Pass 1

## Geometry of the Hilbert space

Hilbert space - complete metric space (vector space) with scalar product

- a) scalar product $\langle f, g\rangle$
i) $\langle f, g\rangle=\overline{\langle g, f\rangle}$
ii) $\left\langle\alpha_{1} f_{1}+\alpha_{2} f_{2}, g\right\rangle=\alpha_{1}\left\langle f_{1}, g\right\rangle+\alpha_{2}\left\langle f_{2}, g\right\rangle$
iii) $\langle f, f\rangle \geq 0,\langle f, f\rangle=0 \Leftrightarrow f=0$.
- b) norm $\|f\|=\sqrt{\langle f, f\rangle}$
i) $\quad\|\alpha f\|=|\alpha|\|f\|$;
ii) $\quad|\langle f, g\rangle| \leq\|f\|\|g\| \quad$ Cauchy - Bunyakovsky - Schwarz inequality,
iii) $\|f+g\| \leq\|f\|+\|g\|$ triangle inequality
- c) Complete: every Cauchy sequence is converging to an element from the space.
Theorem Every metric space can be completed.
- d) Separable - countable $\epsilon$-net exists.


## Distance of a point from a convex set in $H$

Convex set: $f, g \in K \Rightarrow \lambda f+(1-\lambda) g \in K$
Th

$$
\begin{gathered}
\left.\begin{array}{l}
K-\text { closed convex set in } H \\
\delta=\inf \|h-f\|, f \in K
\end{array}\right\} \Rightarrow \exists!g \in K \text { s.t. }\|h-g\|=\delta . \\
\text { Projection into a subspace }
\end{gathered}
$$

$$
P_{G} h=g \in G \text { s.t. }\|h-g\|=\inf \left\|h-g^{\prime}\right\|, g^{\prime} \in G .
$$

Th
$h-P_{G} h$ is orthogonal to $G$.

Othogonalization of a sequence - Schmidt method.

$$
\begin{aligned}
& h_{1}=e_{1} \\
& h_{n}=e_{n}-P_{\mathcal{L}\left\{e_{1}, e_{2}, \ldots, e_{n-1}\right\}} e_{n}
\end{aligned}
$$

Bessel inequality:
$e_{n}$ - othonormal sequence

$$
\sum_{n=1}^{\infty}\left|\left\langle h, e_{k}\right\rangle\right|^{2} \leq\|h\|^{2}
$$

Closed system $\Leftrightarrow$ every vector $h \in H$ can be written in the form $h=$ $\sum_{n=1}^{\infty} h_{n} e_{n}$.

Parseval's equation:

$$
\|h\|^{2}=\sum_{n=1}^{\infty}\left|\left\langle h, e_{k}\right\rangle\right|^{2}
$$

In particular:

$$
\langle g, h\rangle=\sum_{k=1}^{\infty}\left\langle g, e_{k}\right\rangle\left\langle e_{k}, h\right\rangle
$$

provided the ortonormal systme $\left\{e_{k}\right\}$ is closed.

Pass 2

## Compact sets

2003.02.03
(see Kolmogorov, Fomin vol1, p51-57)
We consider arbitrary metric space $R$ with the metric $\rho$.
Compact set - every sequence of elements contains a subsequence converging to some element from the set.

Relatively compact set - every sequence of elements contains a converging subsequence.
i) In $\mathbf{R}^{n}$ every closed bounded set is compact. Every bounded set is relatively compact.
ii) In $l_{2}$ the unit ball is not compact.
iii) The fundamental parallelogram $\Pi$ in $l_{2}$ is compact

$$
\Pi=\left\{x=\left(x_{1}, x_{2}, \ldots\right):\left|x_{j}\right| \leq \frac{1}{2^{j}}, j \in b f N\right\} .
$$

$\epsilon$-net with respect to $M$ - set such that for an arbitrary point $x \in M$ at least one point $a$ from the net can be found such that

$$
\rho(a, x)<\epsilon .
$$

Totally bounded set - for any positive $\epsilon$ a finite $\epsilon$-net can be found (NB! the net depends on $\epsilon$, it does not necessarily belong to the set).
i) Every totally bounded set is bounded. These two notions are equivalent in finite dimensional spaces only.
ii) The fundamental parallelogram in $l_{2}$ is totally bounded.

Theorem A necessary and sufficient condition that a subset $M$ of a complete metric space $R$ be relatively compact is that $M$ be totally bounded.

Proof. Necessity. Consider the sequence $x_{1}, x_{2}, \ldots \in M$ such that $\rho\left(x_{j}, x_{k}\right) \geq \epsilon$. This sequence is not compact.

Sufficiency. Trick with the diagonal subsequence.
Theorem $A$ necessary and sufficient condition that a subset $M$ of a complete metric space $R$ be relatively compact is that for every $\epsilon \geq 0$ there exist in $R$ a compact $\epsilon$-net for $M$.

## Compact set in $C[a, b]$.

Uniformly bounded set - there exists a positive number $C$ such that $|\varphi(x)|<C$ for all $x \in[a, b]$ and all functions $\varphi$ from the set.


$$
\left|x_{1}-x_{2}\right|<\delta \Rightarrow\left|\varphi\left(x_{1}\right)-\varphi\left(x_{2}\right)\right|<\epsilon
$$

for all $\varphi$ from the set.
Theorem (Arzela) A necessary and sufficient condition that a family of continuous functions defined on the closed interval $[a, b]$ be compact in $C[a, b]$ is that this family be uniformly bounded and equicontinuous.
Proof Necessity. Consider any finite $\epsilon / 3$-net $\varphi_{j}, j=1,2, \ldots$. Each function $\varphi_{j}$ is uniformly bounded: $\left|\varphi_{j}(x)\right|<M_{j}$.

Set $M=\max _{j} M_{j}+\epsilon / 3$. By definition of the $\epsilon$-net we have for at least one $\varphi_{j}$

$$
\begin{aligned}
& \max _{x}\left|\varphi(x)-\varphi_{j}(x)\right|<\epsilon / 3 \\
\Rightarrow & |\varphi(x)|<\varphi_{j}(x) \mid+\epsilon / 3<M
\end{aligned}
$$

Each of the functions $\varphi_{j}$ is uniformly continuous, i.e. for any $\epsilon>0$ there exists a $\delta_{j}$ such that

$$
\left|x_{1}-x_{2}\right|<\delta_{j} \Rightarrow\left|\varphi_{j}\left(x_{1}\right)-\varphi_{j}\left(x_{2}\right)\right|<\epsilon / 3
$$

Set $\delta=\min _{j} \delta_{j}$. Then for $\left|x_{1}-x_{2}\right|<\delta$ and any function $\varphi$ from the family and some $\varphi_{j}$ we have
$\left|\varphi\left(x_{1}\right)-\varphi\left(x_{2}\right)\right| \leq\left|\varphi\left(x_{1}\right)-\varphi_{j}\left(x_{1}\right)\right|+\left|\varphi_{j}\left(x_{1}\right)-\varphi_{j}\left(x_{2}\right)\right|+\left|\varphi_{j}\left(x_{2}\right)-\varphi\left(x_{2}\right)\right|<\epsilon$.

Sufficiency. Let $\Phi$ be a uniformly bounded and equicontinuous family of functions. We are going to construct a finite $\epsilon$-net.

Subdivide the interval $[-M, M]$ on the y -axis by means of the points $y_{k}$ such that $y_{k}<y_{k+1}, y_{k+1}-y_{k}<\epsilon / 5$.

Subdivide the interval $[a, b]$ on the x-axis by means of the points $x_{n}$ such that $x_{n}<x_{n+1}, x_{k+1}-x_{k}<\delta=\delta(\epsilon)$.

Claim: continuous functions with the graphs passing through the points $\left(x_{n}, y_{k}\right)$ form a finite $\epsilon$-net for $\Phi$. Obviously this set is finite.

Consider any function $\varphi \in \Phi$. Assign to this function a polygonal arc $\psi(x)$ such that $\left|\psi\left(x_{n}\right)-\varphi\left(x_{n}\right)\right|<\epsilon / 5$. Then by construction:

$$
\begin{gathered}
\left|\psi\left(x_{n}\right)-\psi\left(x_{n+1}\right)\right|<3 \epsilon / 5 \\
\Rightarrow\left|\psi\left(x_{n}\right)-\psi(x)\right| \text { for any } x \in\left[x_{n}, x_{n+1}\right] .
\end{gathered}
$$

Finally:

$$
|\varphi(x)-\psi(x)| \leq\left|\varphi(x)-\varphi\left(x_{n}\right)\right|+\left|\varphi\left(x_{n}\right)-\psi\left(x_{n}\right)\right|+\left|\psi\left(x_{n}\right)-\psi(x)\right|<\epsilon,
$$

where $x_{n}$ is the subdivision point which is closest to $x$ from the left.

## Linear functionals in normed spaces.

Let $R$ be a normed space with the norm $\|\cdot\|$.
Linear functional $f$ - numerical function (values belong to $\mathbf{R}$ or $\mathbf{C}$ ) defined on a normed linear space such that

$$
f(\alpha x+\beta y)=\alpha f(x)+\beta f(y) .
$$

Continuous functional $f$ at point $x_{0}$ -

$$
\forall \epsilon>0 \exists \delta>0:\left\|x-x_{0}\right\|<\delta \Rightarrow\left|f(x)-f\left(x_{0}\right)\right|<\epsilon
$$

Theorem If the linear functional is continuous at some point $x_{0}$, then it is continuous everywhere on its domain.

Bounded functional $f$ - there exists a constant $N$ such that

$$
|f(x)| \leq N\|x\|
$$

Norm of the functional $f$

$$
\|f\|=\sup \{|f(x)| /\|x\| ;\|x\| \neq 0\} .
$$

Theorem For linear functional the conditions of continuity and boundedness are equivalent.
i) In $\mathbf{C}^{n}$ consider the functional

$$
f_{a}(x)=\langle x, a\rangle
$$

parametrized by the vector $a \in \mathbf{C}^{n}$.
ii) In $C([a, b])$ consider the functionals

$$
\begin{gathered}
I(x)=\int_{a}^{b} x(t) d t \\
f_{a}(x)=\int_{a}^{b} x(t) \bar{a}(t) d t
\end{gathered}
$$

where $a \in C([a, b])$.

$$
\delta_{t_{0}}(x)=x\left(t_{0}\right) .
$$

## Hahn-Banach Theorem

Theorem Every linear functional $f(x)$ defined on a linear subspace $G$ of a normed linear space $E$ can be extended to the entire space with preservation of norm, i.e. it is possible to construct a linear functional $F(x)$ such that

$$
\begin{aligned}
& F(x)=f(x) \quad x \in G, \\
& \|F\|_{E}=\|f\|_{G} .
\end{aligned}
$$

## Proof.

i) The theorem will be proved for separable spaces only
ii) It is enough to consider one-dimensional extensions only, since every separable space $E$ contains everywhere dense countable set. One can extend the functional by extending it on one dimensional subspaces.

Consider $G_{1}$ one dimensional extension of $G$ obtained by adding one element $x_{0}$.

$$
y \in G_{1} \Rightarrow y=t x_{0}+x, x \in G, t \in \mathbf{C} .
$$

To extend the functional it is enough to define it on the element $x_{0}$

$$
F(y)=F\left(t x_{0}+x\right)=t F\left(x_{0}\right)+f(x) .
$$

This extension has the same norm as $f$ iff

$$
\left|f(x)+t F\left(x_{0}\right)\right| \leq\|f\|\left\|x+t x_{0}\right\|
$$

holds for all $x \in G, t \in \mathbf{C}$. Putting $z=x / t$ we get

$$
\left|f(z)+F\left(x_{0}\right)\right| \leq\|f\|\left\|z+x_{0}\right\|,
$$

which is equivalent to

$$
\begin{gathered}
-\|f\|\left\|z+x_{0}\right\| \leq f(z)+F\left(x_{0}\right) \leq\|f\|\left\|z+x_{0}\right\| \Rightarrow \\
\Rightarrow-\left(f(z)+\|f\|\left\|z+x_{0}\right\|\right) \leq F\left(x_{0}\right) \leq-f(x)+\|f\|\left\|z+x_{0}\right\|
\end{gathered}
$$

The constant $F\left(x_{0}\right)$ can be chosen iff for any two points $z^{\prime}, z^{\prime \prime} \in G$ the following inequality holds

$$
-\left(f\left(z^{\prime \prime}\right)+\|f\|\left\|z^{\prime \prime}-x_{0}\right\|\right) \leq-f\left(z^{\prime}\right)+\|f\|\left\|z^{\prime}-x_{0}\right\|
$$

The last inequality follows from the boundedness of the original functional and triangle inequality

$$
\begin{gathered}
f\left(z^{\prime}\right)-f\left(z^{\prime \prime}\right) \leq\|f\|\left\|z^{\prime}-z^{\prime \prime}\right\| \\
\leq\|f\|\left\|z^{\prime}-x_{0}\right\|+\|f\|\left\|z^{\prime \prime}-x_{0}\right\| \Rightarrow \\
f\left(z^{\prime}\right)-\|f\|\left\|z^{\prime}-x_{0}\right\| \leq f\left(z^{\prime \prime}\right)+\|f\|\left\|z^{\prime \prime}-x_{0}\right\| .
\end{gathered}
$$

## Linear functionals in the Hilbert space

(AG 58-63 (Dover 30-36))
The following functionals are unbounded in the corresponding Hilbert spaces:

$$
\delta(f)=f(0)
$$

defined on the continuous functions from $L_{2}(\mathbf{R})$.
-

$$
\Phi(f)=\int_{-\infty}^{+\infty} f(t) d t
$$

considered in the space $L_{2}(\mathbf{R})$; this functional is bounded in the space $L_{1}(\mathbf{R})$.

$$
\Phi(f)=\lim _{n \rightarrow \infty}\left\langle f, e_{1}+e_{2}+\ldots+e_{n}\right\rangle,
$$

where $\left\{e_{k}\right\}$ is any orthonormal basis in the Hilbert space.
Example of the functional which is defined on the whole space but not bounded - see AG p 60.

