \in \mathbb{P}^n_{\mathbb{C}} \times (\mathbb{P}^m_{\mathbb{C}})^r \longmapsto \zeta \in \mathbb{P}^n_{\mathbb{C}}.$$

Effective Briançon-Skoda's theorem (Andersson-Götmark)

$$\mathbb{K} \subset \mathbb{C},\ p_0, \dots, p_m, q \in \mathbb{K}[X],\ \deg p_j = d \longrightarrow P_j \text{ sections of } \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(d)$$

$$P_j(\zeta) - P_j(z) = \sum_{k=0}^{n} (\zeta_k - z_k)\, A_{j,k}(z,\zeta)$$

Two formal (bundle - valued) forms

$$\mathbb{H}_P : (z,\zeta) \longmapsto \sum_{j=0}^{m} \Big(\sum_{k=0}^{n} \frac{A_{j,k}(z, w(z,\zeta)\,\zeta)}{w^d(z,\zeta)}\, d_{FS}\zeta_k \Big) \wedge e_j$$

$$\mathbb{I}_d : (z,\zeta) \longmapsto \sum_{j=0}^{m} \frac{e_j^* \wedge e_j}{w^d(z,\zeta)}$$

$$\mu = \min(m+1, n), \quad \mu' = \min(m, n)$$

Theorem (Mats Andersson, E. Götmark, 2011)

One has

$$\frac{|q|}{\|p\|^{\mu}} \in L^1_{\rm loc}(\mathbb{C}^n) \Longrightarrow q \in \sum_{j=0}^{m} q_j\, \mathbb{K}[X]$$

and the such membership is explicitly realized by de-homogenization of the integral reproducing formula

$$z_0^M Q(z) =$$

$$\rfloor_{\sum_j P_j(z) e_j} \left(\int_{\mathbb{P}^n_{\mathbb{C}}} \zeta_0^M\, Q(\zeta) \left(w^{M+n}(z,\zeta) \wedge \Big(\sum_{r=1}^{\mu'} \Big(\mathbb{I}_d(z,\zeta) \wedge \frac{\mathbb{H}_P^{\wedge^{r-1}}(z,\zeta)}{(r-1)!} \wedge U_{\mathrm{P},r}(\zeta) \Big) \Big)_{n,n} \right)\right.$$

which holds as soon as

$$M \geq \max\left(\boxed{\mu' d^{\mu}}, \mu' d - n - \deg q \right).$$

The gate to open questions

- What about the *croftonisation* of toric metrics in the non-archimedean setting? Are there autodual metrics as the Fubini-Study is (Crofton's formula)?
- In the case of codimension > 1, could an averaged global logarithmic height be expressed through a closed formula of the Gualdi type involving a *Ronkin current* instead of the Ronkin function?
- Can one extend the concept of *generalized cycle* (and class?) (as introduced in [ASWY] and [AESWY]) to an adelic point of view within the arithmetic setting; hence attach to such a generalized cycle a logarithmic height?
- Do the so-called *tropical (closed) currents* in $\mathbb{T}^n$ attached to k-tropical cycles (weighted $\mathbb{Z}^n$-rational complexes that fulfill *balancing conditions*) introduced by F. Babaee and J. Huh (2017) admit some factorization that would allow the definition of the concept of *tropical residue current*? One should start even with the codimension one simplest case where the $(1,1)$ current T is of the form $dd^c(p \circ \mathrm{Log})$, where p is the evaluation of a tropical polynomial.
- Do Coleff-Herrera residue currents which are constructed inductively along a multiplicative way, transposing the $\dot{\cap}$ operation to the frame of Coleff-Herrera calculus (following J.E. Solomin, 1977), inherit an arithmetic complexity when defined over $\mathbb{Q}$ over an algebraic variety such as $\mathbb{X}_\Sigma$ (or even just $\mathbb{P}^n$) ?

Thanks for your attention !
and thanks so much once more to Mikael!