Spikes and waves

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Remembering Mikael Passare.

Organization of the talk

- Crystalline measures.
- Multi-layered representation technique for images, David Donoho's discrete model, and Kolountzakis' construction of crystalline measures.
- Sampling and interpolation of Schwartz functions.
- Kepler's problem in dimensions 8 and 24.
- Generalizations of the Riemann zeta function.

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Crystalline measures

- The Dirac measure $\delta_a, \quad a \in \mathbb{R}^n$, (also written $\delta_a(x)$) is a measure supported by $\{a\}$. Its total mass is 1. In signal processing δ_a is often named a spike.
- If $\omega \in \mathbb{R}^n$ the function $\exp(2\pi i \omega \cdot x)$ of $x \in \mathbb{R}^n$ is a wave.
- If $\Lambda, S \subset \mathbb{R}^n$ are **two locally finite sets** and $c(\lambda)$, $a(s)$ are some coefficients can we have

$$\sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda(x) = \sum_{s \in S} a(s)\exp(2\pi i s \cdot x)\,?$$

- If this holds $\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda(x)$ is a crystalline measure.

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Crystalline measures

- The Fourier transform of a function f in $L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi i\, y \cdot x) f(x)\, dx.$$

- A subset $E \subset \mathbb{R}^n$ is **locally finite** if for any $R > 0$ there are only finitely many $x \in E$ such that $|x| \leq R$.
- A locally finite set $\Lambda \subset \mathbb{R}^n$ is either a finite set or a sequence of points $\lambda_j \in \mathbb{R}^n$ tending to infinity.

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Crystalline measures

- An atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if the three following conditions are satisfied:
- μ is supported by a locally finite set Λ,
- μ is a tempered distribution, and
- the distributional Fourier transform $\widehat{\mu}$ of μ is also an atomic measure supported by a locally finite set F.
- The simplest example of a crystalline measure is the Dirac comb $\mu = \sum_{k\in\mathbb{Z}} \delta_k$. We then have $\widehat{\mu} = \mu$.

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Crystalline measures

If μ is a crystalline measure we have

$$\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda(x) = \sum_{s \in S} a(s)\exp(2\pi i s \cdot x)$$

where Λ and S are two locally finite sets.

Pavel Kurasov and Peter Sarnak

- Pavel Kurasov and Peter Sarnak proved the following theorem:

Theorem

There exists a uniformly discrete set of real numbers Λ such that

1. *The vector space over $\mathbb{Q}$ generated by Λ is infinite dimensional.*
2. *$\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is a crystalline measure.*

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Andrew Guinand

The first non trivial crystalline measure was discovered by Andrew Guinand in connection with number theory and the Riemann zeta function (1959). This construction is detailed at the end of this lecture.

Multi-layered representation

We describe a new multi-layered representation technique for images. **An image is parsed into a superposition of coherent layers: smooth-regions layer, textures layer, etc.** The multi-layered decomposition algorithm consists in a cascade of compressions applied successively to the image itself and to the residuals that resulted from the previous compressions. During each iteration of the algorithm, we code the residual part in a lossy way: we only retain the most significant structures of the residual part, which results in a sparse representation.

Multi-layered representation

Each layer is encoded independently with a different transform, or basis, at a different bitrate; and the combination of the compressed layers can always be reconstructed in a meaningful way. **The strength of the multi-layer approach comes from the fact that different sets of basis functions complement each others: some of the basis functions will give reasonable account of the large trend of the data, while others will catch the local transients, or the oscillatory patterns.** This multi-layered representation has a lot of beautiful applications in image understanding, and image and video coding. We have implemented the algorithm and we have studied its capabilities.

Multi-layered representation

- David Donoho and Philip Stark considered a class of signals which can be represented as a sum between a few spikes and a few waves.
- Many images can be modeled as a sum between a cartoon and a texture. A cartoon image is sparse in the wavelet domain. A texture is sparse in the Fourier domain.
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Multi-layered representation

- Let us study a discrete model of the preceding problem and replace $\mathbb{R}^n$ by the ring

$$\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \simeq \{0, 1, \cdots, N-1\}$$

and $\mathcal{S}(\mathbb{R}^n)$ by the Euclidean space $\mathcal{H} = l^2(\mathbb{Z}_N)$.

- For $j \in \mathbb{Z}_N$ the spike $e_j \in \mathcal{H}$ which is located at j is defined by $e_j(k) = 0$ if $k \neq j$ and $e_j(j) = 1$.
- For $k \in \mathbb{Z}_N$ the wave $w_k \in \mathcal{H}$ is defined by

$$w_k(m) = N^{-1/2} \exp(2\pi ikm/N), \qquad 0 \leq m \leq N-1.$$

- The collection $e_j, j \in \mathbb{Z}_N$, is an orthonormal basis of $\mathcal{H}$. The same is true for $w_k, k \in \mathbb{Z}_N$.
- The Fourier coefficients of $f \in \mathcal{H}$ are the inner products $\langle f, w_k \rangle, k \in \mathbb{Z}_N$.