Theorem (F.Riesz) Every linear bounded (=continuous) functional $\Phi$ in the Hilbert space $H$ has the form

$$
\Phi(h)=\langle h, f\rangle,
$$

where $f$ is a certain element of $H$ which is uniquellydefined by $\Phi$; moreover

$$
\|\Phi\|=\|f\|
$$

Proof. Consider the kernel $G$ of the functional

$$
G \equiv\{h \in H: \Phi(h)=0\} .
$$

It is obvious that $G$ is a closed subspace of the Hilbert space $H$ (follows from the linearity and continuity of the functional).

If $G=H$ then the functional $\Phi$ is equal to the zero functional $\Phi(h) \equiv 0$ and the theorem holds for $f=0$.

If $G \neq H$ consider the orthogonal complement of the subspace $G$ in $H$

$$
G^{\perp}=H \ominus G
$$

We are going to prove that the subspace $G^{\perp}$ is in fact one dimensional. Really any two nonzero vectors $h_{1}, h_{2} \in G^{\perp}$ are linear dependent
$\Phi\left(\Phi\left(h_{2}\right) h_{1}-\Phi\left(h_{1}\right) h_{2}\right)=0 \Rightarrow \Phi\left(h_{2}\right) h_{1}-\Phi\left(h_{1}\right) h_{2} \in G=\left(G^{\perp}\right)^{\perp} \Rightarrow \Phi\left(h_{2}\right) h_{1}-\Phi\left(h_{1}\right) h_{2}=0$,
$\left(\Phi h_{j} \neq 0, j=1,2\right)$. Let us chose the basis vector $f$ in $G^{\perp}$ such that

$$
\Phi(f)=\|f\|^{2}
$$

Then the following formula holds

$$
\Phi(h)=\Phi\left(\alpha f+h^{\|}\right)=\alpha \Phi(f)=\alpha\|f\|^{2}=\langle h, f\rangle .
$$

The uniqueness of the representation follows from the fact that the subspace $G^{\perp}$ is one dimensional.

It remains to show that

$$
\|\Phi\|_{\text {as linear functional }}=\|f\|_{\text {as vector from the Hilbert space }} .
$$

We have:

$$
\Phi(h)=\langle h, f\rangle \Rightarrow|\Phi(h)| \leq\|h\|\|f\| \Rightarrow\|\Phi\| \leq\|f\|
$$

On the other hand, putting $h=f$ we obtain

$$
\Phi(f)=\|f\|^{2} \Rightarrow\|\Phi\| \geq\|f\|
$$

It follows that $\|\Phi\|=\|f\|$.
Corollary. Every Hilbert space coincides with its adjoint. Every continuous linear functional defined on a closed subspace can uniquelly be extended to the whole Hilbert space preserving the norm of the functional.

## Bounded linear operators

(AG 67-73 (Dover 39-43))
Linear operator $T$

1) domain $\operatorname{Dom}(T)$ linear subspace of $H$
2) linear transformation $T: f \mapsto T f, f \in \operatorname{Dom}(T), T f \in H$;

$$
T(\alpha f+\beta g)=\alpha T f+\beta T g
$$

Bounded operator:

$$
\begin{gathered}
\sup _{f \in \operatorname{Dom}(T),\|f\|=1}\|T f\|<\infty . \\
\Rightarrow \operatorname{norm}\|T\|=\sup _{f \in \operatorname{Dom}(T),\|f\|=1}\|T f\| .
\end{gathered}
$$

Important facts

1. Every bounded linear operator is continuous.
2. Every continuous linear operator is bounded.
3. Extension by continuity: Suppose that $\overline{\operatorname{Dom}(T)} \neq \operatorname{Dom}(T)$. Then every bounded linear operator can be extended to $\overline{\operatorname{Dom}(T)}$ by continuity. Suppose that $f \in \overline{\operatorname{Dom}(T)}$. Then there exists a certain sequence $f_{n} \rightarrow$ $f, f_{n} \in \operatorname{Dom}(T)$. One can define $T f$ using the following equality

$$
T f=\lim _{n \rightarrow \infty} T f_{n}
$$

Ex. 1 Show that the limit is independent of the chosen sequence $\left\{f_{n}.\right\}$ Prove that the norm of the extended operator is equal to the norm of the original one.

Sum of two linear operators $S$ and $T$ is defined on the common domain $\operatorname{Dom}(S+T)=\operatorname{Dom}(S) \cap \operatorname{Dom}(T)$ using the following equality

$$
(S+T) f=S f+T f
$$

Product of two linear operators $S$ and $T$ is defined on the domain

$$
\operatorname{Dom}(S T)=\{f \in \operatorname{Dom}(T) \subset H: T f \in \operatorname{Dom}(S)\}
$$

by the formula ( $S T) f=S(T f)$.
NB! The domains $\operatorname{Dom}(S T)$ and $\operatorname{Dom}(T S)$ could be different.
Orthogonal sum of two linear operators
Let $T_{1}$ and $T_{2}$ be two linear operators acting in the Hilbert spaces $H_{1}$ and $H_{2}$ respectively. Then the operator $T=T_{1} \oplus T_{2}$ is defined in the Hilbert space $H=H_{1} \oplus H_{2}$ on the domain $\operatorname{Dom}(H)=\operatorname{Dom}\left(H_{1}\right) \oplus \operatorname{Dom}\left(H_{2}\right)$ by the following formula

$$
T\left(h_{1}, h_{2}\right)=\left(T_{1} h_{1}, T_{2} h_{2}\right) .
$$

## Bilinear forms

Bilinear form (sesquilinear) $\Omega$ in $H$ - mapping $H \times H \rightarrow \mathbf{C}$

$$
\Omega:(f, g) \rightarrow \Omega(f, g)
$$

such that:
i) $\Omega\left(\alpha_{1} f_{1}+\alpha_{2} f_{2}, g\right)=\alpha_{1} \Omega\left(f_{1}, g\right)+\alpha_{2} \Omega\left(f_{2}, g\right)$
ii) $\Omega\left(f, \beta_{1} g_{1}+\beta_{2} g_{2}\right)=\overline{\beta_{1}} \Omega\left(f, g_{1}\right)+\overline{\beta_{2}} \Omega\left(f, g_{2}\right)$

Bilinear form is called bounded if

$$
\sup _{\|f\|=1,\|g\|=1}|\Omega(f, g)|<\infty .
$$

Theorem Every bounded bilinear form $\Omega(f, g)$ in $H$ has the form

$$
\Omega(f, g)=\langle A f, g\rangle
$$

where $A$ is a bounded linear operator in $H$ and is uniquelly determined by $\Omega$. Also

$$
\|A\|=\|\Omega\|
$$

Proof. Follows from F.Riesz's theorem.

## Adjoint operator

## Lecture 3

Let $A$ be a bounded linear operator defined on the whole space $H$. Then the adjoint operator $A^{*}$ is the unique operator in $H$, such that for all $f, g \in H$ the following equality holds

$$
\begin{gathered}
\langle A f, g\rangle=\left\langle f, A^{*} g\right\rangle \\
\operatorname{Dom}\left(A^{*}\right)=H
\end{gathered}
$$

The following formula holds

$$
(A B)^{*}=B^{*} A^{*}
$$

Bounded linear operator $A$ defined on the whole Hilbert space is called symmetric or self-adjoint if the following relation holds:

$$
A=A^{*}
$$

The definitions of symmetric and self-adjoint operators coincide in the case of bounded operators only!!!.

Def Let $A$ be an unbounded operator with the domain $\overline{\operatorname{Dom}(A)}=\mathcal{H}$. An element $g \in \mathcal{H}$ is said to belong to the domain $\operatorname{Dom}\left(A^{*}\right)$ of the adjoint operator $A^{*}$ if there exists $h \in \mathcal{H}$, such that

$$
\langle A f, g\rangle=\langle f, h\rangle, \quad \forall f \in \operatorname{Dom}(A)
$$

In this case the adjoint operator $A^{*}$ maps the element $g$ into $h: A^{*} g=h$.
The domain of the adjoint operator is the set of all $g \in \mathcal{H}$ such that $\langle A f, g\rangle \leq C_{g}\|f\|$. The adjoint operator is defined (uniquely) only if the original operator is densely defined.

Def The linear operator $A$ is called symmetric if and only if for any $f, g \in \operatorname{Dom}(A)$ the following equality holds

$$
\langle A f, g\rangle=\langle f, A g\rangle
$$

Theorem Let $A$ be a densely defined symmetric operator in the Hilbert space $\mathcal{H}$. Then the domain of the adjoint operator $A^{*}$ contains the domain $\operatorname{Dom}(A)$ of the original operator

$$
\operatorname{Dom}\left(A^{*}\right) \supset \operatorname{Dom}(A)
$$

and moreover

$$
\left.A\right|_{\operatorname{Dom}(A)}=\left.A^{*}\right|_{\operatorname{Dom}(A)},
$$

i.e. the adjoint operator is an extension of the operator $A$

$$
A \preceq A^{*} .
$$

## Compact operators

(AG 91-97, F.Riesz, B.Sz.-Nagy, 227-244)
A linear operator $A$ defined on the whole Hilbert space $H$ is called compact if it maps every bounded set onto a compact set.

Every compact operator is bounded.
Ex 2 Show that product of any compact operator and a bounded operator defined on the whole space is compact.

Ex 3 Show that the sum of two compact operators is compact.
Theorem If $A$ is a bounded linear operator defined on the whole space $H$, and if the operator $A^{*} A$ is compact, then the operator $A$ is compact.
Proof.
$\left\|A f_{n}-A f_{m}\right\|^{2}=\left\langle A^{*} A\left(f_{n}-f_{m}\right), f_{n}-f_{m}\right\rangle \leq\left\|A^{*} A\left(f_{n}-f_{m}\right)\right\|\left\|f_{n}-f_{m}\right\|$.
Theorem The norm limit of compact operators is a compact operator. Proof. Consider trick with the diagonal sequence.

Theorem

## Absolute norm of an operator

Consider arbitrary orthonormal basis $\varphi_{n_{n} \in \mathbf{N}}$ in the separable Hilbert space $H$. Then the map

$$
\begin{gathered}
\Phi: H \rightarrow \ell_{2} \\
\Phi: f \mapsto\left\{\left\langle f, \varphi_{n}\right\rangle\right\}
\end{gathered}
$$

defines a one-to-one correspondance between all elements from the Hilbert space $H$ and $\ell_{2}$. Every bounded operator can be then represented by its (infinite) matrix

$$
A_{j k}=\left\langle A \varphi_{j}, \varphi_{k}\right\rangle
$$

The absolute norm of the operator $A$ is given by

$$
N(A)=\sqrt{\sum_{j k \in \mathbf{N}}\left|A_{j k}\right|^{2}}=\sqrt{\sum_{j k \in \mathbf{N}}\left|\left\langle A \varphi_{j}, \varphi_{k}\right\rangle\right|^{2}} .
$$

The ordinary norm does not exceed the absolute norm

$$
N(A) \geq\|A\| .
$$

Theorem If the absolute norm of the operator is finite, then the operator is compact.
Proof. Follows from the approximations of the operator $A$ by finite rank operators

$$
A^{n}=\sum_{j, k \leq n} A_{k j}\left\langle\varphi_{k}, \cdot\right\rangle \varphi_{j}
$$

The matrix $\left\{A_{j k}\right\}$ is infinite and its diagonalization even in the symmetric case is a hard problem. We are going to prove that every compact symmetric operator has a basis, where its matrix is diagonal.

## Spectral theorem for compact operators

(F.Riesz, B.Sz.-Nagy, p 227-244)

Theorem
Let $A$ be a symmetric bounded operator acting in the Hilbert space $H$, then:

1) its eigenvalues are real,
2) the eigenfunctions corresponding to different eigenvalues are orthogonal,
3) the quadratic form

$$
Q_{A}(f, f) \equiv\langle A f, f\rangle
$$

is real valued,
4) the smallest constant $N_{A}$ for which

$$
|Q(f, f)| \leq N_{A}\|f\|^{2}
$$

equals to $\|A\|$.
Proof. Points 1-2: the proof follows the same lines as the proof for Hermitian matrices.
3)

$$
\begin{gathered}
Q(f, f)=\langle A f, f\rangle=\overline{\langle f, A f\rangle}=\overline{\langle A f, f\rangle}=\overline{Q(f, f)} \\
\Rightarrow Q(f, f) \in \mathbf{R} .
\end{gathered}
$$

4) The quadratic form of the operator is bounded by the same constant as
the operator.
a) $N_{A} \leq\|A\|$

$$
\begin{gathered}
|Q(f, f)|=|\langle A f, f\rangle| \leq\|A f\|\|f\| \leq\|A\|\|f\|^{2} \\
\Rightarrow N_{A} \leq\|A\|
\end{gathered}
$$

b) $N_{A} \geq\|A\|$

$$
\begin{aligned}
& \|A f\|^{2}=\frac{1}{4}\left[\left\langle A\left(\lambda f+\frac{1}{\lambda} A f\right), \lambda f+\frac{1}{\lambda} A f\right\rangle-\left\langle A\left(\lambda f-\frac{1}{\lambda} A f\right), \lambda f-\frac{1}{\lambda} A f\right\rangle\right] \\
\leq & \frac{1}{4}\left[N_{A}\left\|\lambda f+\frac{1}{\lambda} A f\right\|^{2}+N_{A}\left\|\lambda f-\frac{1}{\lambda} A f\right\|\right]=\frac{1}{2} N_{A}\left[\lambda^{2}\|f\|^{2}+\frac{1}{\lambda^{2}}\|A f\|\right] .
\end{aligned}
$$

Chosing $\lambda^{2}=\frac{\|A f\|}{\|f\|}$ we get

$$
\|A f\|^{2} \leq N_{A}\|A f\|\|f\| \Rightarrow\|A\| \leq N_{A}
$$

## Theorem (Hilbert)

Every nonzero compact operator $A$ has at least one eigenvalue $\mu_{1}$ different from zero, such that $\left|\mu_{1}\right|=\|A\|$.
Proof. Consider the set $F=\{f \in H:\|f\|=1\}$, and a sequence $f_{n} \in F$ such that

$$
\lim _{n \rightarrow \infty}\left\langle A f_{n}, f_{n}\right\rangle=\mu_{1}= \pm\|A\|
$$

Consider the limit

$$
\left\|A f_{n}-\mu_{1} f_{n}\right\|^{2}=\left\|A f_{n}\right\|^{2}-2 \mu_{1}\left\langle A f_{n}, f_{n}\right\rangle+\mu_{1}^{2}\left\|f_{n}\right\|^{2} \rightarrow 0
$$

It follows that

$$
\lim _{n \rightarrow \infty} A f_{n}=\mu f_{n}
$$

The operator $A$ is compact, therefore one can find a subsequence $f_{n_{k}}$ such that $A f_{n_{k}}$ is converging. It follows that the sequence $f_{n_{k}}$ is converging. The limit denoted by $f$ has the following properties:

$$
A f=\mu_{1} f
$$

Theorem Spectral theorem for compact operators (Hilbert)
Every compact operator in the Hilbert space has a finite or infite set of real eigenvalues $\mu_{s}$ of finite mutiplicity approaching zero. The set of corresponding eigenvectors $\varphi_{s}$ can be chosen forming an orthonormal basis in the Hilbert space $H \ominus \operatorname{Ker} A$. The action of the operator $A$ is then given by the following formula

$$
\begin{gathered}
f=\sum_{s \in \mathbf{N}}\left\langle f, \varphi_{s}\right\rangle \varphi_{s}+f_{0}, f_{0} \in \operatorname{Ker} A ; \\
A f=\sum_{s \in \mathbf{N}} \mu_{s}\left\langle f, \varphi_{s}\right\rangle \varphi_{s} .
\end{gathered}
$$

Proof. The previous theorem states that at least one eigenvalue $\mu_{1}$ exists and the corresponding eigenfunction $\varphi_{1}$ can be chosen having unit norm. Consider the space $H_{1}$ of functions from $H$ orthogonal to $\varphi_{1}$. The space $H_{1}$ is a Hilbert space and it is invariant with respect to the operator $A$ : let $f \in H_{1}$ then

$$
\left\langle A f, \varphi_{1}\right\rangle=\left\langle f, A \varphi_{1}\right\rangle=\mu_{1}\langle f, \varphi\rangle=0 .
$$

The restricted operator is compact and the same theorem implies that there exists $\mu_{2},\left|\mu_{2}\right| \leq\left|\mu_{1}\right|$.

Applying this procedure many times we get a set of eigenvalues $\mu_{1}, \mu_{2}, \ldots$; $\left|\mu_{s}\right| \geq\left|\mu_{s+1}\right|$. All these eigenvalues (except zero eigenvalue) have finite multiplicity. Otherwise the operator is not compact. The corresponding eigenvectors form orthonormal basis in $H \ominus \operatorname{Ker} A$.

## Hilbert-Schmidt operators

Consider the function $k(s, t) \in L_{2}\left(\mathbf{R}^{2}\right)$, i.e. such that

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}|K(s, t)|^{2} d s d t<\infty
$$

The corresponding linear operator defined on the whole Hilbert space $L_{2}$ is called Hilbert-Schmidt operator

$$
K f(s)=\int_{-\infty}^{\infty} K(s, t) f(t) d t .
$$

The last formula defines the function $K f$ almost everywhere.
It is a bounded operator, since

$$
\begin{gathered}
\|g\|^{2} \leq\|k\|_{L_{2}\left(\mathbf{R}^{2}\right)}^{2}\|f\|^{2} . \\
\Rightarrow\|K\| \leq\|k\|_{L_{2}\left(\mathbf{R}^{2}\right)} .
\end{gathered}
$$

The absolute norm of this operator is bounded

$$
\|K\| \leq \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}|K(s, t)|^{2} d s d t}=N(K)
$$

Corollary Every Hilbert-Schmidt operator in $L_{2}(\mathbf{R})$ is compact.
Using any orthonormal basis in $L_{2}(\mathbf{R})$ and formula for the absolute norm of an operator in terms of its matrix coefficients we get that

$$
N(K)=\|k\|_{L_{2}\left(\mathbf{R}^{2}\right)}
$$

and that every operator in $L_{2}(\mathbf{R})$ with finite absolute norm is a HilbertSchmidt operator.
Spectral theorem for Sturm-Liouville operator
on a finite interval $[a, b]!!!$
Consider the Hilbert space $L_{2}[a, b]$ and the linear operator

$$
L u=-\frac{d}{d x}\left[p(x) \frac{d}{d x}\right]+q(x) u
$$

defined on the domain of functions having square integrable second derivative and absolutely continuous first derivative and certain boundary conditions at the end points, for example

$$
u(a)=u(b)=0
$$

The coefficients $p, q$ satisfy

$$
p(x), p^{\prime}(x), q(x) \in C[a, b], \quad p(x) \geq p_{0}>0, \quad q(x)>0
$$

The operator $A$ is a positive unbounded operator in $L_{2}[a, b]$.
The energy norm and the energy space
Positive symmetric operator $\Rightarrow$ the energy norm

$$
\|u\|_{A}^{2}=\langle A u, u\rangle
$$

The energy scalar product

$$
\langle u, v\rangle_{A}=\langle A u, v\rangle .
$$

The energy space $H_{A}$ - the completion of $\operatorname{Dom}(A)$ in the energy norm.
Theorem Let $A$ be a positive operator $A \geq a>0$. Then the energy space is a subspace of the original Hilbert space. It is a Hilbert space itself with respect to the energy scalar product.

It is clear that the energy scalar product satisfies all axioms for the scalar product in a Hilbert space. Suppose that $u$ is an element from the energy space. Then there exists a sequence $u_{n} \in \operatorname{Dom}(A)$ converging to $u$ in the energy norm. This sequence is a Cauchy sequence both in the energy and in the original Hilbert spaces:

$$
\left\|u_{n}-u_{m}\right\|_{A} \rightarrow 0 \Rightarrow\left\|u_{n}-u_{m}\right\| \rightarrow 0
$$

It follows that the limit vector $\lim u_{n}=u$ can be associated with a certain vector from the original Hilbert space $H$, i.e. that $H_{A}$ can be embedded into $H$.

$$
\operatorname{Dom}(A) \subset H_{A} \subset H
$$

Example

$$
A u=-\frac{d^{2} u}{d x^{2}} \text { in } L_{2}[0,1]
$$

with Dirichlet boundary conditions at the end points.

$$
\langle u, v\rangle_{A}=\int_{0}^{1} u^{\prime}(x) \overline{v^{\prime}(x)} d x
$$

The energy space consists of absolutely continuous functions equal to zero at the end points and having square integrable first derivatives.

Theorem Let $A$ be a positive definite operator, such that, every set bounded in the energy norm is compact in the original norm. Then the spectrum of this operator is discrete, i.e. there is an infinite sequence of eigenvalues $\lambda_{n}$ and eigenfunctions $\varphi_{n}$ complete in the Hilbert space $H$.
Observation The operator is positive definite $\Rightarrow$ the operator is symmetric $\Rightarrow$ the spectrum is real (and therefore positive) and the eigenfunctions corresponding to different eigenvalues are orthogonal.

Proof.

1. $\lambda_{a}=\inf \|u\|^{2}, u \in H_{A},\|u\|=1$. Minimizing sequence is bounded in the energy norm and therefore a subsequence is converging in the original norm $\Rightarrow$ the first eigenvalue and the first eigenfunction.
2. Consider $H_{A}^{n}=H_{A} \ominus \mathcal{L}\left\{\varphi_{1}, \ldots, \varphi_{n}\right\}$. The same trick to find $\lambda_{2}, \lambda_{2} \geq \lambda_{1}$.
3. Numbers $\lambda_{n}$ tend to $+\infty$. Suppose not $\Rightarrow$ there are two possibilities
a) either the sequence is finite - trivial case
b) or there is an orthonormal sequence of functions, which is bounded in the energy norm $\Rightarrow$ contradiction.
4. The system $\left\{\varphi_{n}\right\}$ is complete in $H_{A}$. Consider the space $H_{A}^{\infty}$ and

$$
\lambda_{\infty}=\inf \|u\|_{A}^{2}, \in u \in H_{A}^{\infty},\|u\|=1
$$

It is an eigenvalue (the proof is identical to 1.$) \Rightarrow$ contradiction.
5. The system $\left\{\varphi_{n}\right\}$ is complete in $H$.

$$
\left\|u-\sum_{k=1}^{N} \alpha_{k} \varphi_{k}\right\|_{A}<\epsilon \Rightarrow\left\|u-\sum_{k=1}^{N} \alpha_{k} \varphi_{k}\right\|<\epsilon / \sqrt{a}
$$

## Sturm-Liouville operator

Theorem The spectrum of the Sturm-Liouville operator $L$ in $L_{2}[a, b]$ is discrete and consists of an infinite number of eigenvalues tending to $+\infty$. The corresponding eigenfunctions form a complete system in $L_{2}[a, b]$.

Consider the energy norm

$$
\|u\|_{L}^{2}=\int_{a}^{b}\left(p(x) u^{\prime 2}+q(x) u^{2}\right) d x \geq p_{0} \int_{a}^{b} u^{\prime 2} d x
$$

It follows that every set bounded in the energy norm is bounded in $W_{2}^{1}$ norm. But every such set is compact in $L_{2}$, since

$$
u(x)=\int_{a}^{x} u^{\prime}(t) d t=\int_{a}^{b} k(x, t) u^{\prime}(t) d t
$$

where

$$
k(x, t)= \begin{cases}1, & a \leq t \leq x \\ 0, & x<t \leq b\end{cases}
$$

is a Hilbert-Schmidt operator.
Ex 4 Write down all details concerning the spectral theorem for SturmLiouville operators.
(AG 103-112 (Dover 63-71))
Spectral projector for Hermitian matrices
Projection operator
Let $G$ be a subspace of the Hilbert space $H$. Then every vector $h \in H$ can be written in the form

$$
h=h^{\|}+h^{\perp},
$$

where $h^{\|} \in G, h^{\perp} \perp G$. The operator

$$
P_{G}: h \mapsto h^{\|}
$$

is called the projector operator on $G$.
Proposition Every projection operator $P$ possesses the following properties

1. $\|P\|=1$;
2. $P^{2}=P$;
3. $P^{*}=P$;
4. $P \geq 0$.