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Multi-layered representation

- A naive definition of a crystalline measure could be given by

$$\sum_{\lambda \in \Lambda} \alpha_\lambda e_\lambda = \sum_{k \in F} \beta_k w_k \qquad (*)$$

  where $\Lambda \subset \mathbb{Z}_N$ and $F \subset \mathbb{Z}_N$
- But this identity is trivial if $\Lambda = F = \mathbb{Z}_N$.

Definition

A crystalline measure is a solution to $(*)$ such that $|\Lambda| \ll N$ and $|F| \ll N$.

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It either means that

$$|\Lambda| \leq \beta N, \; |F| \leq \beta N \qquad (A)$$

for some small $\beta \in (0,1)$ or

$$\Lambda = \Lambda_N, \;\; F = F_N, \;\; |\Lambda_N| = o(N), \;\; |F_N| = o(N) \qquad (B)$$

as $N \to \infty$.

Multi-layered representation

- Donoho and Stark proved the following:
- Let $M = |\Lambda| + |F|$. If the product $|\Lambda||F|$ is strictly less that N then the M vectors $e_\lambda, \lambda \in \Lambda,\ w_k, k \in F$, are linearly independent.
- This is sharp. It does not hold if $|\Lambda| = N$ and $|F| = 1$.

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Multi-layered representation

- Therefore a discrete crystalline measure shall satisfy

$$|\Lambda_N||F_N| \geq N$$

  together with $|\Lambda_N| \ll N$ and $|F_N| \ll N$.
- This discussion leads to an interesting example where $|\Lambda_N| = |F_N| = \sqrt{N}$.
- If $N = N_0^2$ and if $F = \Lambda = \{0, N_0, 2N_0, \cdots, (N_0 - 1)N_0\}$ then we have $e_0 + e_{N_0} + \cdots + e_{(N_0-1)N_0} = f_0 + f_{N_0} + \cdots + f_{(N_0-1)N_0}$.

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Tao's theorem

If $N = p$ is a prime, then $(*)$ implies $|\Lambda| + |F| > p$. A crystalline measure cannot exist if definition (B) is adopted.

Kolountzakis' lemma

Lemma

There exists a non trivial $f_N \in \mathcal{H}$ *supported by* $\Lambda_N = \{\frac{1}{8}N, \ldots, \frac{7}{8}N\}$ *and whose Fourier transform is supported by* $F_N = \Lambda_N$.

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- This would be wrong with $\Lambda_N = F_N = \{\frac{2}{5}N, \ldots, \frac{3}{5}N\}$.
- Using this simple observation Mihalis Kolountzakis constructed a non trivial crystalline measure.
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Sampling and Interpolation of Schwartz functions

- The restriction of a function f to a set E is denoted by $f|_E$.

Definition

Let $\Lambda \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ be two locally finite sets. Then (Λ, F) if a pair of uniqueness if the following property is satisfied:
For any function f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ we have

$$f|_\Lambda = 0 \ \text{and} \ \widehat{f}|_F = 0 \Rightarrow f = 0.$$

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- The knowledge of $f|_\Lambda$ does not suffice to retrieve f. This lack of knowledge is compensated by the information given by $\widehat{f}|_F$.

- If (Λ, F) is a pair of uniqueness then any Schwartz function f is uniquely defined by its restriction to Λ and by the restriction of its Fourier transform to F.
- One assumes that $|(\Lambda \cap \{|x| \leq j\})| \leq Cj^N$ for some exponent N and any $j \geq 1$ and that the distance between two points of $(\Lambda \cap \{|x| \leq j\})$ exceeds cj^{-M} for some exponent M.
- Then the restriction to Λ of $\mathcal{S}(\mathbb{R}^n)$ is exactly the space $\mathcal{S}(\Lambda)$ of sequences indexed by Λ which have a fast decay at infinity. The same for F.
- We aim at constructing a linear operator $B : \mathcal{S}(\Lambda) \oplus \mathcal{S}(F) \mapsto \mathcal{S}(\mathbb{R}^n)$ such that for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$u = f|_\Lambda, v = \widehat{f}|_F \Rightarrow f = B(u, v).$$

- If (Λ, F) is a pair of uniqueness then any Schwartz function f is uniquely defined by its restriction to Λ and by the restriction of its Fourier transform to F.
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In one dimension Maryna Viazovska et al. solved this problem when f is an even function and when $\Lambda = F = \{\pm\sqrt{k}, k = 0, 1, \ldots\}$.

A conjecture

- If (Λ, F) is a pair of uniqueness, then every tempered distribution S can be written as a series of spikes located on Λ and of waves whose frequencies belong to F: $S(x) = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda(x) + \sum_{s \in S} a(s) \exp(2\pi i s \cdot x)$.
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- More precisely there exist two families $c_\lambda(x), \lambda \in \Lambda$, and $a_\omega(x), \omega \in F$, of Schwartz functions such that for any Schwartz function f we have

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There exists a sequence $a_k(x)$, $k = 0, 1, \ldots$ of even real-valued Schwartz functions in the real variable x with the property that for any **even** *Schwartz function f and any real number x we have*

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- This cannot be true if f is odd. Indeed the function $g(x) = \sin(\pi x^2)/\sinh(\pi x)$ is a counter example: g vanishes on $\pm k^{1/2}, k = 0, 1, \ldots$ and the Fourier transform of g is $-ig$.