Theorem If $P$ is an operator defined everywhere in $H$, and such that, for all $h_{1}, h_{2} \in H$

1) $\left\langle P^{2} h_{1}, h_{2}\right\rangle=\left\langle P h_{1}, h_{2}\right\rangle$
2) $\left\langle P h_{1}, h_{2}\right\rangle=\left\langle h_{1}, P h_{2}\right\rangle$,
then there is a sub-space $G \in H$ such that $P=P_{G}$.
Proof The operator $P$ is bounded by 1 , since

$$
\|P h\|^{2}=\langle P h, P h\rangle=\left\langle P^{2} h, h\right\rangle=\langle P h, h\rangle \leq\|P h\|\|h\|
$$

Let $G$ be the eigensubspace corresponding to the eigenvalue 1

$$
g \in G \Leftrightarrow P g=g .
$$

The subspace $G$ is closed linear subspace. Moreover Range $(P)=G$. The same is true for the projector $P_{G}$. Therefore let us prove that $\langle P f, g\rangle=$ $\left\langle P_{G} f, g\right\rangle$ for any $g \in G$

$$
\langle P f, g\rangle=\langle f, P g\rangle=\left\langle f, P_{G} g\right\rangle=\left\langle P_{G} f, g\right\rangle \Rightarrow P=P_{G} .
$$

## Theorems

1. The product of two projection operators $P_{G_{1}}$ and $P_{G_{2}}$ is a projection operator iff these operators commute

$$
P_{G_{1}} P_{G_{2}}=P_{G_{2}} P_{G_{1}}
$$

if this condition is satisfied, then

$$
P_{G_{1}} P_{G_{2}}=P_{G}
$$

where $G=G_{1} \cap G_{2}$.
2. The two sub-spaces $G_{1}$ and $G_{2}$ are orthogonal iff

$$
P_{G_{1}} P_{G_{2}}=0 .
$$

3. A sum of projections is a projection operator iff the subspaces are pairwise orthogonal
4. The difference of two projectors

$$
P_{G_{1}}-P_{G_{2}}
$$

is a projector iff $G_{2} \in G_{1}$, then $G=G_{1} \ominus G_{2}$.

## Proof

1. a) Let $P_{G_{1}} P_{G_{2}}$ be a projector $\Rightarrow$

$$
P_{G_{1}} P_{G_{2}}=\left(P_{G_{1}} P_{G_{2}}\right)^{*}=P_{G_{2}}^{*} P_{G_{1}}^{*}=P_{G_{2}} P_{G_{1}} .
$$

b) Suppose that $P_{G_{1}} P_{G_{2}}=P_{G_{2}} P_{G_{1}} \Rightarrow$

$$
\begin{aligned}
& \left(P_{G_{1}} P_{G_{2}}\right)^{2}=\ldots=P_{G_{1}} P_{G_{2}} \\
& \left(P_{G_{1}} P_{G_{2}}\right)^{*}=\ldots=P_{G_{1}} P_{G_{2}}
\end{aligned}
$$

c)

$$
\left.\begin{array}{l}
g \in \operatorname{Range}\left(P_{G_{1}} P_{G_{2}}\right) \Rightarrow g \in G_{1} \\
g \in \operatorname{Range}\left(P_{G_{2}} P_{G_{1}}\right) \Rightarrow g \in G_{2}
\end{array}\right\} \Rightarrow g \in G_{1} \cap G_{2}
$$

2. $G_{1} \perp G_{2} \Rightarrow P_{G_{1}} P_{G_{2}}=0$.

$$
P_{G_{1}} P_{G_{2}}=0: h \xrightarrow{P_{G_{1}}} g \in G_{1} \text { arbitrary } \xrightarrow{P_{G_{2}}} 0 \Rightarrow G_{1} \perp G_{2}
$$

3. Consider the operator

$$
Q=P_{G_{1}}+P_{G_{2}}+\ldots+P_{G_{n}} .
$$

Suppose that $G_{j} \perp G_{k} \Rightarrow P_{G_{j}} P_{G_{k}}=0$

$$
\Rightarrow\left\{\begin{array}{l}
Q^{2}=\ldots=Q \\
Q^{*}=\ldots=Q
\end{array} \Rightarrow Q\right. \text { is a projection. }
$$

Theorem If $\left\{P_{k}\right\}$ is an infinite monotonic sequence of projection operators, then as $k \rightarrow \infty P_{k}$ converges strongly to some projector $P$.
NB! The theorem is not true for the convergence in the operator norm.
Proof Suppose that the sequence is increasing $P_{k+1} \geq P_{k}$. Then the sequence of subspaces $G_{k}$ is increasing as well: $G_{k+1} \supset G_{k}$. Let us denote by $G$ the closure of the union of all $G_{k}: G=\overline{\cup G_{k}}$. Then $G$ possesses the following decomposition

$$
G=G_{1} \oplus\left(G_{2} \ominus G_{1}\right) \oplus\left(G_{3} \ominus G_{2}\right) \oplus \ldots
$$

Then every vector $f \in \mathcal{H}$ can be presented as an orthogonal sum

$$
f=f_{0}+f_{1}+f_{2}+\ldots, f_{0} \in G^{\perp}, f_{j} \in G_{j} \ominus G_{j-1}
$$

The norm $f$ can be calculated using

$$
\|f\|^{2}=\left\|f_{1}\right\|^{2}+\left\|f_{2}\right\|^{2}+\ldots
$$

Then the sequence $P_{k} f$ converges strongly to $P_{G} f$

$$
\left\|P_{g} f-P_{k} f\right\|^{2}=\left\|\sum_{j=k+1}^{\infty} P_{j} f\right\|^{2}=\sum_{j=k+1}^{\infty}\left\|f_{j}\right\|^{2} \rightarrow 0, k \rightarrow \infty
$$

## Introduction into the theory of unbounded linear operators

(Birman, Solomyak 60-72)
Linear operator:
domain of the operator $\operatorname{Dom}(A)$ - linear subset of the Hilbert space $\mathcal{H}$;
linear mapping

$$
\begin{aligned}
A: & \operatorname{Dom}(A) \rightarrow \mathcal{H} \\
& x \mapsto A x \in \operatorname{Range}(A) \subset \mathcal{H} .
\end{aligned}
$$

Two operators $A_{1}$ and $A_{2}$ are equal if and only if their domains are equal as well as the corresponding mappings.

## Graph norm

The domain of the operator is a pre-Hilbert space (=not necessarily complete) with respect to the inner product

$$
\langle x, y\rangle_{A}=\langle x, y\rangle+\langle A x, A y\rangle .
$$

Kernel of the operator $N(A)=\{x \in \operatorname{Dom}(A): A x=0\}$.
Theorem
Let $A$ be a linear operator on $\mathcal{H}$. Then $A$ has a bounded inverse if and only if

$$
\|T x\| \geq c\|x\|, c>0, \forall x \in \operatorname{Dom}(A) .
$$

Graph of an operator $\Gamma(A)$ - subset of $\mathcal{H} \oplus \mathcal{H}$

$$
\{(x, y): x \in \operatorname{Dom}(A), y=A x\} .
$$

The scalar product between any two vectors in $\Gamma(A)$ is equal to the corresponding graph inner product

$$
\langle(x, A x),(y, A y)\rangle_{\mathcal{H} \oplus \mathcal{H}}=\langle x, y\rangle+\langle A x, A y\rangle=\langle x, y\rangle_{A} .
$$

## Theorem

Linear set $M \subset \mathcal{H} \oplus \mathcal{H}$ is a graph of a linear operator iff

$$
N\left(\left.P_{1}\right|_{M}\right)=\{0\}
$$

where $P_{1}$ denotes the projector operator onto $\mathcal{H} \oplus\{0\}$ in $\mathcal{H} \oplus \mathcal{H}$.

## Closed operator

Three equivalent definitions
I An operator $A$ is closed iff the domain the operator is complete with respect to the graph norm $\langle x, y\rangle_{A}$.
II An operator $A$ is closed iff $\Gamma(A)$ is closed in $\mathcal{H} \oplus \mathcal{H}$.
III An operator $A$ is closed if relations

$$
x_{n} \in \operatorname{Dom}(A), \lim _{n \rightarrow \infty} x_{n}=x, \lim _{n \rightarrow \infty} A x_{n}=y
$$

imply that

$$
x \in \operatorname{Dom}(A), \quad A x=y
$$

These definitions are equivalent. Note the difference between bounded and closed operators.

Theorem
A bounded operator $A$ is closed if and only if the domain of the operator is closed.

## Theorem

Let $A_{1}, A_{2}, A_{3}$ be linear operators, $A_{1} \subset A_{2} \subset A_{3}, A_{1}$ and $A_{3}$ are closed operators, and

$$
\operatorname{dim}\left[\operatorname{Dom}\left(A_{3}\right) / \operatorname{Dom}\left(A_{1}\right)\right]<\infty
$$

Then $A_{2}$ is closed.
Theorem (Important!)
If $A$ is closed and $\operatorname{Dom}(A)$ is a subspace of the Hilbert space $\mathcal{H}$, then $A$ is a bounded operator.
(to be proven later)

## Theorem

Let $A$ be a closed operator having an inverse and a closed range. Then $A^{-1}$ is bounded.
Proof The graph norms of the operator and its inverse coincide. Hence the inverse operator is closed and its domain $\operatorname{Dom}\left(A^{-1}\right)=\operatorname{Range}(A)$ is a subspace of $\mathcal{H}$.

## Closure of an operator

Suppose that $\overline{\Gamma(A)}$ is a graph of an operator. Then this operator is called the closure of $A$ and denoted by $\bar{A}$.

An operator $A$ is closable iff for any sequence $\left\{x_{n}\right\}$ in $\operatorname{Dom}(A)$ satisfying $\lim _{n \rightarrow \infty} x_{n}=0$ and $\lim _{n \rightarrow \infty} T x_{n}=y$ we have $y=0$.

Example 1.
$\mathcal{H}=L_{2}(0,1), B x(t)=\frac{1}{i} \frac{d}{d t} x(t), \operatorname{Dom}(B)=C_{0}^{\infty}(0,1)$. Then the closure of $B$ is the second derivative operator with the domain

$$
x \in W_{2}^{1}(0,1), x(0)=0
$$

Example 2.
$\mathcal{H}=L_{2}(0,1), A x(t)=x(0) \mathbf{1}(t), \operatorname{Dom}(A)=W_{2}^{1}(0,1)$. The operator $A$ is not closable

$$
x_{n}(t)=(1-t)^{n} .
$$

The inner product is given by

$$
\langle x, y\rangle=\int_{0}^{1} x(t) \overline{y(t)} d t+x(0) \overline{y(0)}
$$

Closure of this operator can be defined in the space $L_{2}(0,1) \oplus \mathbf{C}$.

## Adjoint operator

The domain $\operatorname{Dom}\left(A^{*}\right)$ of the adjoint operator $A^{*}$ consists of all vectors $y \in \mathcal{H}$ such that $\langle A x, y\rangle$ is a bounded linear functional with respect to $x$, i.e.

$$
\langle A x, y\rangle \leq C_{y}\|x\|
$$

Every such functional is given by a certain element $h \in \mathcal{H}$. The mapping $y \mapsto h$ is linear and determines the adjoint operator.

1. Adjoint operators are defined for densely defined operators only, otherwise the element $h$ is not unique.
2. The domain $\operatorname{Dom}\left(A^{*}\right)$ is never empty ( $\left.\ni 0\right)$.

Let us introduce the operator $W: \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H} ;(x, y) \mapsto(-y, x)$.

# Introduction into the theory of unbounded linear operators 

(continuation)

(Birman, Solomyak 68-75)

## Theorem

Let $A$ be a densely defined linear operator. Then

$$
[W \Gamma(A)]^{\perp}=\Gamma\left(A^{*}\right) .
$$

## Corollary

1. The operator $A^{*}$ is linear and closed.
2. If $A$ is closable, then $(\bar{A})^{*}=A^{*}$.

Theorem
a) $B$ bounded operator $\Rightarrow(A+B)^{*}=A^{*}+B^{*}$.
b) $B, B^{-1}$ - bounded operators $\Rightarrow(A B)^{*}=B^{*} A^{*},(B A)^{*}=A^{*} B^{*}$.

Proof. a) Let $y \in \operatorname{Dom}(A)$

$$
\langle A x+B x, y\rangle=\langle A x, y\rangle+\langle B x, y\rangle .
$$

$\langle B x, y\rangle$ is a bounded linear functional with respect to $x$ for any $y$. Therefore $\langle A x+B x, y\rangle$ is a bounded linear functional iff $\langle A x, y\rangle$ is bounded with respect to $x$. Hence $\operatorname{Dom}\left((A+B)^{*}\right)=\operatorname{Dom}\left(A^{*}\right)$. Let $x \in \operatorname{Dom}(A), y \in \operatorname{Dom}\left(A^{*}\right)$, then equality

$$
\langle A x+B x, y\rangle=\left\langle x, A^{*} y\right\rangle+\left\langle x, B^{*} y\right\rangle
$$

implies that the proposition.
b) We note first that $\operatorname{Dom}(A B)=B^{-1} \operatorname{Dom}(A)$, i.e. every element $x \in$ $\operatorname{Dom}(A B)$ possesses the representation $x=B^{-1} z, z \in \operatorname{Dom}(A)$. The norms of $x$ and $z$ are equivalent

$$
c\|x\| \leq\|z\| \leq C\|x\| .
$$

Consider any $x=B^{-1} z \in \operatorname{Dom}(A B)$, then

$$
\langle A B x, y\rangle=\langle A z, y\rangle
$$

determines a bounded linear functional with respect to $x$ iff it determines a bounded linear functional with respect to $z$. It follows that $\operatorname{Dom}\left((A B)^{*}\right)=$
$\operatorname{Dom}\left(A^{*}\right)$. Then the following equality proofs the theorem

$$
\langle A B x, y\rangle=\langle A(B x), y\rangle=\left\langle B x, A^{*} y\right\rangle=\left\langle x, B^{*} A^{*} y\right\rangle .
$$

## Theorem

The subspaces Range $(A)$ and $\operatorname{Kernel}\left(A^{*}\right)$ are orthogonal in $\mathcal{H}$ and

$$
\mathcal{H}=\overline{\operatorname{Range}(A)} \oplus \operatorname{Kernel}\left(A^{*}\right) .
$$

In particular:

$$
\mathcal{H}=\overline{\operatorname{Range}(A-\lambda I)} \oplus \operatorname{Kernel}\left(A^{*}-\bar{\lambda} I\right)
$$

Without proof:

## Theorem

Let $\overline{\operatorname{Dom}(A)}=\overline{\text { Range }(A)}=\mathcal{H}$ and let $A$ have an inverse. Then the adjoint $A^{*}$ also has an inverse and

$$
\left(A^{*}\right)^{-1}=\left(A^{-1}\right)^{*} .
$$

## Theorem

Let $A$ be a densely defined and closable, then

$$
A^{* *}=\bar{A} .
$$

The proof is based on the formula valid for closed operators:

$$
W \Gamma(T) \oplus \Gamma\left(A^{*}\right)=\mathcal{H} \oplus \mathcal{H}
$$

## Theorem (The closed graph theorem)

Let $A$ be a closed operator in the Hilbert space $\mathcal{H}$ and its domain is a subspace of $\mathcal{H}$. Then $A$ is bounded.

## Proof

1. There is a ball in $\operatorname{Dom}(A)$, such that the vectors mapped to a bounded set form a dense subset. Let us denote by $S$ the following subset of $\operatorname{Dom}(A)$

$$
S=\{u \in \operatorname{Dom}(A):\|A u\|<1\} .
$$

Then $\operatorname{Dom}(A)$ is the union of the open sets $S, 2 S, 3 S, \ldots$. It follows from Bair's theorem ${ }^{1}$ that $\bar{S}$ contains a ball, say $B\left(u_{0}, r\right)$.
2. This ball can be chosen having center at the origin. Any $u \in \operatorname{Dom}(A), \|$ $u \|<2 r$ can be written in the form $u=u^{\prime}-u^{\prime \prime}$, with $u^{\prime}, u^{\prime \prime} \in B\left(u_{0}, r\right)$. Since $B \subset \bar{S}$ there are two sequences

$$
u_{n}^{\prime} \rightarrow u^{\prime}, \quad u_{n}^{\prime \prime} \rightarrow u^{\prime \prime}, u_{n}^{\prime}, u_{n}^{\prime \prime} \in S
$$

From $\left\|A\left(u_{n}^{\prime}-u_{n}^{\prime \prime}\right)\right\| \leq\left\|A u_{n}^{\prime}\right\|+\left\|A u_{n}^{\prime \prime}\right\| \leq 2$ it follows that $u=\lim _{n \rightarrow \infty}\left(u_{n}^{\prime}-\right.$ $\left.u_{n}^{\prime \prime}\right) \in 2 \bar{S}$. Using homogeneity we conclude that the ball $B(0, \lambda r)$ is a subset of $\lambda \bar{S}$ for any $\lambda>0$.
3. The operator $A$ maps the whole ball to a bounded set. Consider arbitrary $u \in \operatorname{Dom}(A),\|u\|<r$, we shall prove that $\|A u\|$ is uniformly bounded, i.e. that the operator $A$ maps a unit ball to a bounded set. Take any $\epsilon, 0<\epsilon<1$. $u \in \bar{S} \Rightarrow$ there exists $u_{1} \in S$, such that $\left\|u-u_{1}\right\| \leq \epsilon r$. The difference $u-u_{1}$ belongs to the set $\epsilon \bar{S}$ and therefore there exists $u_{2} \in \epsilon S\left(\left\|A u_{2}\right\|<\epsilon\right)$ such that $\left\|u-u_{1}-u_{2}\right\| \leq \epsilon^{2} r$. Proceeding in this way we get the sequence $u_{n}$ with the properties

$$
\left\|u-u_{1}-u_{2}-\ldots-u_{n}\right\|<\epsilon^{n} r, \quad\left\|A u_{n}\right\|<\epsilon^{n-1}
$$

Both sequences $\sum_{k=1}^{n} u_{k}$ and $A \sum_{k=1}^{n} u_{k}$ are Cauchy sequences. Moreover $\sum_{k=1}^{n} u_{k} \rightarrow u$ and it follows that

$$
\|A u\|=\left\|\lim _{n \rightarrow \infty} \sum_{k=1}^{n} A u_{k}\right\| \leq \sum_{k=1}^{n} \epsilon^{k-1}=(1-\epsilon)^{-1}
$$

It follows that the operator $A$ is bounded and

$$
\|A\| \leq 1 / r
$$

since the real number $\epsilon$ is arbitrary between 0 and 1 .

## Perturbation theory for closed operators

Perturbation by bounded operators

[^0]
## Theorem

Let $A$ be a closed operator and $B$ be a bounded operator, such that $\operatorname{Dom}(B) \supset$ $\operatorname{Dom}(A)$. Then the operator $A+B$ defined on the domain $\operatorname{Dom}(A)$ is closed. Proof Let $x_{n}$ be an arbitrary sequence from $\operatorname{Dom}(A)$ such that

$$
\lim _{n \rightarrow \infty} x_{n}=x, \lim _{n \rightarrow \infty}(A+B) x_{n}=y
$$

Since the operator $B$ is bounded and $\lim _{n \rightarrow \infty} x_{n}=x$, the sequence $B x_{n}$ converges to the vector $B x$. Hence the sequence $x_{n}$ satisfies the two conditions

$$
\lim _{n \rightarrow \infty} x_{n}=x, \lim _{n \rightarrow \infty} A x_{n}=y-B x .
$$

Using that the operator $A$ is closed one concludes that $x \in \operatorname{Dom}(A)$ and $y-B x=A x$. The theorem is proven.

Perturbation by dominated operators
Def Operator $B$ is called $A$-bounded iff
$\operatorname{Dom}(B) \supset \operatorname{Dom}(A)$;
$\|B u\|^{2} \leq a^{2}\|A u\|^{2}+b^{2}\|u\|^{2}$.
The operator $B$ is called strongly dominated by $A$ if the constant $a$ can be chosen less than 1.

Equivalent condition

$$
\begin{equation*}
\|B u\| \leq \alpha\|A u\|+\beta\|u\| \tag{*}
\end{equation*}
$$

## Theorem

Let $A$ be a closed operator and $B$ be strongly dominated by $A$. Then the operator $A+B$ defined on $\operatorname{Dom}(A)$ is closed.
Proof. Suppose that $\beta \leq \alpha$ in (*), then the graph norms for the operator $A$ and $A+B$ are equivalent

$$
\begin{aligned}
& \|A x+B x\|+\|x\| \geq\|A x\|-\|B x\|+\|x\| \geq(1-\alpha)(\|A x\|+\|x\|) . \\
& \|A x+B x\|+\|x\| \leq\|A x\|+\|B x\|+\|x\| \leq(1+\alpha)(\|A x\|+\|x\|) .
\end{aligned}
$$

Since the operator $A$ is closed, its domain is closed with respect to the graph norm $\|x\|_{A}$, but then it is closed with respect to the graph norm $\|x\|_{A+B}$, i.e. the operator $A+B$ is closed on the domain $\operatorname{Dom}(A)$.

The case $\beta>\alpha$ can be treated using the operators $q A$ and $q B$ with $q=\alpha / \beta$.

Ex 5 Prove that the inverse operator to any invertible closed operator $A$ with Range $(A)$ being a subspace of $\mathcal{H}$ is bounded.

## Lecture 7

03.03.10

Please read paragraphs 5 and 6 from Birman, Solomyak yourself. (Pages 75-81)

## Defect Number, Spectrum and Resolvent

(Birman, Solomyak 81-86, AG 124-139(Dover 88-93))
All operators in this section are closed and densely defined!!
Def Defect number - the dimension of the orthogonal complement to Range ( $A$ )

$$
d_{A}=\operatorname{dim}(\mathcal{H} \ominus \text { Range }(A))=\operatorname{dim} \operatorname{Kernel}\left(A^{*}\right)
$$

The defect number is the number of conditions one needs to impose in order to guarantee that the equation $A u=f$ is solvable, provided that the operator $A$ is invertible. In particular, if $d_{A}=0$, then the inverse operator is defined on the whole Hilbert space if $\|A u\| \geq c\|u\|$.

Theorem
Let $A$ be a closed operator satisfying

$$
\|A u\| \geq c\|u\|, \quad \forall u \in \operatorname{Dom}(A)
$$

, $\operatorname{Dom}(B) \supset \operatorname{Dom}(A)$ and

$$
\|B u\| \leq a\|A u\|, \quad a<1, u \in \operatorname{Dom}(A)
$$

Then $A+B$ is closed on $\operatorname{Dom}(A)$ and the defect numbers of $A$ and $A+B$ are equal

$$
d_{A+B}=d_{A}
$$

Proof The operator $B$ is strongly dominated by $A$ and thus the operator sum $A+B$ is closed on the domain of the original operator. Moreover

$$
\|(A+B) u\| \geq\|A u\|-\|B u\| \geq(1-a) c\|u\|,
$$

and it follows that Range $(A+B)$ is a subspace. To prove that $d_{A+B}=d_{A}$ consider two possibilities:
$1 d_{A+B}<d_{A}$ Then there exists a certain vector $u \in \operatorname{Dom}(A)$ such that $A u$ is orthogonal to Range $(A+B)$ and in particular $A u \perp(A+B) u$

$$
\rightarrow\langle(A+B) u, A u\rangle=0 \Rightarrow\|A u\|^{2}=|\langle A u, B u\rangle| \leq c\|A u\|^{2},
$$

which is possible only if $u=0$.
$2 d_{A+B}>d_{A}$ Then there exists a certain vector $u \in \operatorname{Dom}(A)$ such that $(A+B) u$ is orthogonal to Range $(A)$ and in particular $(A+B) u \perp A u$. The same calculations lead to a contradiction.

It follows that $d_{A+B}=d_{A}$.
Consider the family of operator $A-\lambda I, \lambda \in \mathbf{C}$.
Def The defect number of $A$ at $\lambda$

$$
d_{A}(\lambda)=d_{A-\lambda I}
$$

Def The quasiregular set of $A \hat{\rho}(A)$ - the set of points $\lambda$ for which $A-\lambda I$ has continuous inverse on Range $(A-\lambda I)$

$$
\|(A-\lambda) u\| \geq c\|u\|, \forall u \in \operatorname{Dom}(A)
$$

## Theorem

The set $\hat{\rho}(A)$ is open. The function $d_{A}(\lambda)$ is constant on each connected component of $\hat{\rho}(A)$.
Proof We prove first that each point $\lambda_{0}$ belongs to $\hat{\rho}$ together with a certain neighborhood. Really $\lambda_{0} \in \hat{\rho}(A)$ implies that there exists a certain constant $c_{0}$ such that the following estimate holds

$$
\left\|\left(A-\lambda_{0}\right) u\right\| \geq c_{0}\|u\|, \forall u \in \operatorname{Dom}(A)
$$

Take any $\lambda$ from the disk $\left|\lambda-\lambda_{0}\right|<c_{0}$ and consider the decomposition

$$
A-\lambda I=\left(A-\lambda_{0} I\right)+\left(\lambda_{0}-\lambda\right) I
$$

It follows that

$$
\|(A-\lambda) u\| \geq\left(c_{0}-\left|\lambda_{0}-\lambda\right|\right)\|u\|
$$

which implies that the point $\lambda$ is inside $\hat{\rho}$.
Consider any two points belonging to the same connected component of $\hat{\rho}$. The path connected these points can be covered by a finite number of disks. Therefore the defect number is constant on each connected component.

Def Regular points - the values of the parameter $\lambda$ for which the inverse operator $(A-\lambda I)^{-1}$ exists and is a bounded operator defined everywhere in H

$$
d_{A}(\lambda)=0 .
$$

The set of all regular points will be denoted by $\rho(A)$.
Def Spectrum of the operator $A$ - all points from $\mathbf{C}$ which are not regular

$$
\sigma(A)=\mathbf{C} \backslash \rho(A)
$$

Theorem The correspondence between $\operatorname{Dom}(A)$ and $R(A-\lambda I)$ determined by the operator $A-\lambda I$ is one-to-one iff $\lambda$ is not an eigenvalue of the operator $A$.