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Kepler and piles of oranges

Kepler raised the following issue: what is the optimal (in terms of density) packing of balls with radius 1? This problem can be raised in any dimension.

Kepler and oranges

Kepler's problem

- Maryna Viazovska discovered the optimal sphere packing in dimension 8 and 24. Viazovska started from the linear programming approach by Cohn and Elkies. We focus on dimension 8.
- Using the tools needed to prove Theorem 2 Maryna constructed a radial Schwartz function ϕ on $\mathbb{R}^8$ such that $\phi(x) \leq 0$ if $|x| \geq \sqrt{2}$, $\widehat{\phi} \geq 0$ and $\phi(0) = \widehat{\phi}(0) = 1$.
- Then $\phi = \widehat{\phi} = 0$ on $\Lambda_8 \setminus \{0\}$.
- This implies that the density of a packing of $\mathbb{R}^8$ with balls of radius r cannot exceed $\frac{\pi^4}{384}$. But $\frac{\pi^4}{384}$ is exactly the density of the packing with balls of radius $\sqrt{2}/2$ centered on the Λ_8 lattice. This ends the proof.

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- This optimal lattice Λ_8 is defined by $\{x \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8;\ x_1 + \cdots + x_8 \in 2\mathbb{Z}\}$.
- The optimal packing is the collection of all balls of radius $\sqrt{2}/2$ centered at $\lambda \in \Lambda_8$.
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- This configuration is optimal since 240 is the kissing number in dimension 8. It is the largest number of disjoint balls of radius r which are tangent to a given ball of radius r.

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Crystalline measures and new ζ functions

- We now start from a crystalline measure and construct the corresponding ζ function.
- Let $\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$ be a crystalline measure. Let us assume that there exists a $s_0 > 0$ such that $\sum_{\{\lambda \in \Lambda, \lambda \neq 0\}} |c(\lambda)||\lambda|^{-s_0}$ is finite.
- The corresponding zeta function is the sum of the Dirichlet series $\zeta(\mu, s) = \sum_{\{\lambda \in \Lambda, \lambda \neq 0\}} c(\lambda)|\lambda|^{-s}$, $s \in \mathbb{C}$.
- $\zeta(\mu, s)$ is obviously analytic in the open half plane defined by $s \in \mathbb{C}$, $\Re s > s_0$.
- If $n = 1$ and if μ is the Dirac comb then $\zeta(\mu, s)$ is two times the Riemann zeta function. If μ is a sum of Dirac measures on a lattice then $\zeta(\mu, s)$ is the Epstein zeta function.

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Theorem

- *Let $\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$ be a crystalline measure on $\mathbb{R}^n$ and let $\widehat{\mu} = \sum_{\omega \in F} a(\omega)\delta_\omega$ be the Fourier transform of μ.*
- *Let $\xi(\mu, s) = \pi^{-s/2}\Gamma(s/2)\zeta(\mu, s)$.*
- *Then $\xi(\mu, s) - \frac{2a(0)}{n-s} - \frac{2c(0)}{s} = E(\mu, s)$ is an entire function.*
- *We have $\xi(\mu, s) = \xi(\widehat{\mu}, n - s)$.*
- *In the general case $\zeta(\mu, s)$ is a meromorphic function in the complex plane with a unique pole at $s = n$.*

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Andrew Guinand

This connection between crystalline measures, the Poisson summation formula, and the ζ function goes back to Riemann himself.

- Here is a one dimensional example proposed by Andrew Guinand.
- A sequence $\gamma_k, k = 0, 1, \ldots,$ is defined by

$$\sum_0^\infty \gamma_k q^k = \prod_1^\infty (1 - q^n)(1 + q^{2n})^{2/3}(1 + q^n)^{1/3}$$

where $|q| < 1$.
- $\gamma_0 = 1, \gamma_1 = -2/3, \gamma_2 = -4/9, \gamma_3 = -40/81, \gamma_4 = -160/243, \gamma_5 = 268/729, \ldots$. We have $|\gamma_k| \leq Ck^{1/3}$.
- If $\lambda_k = \sqrt{k + 1/9}, k = 0, 1, \ldots,$ the Guinand measure is

$$\mu_G = \sum_0^\infty \gamma_k(\delta_{\lambda_k} + \delta_{-\lambda_k})$$

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The Guinand ζ function is defined by the series

$$\zeta_G(s) = \sum_0^{\infty} \gamma_k (k + 1/9)^{-s/2}$$

which converges if $\Re s > 8/3$. Then we have (Guinand, Y.M.)

Theorem

The Guinand measure μ_G satisfies $\widehat{\mu}_G = \mu_G$.

The Guinand ζ function $\zeta_G(s)$ is an entire function in the complex plane.

We have

$$\pi^{-s/2}\Gamma(s/2)\zeta_G(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta_G(1-s).$$

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