Def Self-adjoint operator - $A=A^{*}$.
Theorem A number $\lambda$ is an eigenvalue of a self-adjoint operator $A$ iff

$$
\overline{R(A-\lambda I)} \neq H
$$

The eigensubspace corresponding to the eigenvalue $\lambda$ can be calculated as follows

$$
G(\lambda)=H \ominus \overline{R(A-\lambda I)}
$$

Proof. The proof is based on the formula

$$
N(A-\lambda I)=H \ominus \overline{R(A-\lambda I)}
$$

Theorem (Boundedness of the resolvent of a self-adjoint operator) Non-real points in the complex $\lambda$-plane are regular points for any self-adjoint operator $A$.
Proof.

$$
g=(A-\lambda I) f \Rightarrow\|g\|^{2} \geq(\Im \lambda)^{2}\|f\|^{2}
$$

Hence

$$
\begin{aligned}
\left\|(A-\lambda I)^{-1} g\right\| & \leq \frac{1}{|\Im \lambda|}\|g\| \\
\sigma(A) & \subset \mathbf{R}
\end{aligned}
$$

Corollary The set of regular points of a self-adjoint operator $A$ coincide with the set of points for which $R(A-\lambda I)=H$.

## Def

Point spectrum of a self-adjoint operator $A$ - the set of points for which $\overline{\overline{R(A-\lambda I)} \neq H}$.
Continuous spectrum of a self-adjoint operator $A$ - the of point for which $R(A-\lambda I) \neq \overline{R(A-\lambda I)}$.

NB Sometimes the eigenvalues of infinite multiplicity are included into continuous spectrum (like in AG book).

Theorem The spectrum of a self-adjoint operator is closed.

## The resolvent

## Lecture 8

## Resolvent

Def The operator-valued function $R_{\lambda}(A)=(A-\lambda I)^{-1}$ defined for $\lambda \in$ $\rho(A)$ is called the resolvent of $A$.

Hilbert identity:

$$
R_{\lambda}-R_{\mu}=(\lambda-\mu) R_{\lambda} R_{\mu}
$$

Theorem The resolvent $R_{\lambda}(A)$ depends analytically on $\lambda \in \rho(A)$. In a neighborhood of each point $\lambda_{0} \in \rho(A)$ the resolvent is represented by the power series

$$
R_{\lambda}(A)=\sum_{0}^{\infty}\left(\lambda-\lambda_{0}\right)^{k} R_{\lambda_{0}}^{k+1}
$$

converging uniformly in the disk $\left|\lambda-\lambda_{0}\right|<\left\|R_{\lambda_{0}}\right\|$.
Ex 6 Calculate the resolvents of the following operators:

$$
\begin{array}{cl}
L_{1} & =-\frac{d^{2}}{d x^{2}} \text { in } L_{2}(0, \pi) \\
\operatorname{Dom}\left(L_{1}\right) & =\left\{\psi \in W_{2}^{2}(0, \pi): \psi(0)=\psi(\pi)=0\right\} \\
L_{2} & =-\frac{d^{2}}{d x^{2}} \text { in } L_{2}(0, \pi) \\
\operatorname{Dom}\left(L_{2}\right) & =\left\{\psi \in W_{2}^{2}(0, \pi): \psi^{\prime}(0)=\psi^{\prime}(\pi)=0\right\}
\end{array}
$$

Check that the singularities of the resolvents coincide with the eigenvalues of the corresponding operators. Show that the difference between the resolvents is an operator of rank two.

## Symmetric and self-adjoint operators

(Birman, Solomyak 97-100, AG 128-131(Dover 85-87))
Operator $A$ is called symmetric iff

$$
\langle A x, y\rangle=\langle x, A y\rangle
$$

for any $x, y \in \operatorname{Dom}(A)$.
Operator $A$ is called self-adjoint iff $A=A^{*}$.
Every self-adjoint operator is symmetric, but the opposite is not true in general.

The adjoint operator to any symmetric operator is an extension of the symmetric operator:

$$
A-\text { symmetric } \Rightarrow A^{*} \supset A .
$$

## Lemma

Let $A$ be a symmetric operator, $\tilde{A}$ - its symmetric extension, then the following relation holds

$$
A \subset \tilde{A} \subset \tilde{A}^{*} \subset A^{*}
$$

Maximal symmetric operator - the operator that cannot be extended to another different symmetric operator. Such operator is not necessarily self-adjoint.

## Theorem

Let $A$ be a symmetric operator. Then the upper half-plane $\Im \lambda>0$ and the lower half-plane $\Im \lambda<0$ are contained in the quasiregular set $\hat{\rho}(A)$.

## Proof

$$
\begin{aligned}
\|(A-\lambda) f\|^{2} & =\|(A-a-i b) f\|^{2} \\
& =\|(A-a) f\|^{2}+|b|^{2}\|f\|^{2}+2 \Re i b\langle(A-a) f, f\rangle \\
& =\|(A-a) f\|^{2}+|b|^{2}\|f\|^{2} \\
& \geq \mid \Im \lambda\|f\|^{2}
\end{aligned}
$$

The operator $A-\lambda$ is boundedly invertible if $\Im \lambda \neq 0$.
Corollary
The core of the spectrum of a closed symmetric operator is contained in $\mathbf{R}$.
Deficiency indices $n_{ \pm}(A)$ for a symmetric operator $A$ - the defect numbers of the operator in the upper and lower half-planes.

$$
n_{ \pm}(A)=\operatorname{dim}\left(A^{*} \mp i\right)
$$

## Lemma

If a symmetric operator $A$ has at least one real quasiregular point, then $n_{+}(A)=n_{-}(A)$.

In particular: any semibounded symmetric operator has equal deficiency indices.

Theorem
In order that a closed symmetric operator $A$ be self-adjoint it is necessary and sufficient that $n_{+}(A)=n_{-}(A)=0$.
Proof.
Necessity
Let $A$ be self-adjoint, $\lambda \notin \mathbf{R}$ then

$$
\operatorname{Kernel}\left(A^{*}-\lambda\right)=\operatorname{Kernel}(A-\lambda)=0 \Rightarrow n_{ \pm}(A)=0
$$

Sufficiency. Take any $y \in \operatorname{Dom}\left(A^{*}\right)$. Calculate $h=\left(A^{*}+i\right) y$. Since $n_{ \pm}(A)=$ $0 \Rightarrow$ Range $(A \mp i)=\mathcal{H}$, there exists certain $y_{0} \in \operatorname{Dom}(A)$ such that $(A+$ i) $y_{0}=h$. For any $x \in \operatorname{Dom}(A)$ the following chain of equalities holds

$$
\langle(A-i) x, y\rangle=\left\langle x,\left(A^{*}+i\right) y\right\rangle=\langle x, h\rangle=\left\langle x,(A+i) y_{0}\right\rangle=\left\langle(A-i) x, y_{0}\right\rangle
$$

It follows that $y_{0}=y$ since $(A-i) x$ runs over the whole Hilbert space $\mathcal{H}$.
Corollary
A symmetric operator $A$ such that the range of $A-\lambda$ is all of $\mathcal{H}$ is self-adjoint.
Ex 7 Prove that one of the following operators is self-adjoint

$$
\begin{aligned}
L_{3} & =-\frac{d^{2}}{d x^{2}} \text { in } L_{2}(0, \infty) \\
\operatorname{Dom}\left(L_{3}\right) & =\left\{\psi \in W_{2}^{2}(0, \infty): \psi(0)=\psi^{\prime}(0)=0\right\} \\
L_{4} & =-\frac{d^{2}}{d x^{2}} \text { in } L_{2}(0, \infty) \\
\operatorname{Dom}\left(L_{4}\right) & =\left\{\psi \in W_{2}^{2}(0, \infty): \psi(0)=0\right\}
\end{aligned}
$$

## Lecture 9

03.03.31

## Isometric operators

## Def

Linear operator $V$ is called isometric iff for any $x \in \operatorname{Dom}(V)$ the following equality holds

$$
\|V x\|=\|x\| .
$$

The kernel of any isometric operator is trivial $\Rightarrow$ every isometric operator is invertible and the inverse operator $V^{-1}$ defined on $\operatorname{Dom}\left(V^{-1}\right)=$ Range $(V)$ is isometric.

## Theorem

The core of the spectrum of an isometric operator belongs to the unit circle.
Proof Consider any $z,|z| \neq 1$.

$$
\|(V-z) x\| \geq|\|V x\|-z\|x\||=|1-|z||\|x\|
$$

$\Rightarrow$ the operator $(V-z)^{-1}$ is bounded and $z$ belongs to the quasiregular set.

## Deficiency indices

$n_{i}(V)=\operatorname{def} R(V-z I),|z|<1$,
$n_{e}(V)=\operatorname{def} R(V-z I),|z|>1$.

## Theorem

Let $V$ be an isometric operator.

$$
n_{i}(V)=\operatorname{def} \operatorname{Range}(V), \quad n_{e}(V)=\operatorname{def} \operatorname{Dom}(V)
$$

## Proof

$$
\begin{gathered}
n_{i}(V)=\operatorname{def} \operatorname{Range}(V-0)=\operatorname{def} \operatorname{Range}(V) . \\
\operatorname{def} \operatorname{Dom}(V)=d_{I_{\operatorname{Dom}(V)}}=d_{z I_{\operatorname{Dom}(V)}}=d_{z I_{\operatorname{Dom}(V)}-V}
\end{gathered}
$$

Vi use the fact that perturbation of the operator $z I_{\operatorname{Dom}(V)}$ by the operator $V$ does not change the defect, since

$$
\|V x\| \leq(1-\epsilon)\left\|z I_{\operatorname{Dom}(V)} x\right\|=(1-\epsilon)|x|\|x\|
$$

## Lemma

The eigenvectors corresponding to different eigenvalues of an isometric operator are orthogonal.

## Def

Linear operator is called unitary iff

1. $\operatorname{Dom}(V)=\mathcal{H}$, Range $(V)=\mathcal{H}$;
2. it is isometric $\|V x\|=\|x\|$.

## Lemma

A linear operator is unitary iff

$$
V^{*} V=V V^{*}=I
$$

## Theorem

Let $V$ be an isometric operator. Then the adjoint operator defined originally on the whole $\mathcal{H}$ is isometric between the spaces $\operatorname{Dom}\left(V^{*}\right)=$ Range $(V)$ and Range $\left(V^{*}\right)=\operatorname{Dom}(V)$. Moreover

$$
V^{*} V=P_{\operatorname{Dom}(V)}, \quad V V^{*}=P_{\text {Range }(V)} .
$$

## Cayley transform

$$
\begin{gathered}
V=(A-\lambda I)(A-\bar{\lambda} I)^{-1} \\
h=(A-\bar{\lambda}) f, \quad V h=(A-\lambda) f \\
\|V h\|^{2}=\|(A-\lambda) f\|^{2}=\|(A-\alpha) f\|^{2}+\beta^{2}\|f\|^{2} \\
\|h\|^{2}=\|(A-\bar{\lambda}) f\|^{2}=\|(A-\alpha) f\|^{2}+\beta^{2}\|f\|^{2}
\end{gathered}
$$

The inverse Cayley transform

$$
A=(\bar{\lambda} V-\lambda I)(V-I)^{-1}
$$

It is necessary that $\overline{\operatorname{Range}(V-I)}=\mathcal{H}$. Point 1 is not an eigenvalue of V.

## Theorem

The Cayley transform

$$
V=(A-\lambda I)(A-\bar{\lambda} I)^{-1}
$$

is a one-to-one correspondence between the set of closed symmetric operators and the set of isometric operators satisfying

$$
\overline{\operatorname{Range}(V-I)}=\mathcal{H} \text {. }
$$

## Theorem

A symmetric operator $A$ is self-adjoint if and only if its Cayley transform is a unitary operator.
$\lambda=i$

$$
V=(A-i)(A+i)^{-1}, \quad A=-i(V+1)(V-1)^{-1}
$$

## Lecture 10

### 03.04 .06 <br> Extensions of Symmetric operators, von Neumann formulas

(Birman Solomyak 105-122, Akhiezer, Glazman vol 2, 91-101)
$V$ - closed isometric operator with the domain $\operatorname{Dom}(V)$ and range Range $(V)$.
Consider the subspaces $D_{0} \subset \mathcal{H} \ominus \operatorname{Dom}(V), R_{0} \subset \mathcal{H} \ominus$ Range $(V)$ and isometric operator

$$
V_{0}: D_{0} \rightarrow R_{0}\left(\Rightarrow \operatorname{dim} D_{0}=\operatorname{dim} R_{0}\right)
$$

Then the operator $\tilde{V}=V \oplus V_{0}$ is an isometric extension of $V$.
Moreover

$$
\begin{gathered}
n_{i}(\tilde{V})=\operatorname{dim}\left(\mathcal{H} \ominus \operatorname{Range}(\tilde{V})=n_{i}(V)-\operatorname{dim}\left(D_{0}\right)\right. \\
n_{e}(\tilde{V})=\operatorname{dim}\left(\mathcal{H} \ominus \operatorname{Dom}(\tilde{V})=n_{e}(V)-\operatorname{dim}\left(D_{0}\right)\right.
\end{gathered}
$$

Conclusions:
Isometric operator $V$ has a nontrivial extension iff both deficiency indices are different from zero;
Isometric operator $V$ can be extended to a unitary operator iff the deficiency indices are equal;
Every nontrivial extension of isometric operator $V$ is unitary if $n_{i}(V)=$ $n_{e}(V)=1$.

The Cayley transform makes it possible to translate these conclusions to the language of symmetric- self-adjoint operators
Symmetric operator $A$ has a nontrivial extension iff both deficiency indices are different from zero;
Symmetric operator $A$ can be extended to a self-adjoint operator iff the deficiency indices are equal;
Every nontrivial extension of symmetric operator $A$ is self-adjoint if $n_{+}(V)=$ $n_{-}(V)=1$.

## von Neumann formulae

Theorem
$A$ - closed symmetric operator; $\lambda$ - nonreal complex number. Then the domain of the adjoint operator possesses the decomposition

$$
\operatorname{Dom}\left(A^{*}\right)=\operatorname{Dom}(A) \dot{+} \operatorname{Kernel}\left(A^{*}-\lambda I\right) \dot{+} \operatorname{Kernel}\left(A^{*}-\bar{\lambda} I\right)
$$

Proof Let $y$ be an element from $\operatorname{Dom}\left(A^{*}\right)$. Then equality

$$
\mathcal{H}=\text { Range }(A-\lambda I) \oplus \operatorname{Kernel}\left(A^{*}-\bar{\lambda} I\right)
$$

implies that vector $\left(A^{*}-\lambda\right) y$ can be presented as follows

$$
\begin{gathered}
\left(A^{*}-\lambda\right) y=(A-\lambda) x+(\bar{\lambda}-\lambda) x_{\bar{\lambda}} . \\
\Rightarrow\left(A^{*}-\lambda\right)\left(y-x-x_{\bar{\lambda}}\right)=0 \\
\Rightarrow y-x-x_{\bar{\lambda}} \in \operatorname{Kernel}\left(A^{*}-\lambda\right) \\
\Rightarrow y=x+x_{\bar{\lambda}}+x_{\lambda}
\end{gathered}
$$

where

$$
x \in \operatorname{Dom}(A), x_{\bar{\lambda}} \in \operatorname{Kernel}\left(A^{*}-\bar{\lambda}\right), x_{\lambda} \in \operatorname{Kernel}\left(A^{*}-\lambda\right) .
$$

Action of the adjoint operator:

$$
A^{*}\left(x+x_{\lambda}+x_{\bar{\lambda}}\right)=A x+\lambda x_{\lambda}+\bar{\lambda} x_{\bar{\lambda}} .
$$

## Theorem

Let $D_{0} \subset \operatorname{Kernel}\left(A^{*}-\lambda I\right), R_{0} \subset \operatorname{Kernel}\left(A^{*}-\bar{\lambda} I\right)$ be subspaces of the same dimension. Let $V_{0}$ be a unitary isometric mapping of $D_{0}$ onto $R_{0}$. Then the restriction of the adjoint operator $A^{*}$ to the domain

$$
\operatorname{Dom}(\tilde{A})=\operatorname{Dom}(A) \dot{+}\left(V_{0}-I\right) D_{0}
$$

is a closed symmetric extension of the operator $A$.
Proof is based on the formula

$$
\operatorname{Dom}(A)=\operatorname{Range}(V-I)
$$

## Birman-Krein-Vishik theory for semibounded operators

$A$ - positive symmetric operator
A - any positive self-adjoint extension of $A$
Theorem
Then the domain of the adjoint operator possesses the decomposition

$$
\operatorname{Dom}\left(A^{*}\right)=\operatorname{Dom}(\mathbf{A}) \dot{+} \operatorname{Kernel}\left(A^{*}+I\right) .
$$

Action of the adjoint operator:

$$
A^{*}\left(x+x_{-1}\right)=\mathbf{A} x-x_{-1} .
$$

## Theorem

Self-adjoint extensions of the operator $A$ can be described as restrictions of the operator $A^{*}$ to the set of functions satisfying the following boundary conditions

$$
U x_{-1}=(\mathbf{A}+1) x
$$

where $U$ is a certain isometric operator.
(see the paper by Simon)

## I. Rank one perturbations

(after the book S.Albeverio and P.Kurasov Singular perturbations of differential operators, Cambridge 2000, chapter 1)

## Bounded perturbations

## Resolvent analysis

Let us start our investigation of finite rank perturbations of self-adjoint operators with the simplest sort of perturbation - a rank one bounded perturbation. Let $A$ be a self-adjoint (perhaps unbounded) operator in the Hilbert space $H$ with domain $\operatorname{Dom}(A)$. Let $\varphi$ be a vector from the Hilbert space, $\varphi \in H$ and $\alpha$ be a real number, $\alpha \in \mathbf{R}$. A symmetric rank one bounded perturbation of $A$ is the operator defined by the following formula

$$
\begin{equation*}
A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi, \tag{1}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ denotes the scalar product in the Hilbert space $H$. The rank one operator $\alpha\langle\varphi, \cdot\rangle \varphi$ is a bounded operator in the Hilbert space and the operator sum $A_{\alpha}$ is well defined. Actually the operator $A_{\alpha}$ is self-adjoint on the domain $\operatorname{Dom}(A)$ of the operator $A$. The spectral properties of the perturbed operator can be obtained using its resolvent, which can be calculated using Krein's formula connecting the resolvents of two self-adjoint extensions of one symmetric operator with finite deficiency indices. In fact the operators $A_{\alpha}$ and $A$ are two self-adjoint extensions of the symmetric operator $A^{0}$ being the restriction of the operator $A$ to the set of all functions orthogonal to the vector $\varphi$ :

$$
\operatorname{Dom}\left(A^{0}\right)=\{\psi \in \operatorname{Dom}(A):\langle\varphi, \psi\rangle=0\}
$$

The operator $A^{0}$ is a symmetric nondensely defined operator, since $\varphi \in H$. Self-adjoint extensions of such symmetric operators have been studied by M. A. Krasnosel'skiĭ [?, ?]. The resolvent of the operator $A_{\alpha}$ can be calculated in this case without using the extension theory for symmetric operators.

Theorem 1 Let $A$ be a self-adjoint operator acting in the Hilbert space $H$ and let $\varphi$ be arbitrary vector from the Hilbert space, $\varphi \in \mathcal{H}$. Then the resolvents of the original operator $A$ and its rank one perturbation $A_{\alpha}=$ $A+\alpha\langle\varphi, \cdot\rangle \varphi, \alpha \in \mathbf{R}$, are related as follows for arbitrary $z, \Im z \neq 0$,

$$
\begin{equation*}
\frac{1}{A_{\alpha}-z}-\frac{1}{A-z}=-\frac{\alpha}{1+\alpha F(z)}\left\langle\frac{1}{A-\bar{z}} \varphi, \cdot\right\rangle \frac{1}{A-z} \varphi \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
F(z)=\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle . \tag{3}
\end{equation*}
$$

Proof To calculate the resolvent of the self-adjoint operator $A_{\alpha}$ we have to solve the following equation

$$
h=\left(A_{\alpha}-z\right) f
$$

for a given $h \in H$ and $f \in \operatorname{Dom}\left(A_{\alpha}\right)=\operatorname{Dom}(A)$. We assume that the imaginary part of the spectral parameter $z$ is positive $\Im z>0$. We apply the operator $A_{\alpha}-z$ to the latter equality

$$
\begin{aligned}
h & =(A+\alpha\langle\varphi, \cdot\rangle \varphi-z) f \\
& =A f-z f+\alpha\langle\varphi, f\rangle \varphi .
\end{aligned}
$$

By applying the resolvent of the original operator we get

$$
\frac{1}{A-z} h=f+\alpha\langle\varphi, f\rangle \frac{1}{A-z} \varphi .
$$

Projection on the vector $\varphi$ leads to the following formula for $\langle\varphi, f\rangle$

$$
\langle\varphi, f\rangle=\frac{\left\langle\varphi, \frac{1}{A-z} h\right\rangle}{1+\alpha\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle} .
$$

It follows that

$$
f=\frac{1}{A-z} h-\frac{\alpha}{1+\alpha\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle}\left\langle\varphi, \frac{1}{A-z} g\right\rangle \frac{1}{A-z} \varphi,
$$

which is exactly formula (2). The theorem is proven.

Formula (2) can be used to calculate the resolvent of the operator $A_{\alpha}=$ $A+\alpha\langle\varphi, \cdot\rangle \varphi$ even in the case where the vector $\varphi$ is not an element from the Hilbert space, but a linear functional on the domain $\operatorname{Dom}(A)$. The perturbation can then be defined using the quadratic form $\alpha\langle\varphi, \cdot\rangle \varphi$, where
the scalar product is understood as the action of linear functionals. Such generalized perturbations will be studied in the following sections of this chapter. Formula (2) can even be used to define the perturbed operator $A_{\alpha}$ in the case where the perturbation $\alpha\langle\varphi, \cdot\rangle$ is not a bounded operator in the Hilbert space. In the latter case the domains of the operators $A$ and $A_{\alpha}$ are different in the general situation and the extension theory for symmetric operators starts to play an important role during the investigation of such perturbations. (In fact we have not used the extension theory to derive the resolvent formula (2).) See the following section where the case of infinite coupling constant $\alpha$ is considered.

Let us study first the spectral properties of the bounded perturbations defined above. These properties are described by the function

$$
\begin{equation*}
F_{\alpha}(z)=\left\langle\varphi, \frac{1}{A_{\alpha}-z} \varphi\right\rangle \tag{4}
\end{equation*}
$$

The function $F_{0}(z) \equiv F(z)$ appears in the denominator in formula (2). This function is related to Krein's $Q$-function

$$
Q_{\alpha}(z)=\left\langle\varphi, \frac{1+A_{\alpha} z}{A_{\alpha}-z} \frac{1}{A_{\alpha}^{2}+1} \varphi\right\rangle
$$

which will be defined later

$$
F_{\alpha}(z)=\left\langle\varphi, \frac{1}{A_{\alpha}^{2}+1} \varphi\right\rangle+Q_{\alpha}(z)
$$

The function $F_{\alpha}$ is a Nevanlinna function, i.e. a holomorphic function in $\mathbf{C} \backslash \mathbf{R}$ satisfying the following conditions

$$
\begin{gather*}
\overline{F(z)}=F(\bar{z})  \tag{5}\\
\frac{\Im F(z)}{\Im z} \geq 0, \quad z \in \mathbf{C} \backslash \mathbf{R} . \tag{6}
\end{gather*}
$$

Such functions are also called Herglotz and $R$-functions. Every Nevanlinna function $R$ possesses the representation

$$
\begin{equation*}
R(z)=a+b z+\int_{\mathbf{R}} \frac{1+\lambda z}{\lambda-z} \frac{1}{\lambda^{2}+1} d \sigma(\lambda) \tag{7}
\end{equation*}
$$

where $a \in \mathbf{R}, b \geq 0$ and the positive measure $d \sigma(\lambda)$ satisfies

$$
\begin{equation*}
\int_{\mathbf{R}} \frac{d \sigma(\lambda)}{\lambda^{2}+1}<\infty \tag{8}
\end{equation*}
$$

The operator $A_{\alpha}$ is self-adjoint and the vector $\varphi$ is an element from the Hilbert space. Therefore there exists a spectral measure $d \mu_{\alpha}$ such that the function $F_{\alpha}(z)$ is given by the integral

$$
F_{\alpha}(z)=\int_{\mathbf{R}} \frac{d \mu_{\alpha}(\lambda)}{\lambda-z}
$$

where the measure $\mu_{\alpha}$ is finite, i.e.

$$
\int_{\mathbf{R}} d \mu_{\alpha}<\infty
$$

The functions $F_{\alpha}(z)$ belong to the class $\mathcal{R}_{0}$ of Nevanlinna functions [?]. The class $\mathcal{R}_{0}$ is the subset of Nevanlinna functions $R$ with the following properties

$$
\begin{gathered}
\sup _{y>0} y \Im R(i y)<\infty \\
\lim _{y \rightarrow \infty} R(i y)=0
\end{gathered}
$$

Every Nevanlinna function from the class $\mathcal{R}_{0}$ possesses the following representation

$$
R(z)=\int_{\mathbf{R}} \frac{d \sigma(\lambda)}{\lambda-z}
$$

where the measure $d \sigma$ is finite, $\int_{\mathbf{R}} d \sigma(\lambda)<\infty$.
Formula (2) implies that the functions $F_{\alpha}(z)$ and $F_{0}(z)$ are related by the following rational transformation

$$
\begin{equation*}
F_{\alpha}(z)=\frac{F_{0}(z)}{1+\alpha F_{0}(z)} \tag{9}
\end{equation*}
$$

The difference of the resolvents of the original and perturbed operators is a rank one operator and its trace can easily be calculated

$$
\operatorname{Tr}\left[\frac{1}{A-z}-\frac{1}{A_{\alpha}-z}\right]=\frac{\alpha}{1+\alpha F_{0}(z)}\left\langle\varphi, \frac{1}{(A-z)^{2}} \varphi\right\rangle
$$

We use the relation

$$
\left\langle\varphi, \frac{1}{(A-z)^{2}} \varphi\right\rangle=\frac{d F_{0}(z)}{d z}
$$

to get the following formula

$$
\begin{equation*}
\operatorname{Tr}\left[\frac{1}{A-z}-\frac{1}{A_{\alpha}-z}\right]=\frac{d}{d z} \ln \left(1+\alpha F_{0}(z)\right) \tag{10}
\end{equation*}
$$

The branch of the logarithm can be fixed arbitrarily. Different branches of the $\ln$ function lead to the same result after differentiation.

The family of measures $\mu_{\alpha}, \alpha \in \mathbf{R}$, is characterized by the following lemma.

Lemma 1 Let $A$ be a self-adjoint operator in $H, A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi$ be its bounded rank one perturbation. Let $d \mu_{\alpha}(E)$ be the corresponding spectral measure. Let $f \in L_{1}(\mathbf{R})$. Then $f \in L_{1}\left(\mathbf{R}, d \mu_{\alpha}\right)$ for almost every $\alpha$ and we have

$$
\alpha \mapsto \int_{\mathbf{R}} f(E) d \mu_{\alpha}(E) \in L_{1}(\mathbf{R}, d \alpha)
$$

and

$$
\begin{equation*}
\int_{\mathbf{R}}\left(\int_{\mathbf{R}} f(E) d \mu_{\alpha}(E)\right) d \alpha=\int_{\mathbf{R}} f(E) d E \tag{11}
\end{equation*}
$$

Proof We prove the lemma first for functions of the form

$$
f_{z}(E)=\frac{1}{E-z}-\frac{1}{E+i}
$$

where $z \in \mathbf{C} \backslash \mathbf{R}$. The integral on right hand side of (11) can be calculated by closing the contour in the upper half plane

$$
\int_{\mathbf{R}} f_{z}(E) d E= \begin{cases}0 & \Im z<0 \\ 2 \pi i & \Im z>0\end{cases}
$$

On the other hand

$$
\begin{aligned}
h_{z}(\alpha) & =\int_{\mathbf{R}} f_{z}(E) d \mu_{\alpha}(E) \\
& =F_{\alpha}(z)-F_{\alpha}(-i) \\
& =\frac{1}{\alpha+F_{0}(z)^{-1}}-\frac{1}{\alpha+F_{0}(-i)^{-1}} .
\end{aligned}
$$

The function $F_{0}(z)$ belongs to the Nevanlinna class and the poles of the function $h_{z}(\alpha)$ are situated in the same half planes as those of the function $f_{z}(E)$. Integration of the function $h_{z}(\alpha)$ with respect to $\alpha$ gives the same result as the integration of $f_{z}(E)$ with respect to $E$. The result is proven for every function $f_{z}(E)$. The statement of the lemma follows from the StoneWeierstrass approximation theorem.

The results obtained here for bounded rank one perturbations will be generalized in what follows for arbitrary perturbations having finite and even infinite rank.

## Infinite coupling

We have just considered rank one bounded perturbations of self-adjoint operators given by (1). Only finite real parameters $\alpha$ have been considered. If the coupling constant $\alpha$ is infinite then formula (1) has only a heuristic meaning. We use instead the resolvent formula to define the perturbation in this case. We are going to show that such a perturbation determines a self-adjoint operator relation, not a self-adjoint operator.

The operator corresponding to the formal expression (1) when $\alpha=\infty$ is well defined on the domain $\operatorname{Dom}\left(A^{0}\right)=\{\psi \in D(A):\langle\varphi, \psi\rangle=0\}$. The domain $\operatorname{Dom}\left(A^{0}\right)$ is not dense in the Hilbert space. One can define the perturbed operator in this case using the resolvent formula (2). This formula gives the following expression for the perturbed resolvent in the case $\alpha=$ $\infty \Rightarrow 1 / \alpha=0$

$$
\begin{align*}
\frac{1}{A_{\infty}-z} & =\frac{1}{A-z}-\frac{1}{1 / \alpha+F_{0}(z)}\left\langle\frac{1}{A-\bar{z}} \varphi, \cdot\right\rangle \frac{1}{A-z} \varphi  \tag{12}\\
& =\frac{1}{A-z}-\frac{1}{F_{0}(z)}\left\langle\frac{1}{A-\bar{z}} \varphi, \cdot\right\rangle \frac{1}{A-z} \varphi .
\end{align*}
$$

The latter expression defines a self-adjoint relation $A_{\infty}$ in the Hilbert space $H$, not a self-adjoint operator. To prove this we apply the latter operator
equality to the vector $\varphi$ and get the following equation

$$
\begin{aligned}
\frac{1}{A_{\infty}-z} \varphi & =\frac{1}{A-z} \varphi-\frac{1}{F_{0}(z)}\left\langle\frac{1}{A-\bar{z}} \varphi, \varphi\right\rangle \frac{1}{A-z} \varphi \\
& =\frac{1}{A-z} \varphi-\frac{1}{F_{0}(z)} F_{0}(z) \frac{1}{A-z} \varphi \\
& =0
\end{aligned}
$$

due to the formula (4). It follows that formula (12) does not define the resolvent of any self-adjoint operator. It gives the formula for the resolvent of the self-adjoint relation $A^{0} \dot{+}(0, \varphi)$. The latter formula implies that $F_{\infty}(z) \equiv$ 0 , which coincides with the limit

$$
\lim _{\alpha \rightarrow \infty} F_{\alpha}(z)=\lim _{\alpha \rightarrow \infty} \frac{F_{0}(z)}{1+\alpha F_{0}(z)}=0
$$

In what follows we are going to consider only self-adjoint operators and we try to avoid discussing self-adjoint relations. But we have to keep in mind that using Krein's formula one obtains not only operators, but also operator relations. The operator relations in connection with the finite rank perturbations have been recently studied by H. de Snoo and S. Hassi [?, ?, ?, ?, ?, ?].

Krein's formula

## Bounded and singular perturbations

We have mentioned in the previous section that not only can bounded rank one perturbations be defined in the framework of the theory of selfadjoint operators, but formula (2) can define a rank one perturbation of the self-adjoint operator $A$ even if $\varphi$ is not an element from the Hilbert space. We start with two examples. Consider first the formal linear differential operator

$$
\begin{equation*}
B_{\alpha} \psi=-\frac{d^{2}}{d x^{2}} \psi+\alpha\langle\delta, \psi\rangle \delta, \tag{13}
\end{equation*}
$$

defined on functions on the real line, where the symbol $\delta$ denotes Dirac's delta function and the scalar product $\langle\delta, \psi\rangle$ is defined as the action of the linear functional $\delta$ on the function $\psi$. The standard norm on the domain of the operator $B_{0}$ is equal to the norm in the Sobolev space $W_{2}^{2}(\mathbf{R})$. Therefore
the scalar product $\langle\delta, \psi\rangle$ is well defined since the function $\psi$ is continuous at the origin. Suppose that there exists a self-adjoint operator acting in $L_{2}(\mathbf{R})$ corresponding to the formal differential expression (13). Then this operator coincides with the unperturbed operator $B=-d^{2} / d x^{2}$ on the set of functions vanishing at the origin, i.e. on the domain

$$
\operatorname{Dom}\left(B^{0}\right)=\left\{\psi \in W_{2}^{2}(\mathbf{R}): \psi(0)=0\right\}
$$

The restricted operator $B^{0}=\left.B\right|_{\operatorname{Dom}\left(B^{0}\right)}$ is a symmetric operator with the deficiency indices $(1,1)$. The restricted operator is densely defined and all its self-adjoint extensions can be calculated using the von Neumann theory. The adjoint operator $B^{0 *}$ coincides with the second derivative operator and has domain $\operatorname{Dom}\left(B^{0 *}\right)=\left\{\psi \in W_{2}^{2}(\mathbf{R} \backslash\{0\}): \psi(-0)=\psi(+0)\right\}$. To define the self-adjoint operator $B_{\alpha}$ we apply the linear operator (13) to an arbitrary function $\psi$ from the domain of the adjoint operator $B^{0 *}$

$$
\psi \in \operatorname{Dom}\left(B^{0 *}\right) .
$$

The expression

$$
B_{\alpha} \psi=\left(-\frac{d^{2}}{d x^{2}}+\alpha\langle\delta, \cdot\rangle \delta\right) \psi
$$

is well defined in the distributional sense, since every $\psi \in \operatorname{Dom}\left(B^{0 *}\right)$ is a continuous square integrable function due to the Sobolev embedding theorem (see [?]). The result in general is equal to the sum of a square integrable function and a distribution with support at the origin. From the condition that the range of the function $\psi$ belongs to the Hilbert space $L_{2}(\mathbf{R})$ we get

$$
-\psi^{\prime}(+0)+\psi^{\prime}(-0)+\alpha \psi(0)=0
$$

If the function $\psi$ satisfies the latter condition, then $B_{\alpha} \psi \in L_{2}(\mathbf{R})$. Consider the restriction of the operator $B_{\alpha}$ to the domain

$$
\begin{aligned}
\operatorname{Dom}\left(B_{\alpha}\right)= & \left\{\psi \in W_{2}^{2}(\mathbf{R} \backslash\{0\}): \psi(-0)=\psi(+0) \equiv \psi(0)\right. \\
& \left.\psi^{\prime}(+0)-\psi^{\prime}(-0)=\alpha \psi(0)\right\}
\end{aligned}
$$

The restrictions of the operators $B_{\alpha}$ and $B^{0 *}$ to this domain coincide. This operator is a self-adjoint extension of the operator $B^{0}$ and this operator can
be considered as a natural definition for the operator $B_{\alpha}$ in the framework of the theory of self-adjoint operators.

The above example shows that formula (1) can define a rank one perturbation of the self-adjoint operator $A$ even if the vector $\varphi$ does not belong to the corresponding Hilbert space. But the vector $\varphi$ cannot be arbitrary. To define the perturbed operator we used the restriction of the original operator $B$ to the domain $\operatorname{Dom}\left(B^{0}\right)$. Therefore the vector $\varphi$ should be a linear bounded functional on the domain of the operator $A$ with the graph norm. Only in this case does the restriction of the self-adjoint operator $A$ have nontrivial deficiency indices. But not every vector $\varphi \in \operatorname{Dom}(A)^{*}$ defines a rank one perturbation in a unique way. Consider the following linear differential operator

$$
\begin{equation*}
B_{\alpha}^{\prime} \psi=-\frac{d^{2}}{d x^{2}} \psi+\alpha\left\langle\delta^{(1)}, \psi\right\rangle \delta^{(1)} \tag{14}
\end{equation*}
$$

where $\delta^{(1)}$ denotes the first derivative of the delta function. If the self-adjoint operator in $L_{2}(\mathbf{R})$ corresponding to the formal expression (14) exists, then it coincides with the operator $B=-d^{2} / d x^{2}$ restricted to the domain of functions from $W_{2}^{2}(\mathbf{R} \backslash\{0\})$, satisfying certain boundary conditions at the origin. Consider the formal expression $B_{\alpha}^{\prime} \psi$, where $\psi \in W_{2}^{2}(\mathbf{R} \backslash\{0\})$. The scalar product $\left\langle\delta^{(1)}, \psi\right\rangle$ is well defined only if the function $\psi$ is continuous at the origin and has continuous first derivative at this point. But if these two conditions are satisfied, then $B_{\alpha}^{\prime} \psi$ belongs to the Hilbert space $L_{2}(\mathbf{R})$ if and only if $\psi^{\prime}(0)=0$. The second derivative operator defined on the domain of $C^{1}(\mathbf{R})$ functions from $W_{2}^{2}(\mathbf{R} \backslash\{0\})$ satisfying the latter condition is symmetric but not self-adjoint. Therefore the heuristic rank one perturbation (14) does not determine any self-adjoint operator, but only a symmetric operator. The corresponding family of self-adjoint extensions of this symmetric operator is described by one real parameter, which is not determined by the heuristic expression (14). One needs additional assumptions on the interaction to define the unique self-adjoint operator in this case.

The two examples considered show that the rank one perturbations can be defined by vectors $\varphi \in \operatorname{Dom}(A)^{*}$. We are going to refer to the perturbations defined by vectors $\varphi$ which do not belong to the Hilbert space, $\varphi \notin H$ singular. The perturbations defined by vectors from the Hilbert space will be called bounded. The main difference between singular and bounded rank one perturbations is that the domains of any self-adjoint operator and its
rank one singular perturbation are in general different, while the domain of the operator does not change under rank one bounded perturbations. We are going to consider differential operators with singular interactions. The first mathematically rigorous study of such operators was carried out by F.A.Beresin and L.D.Faddeev [?]. The Laplace operator with delta interaction in $L_{2}\left(\mathbf{R}^{3}\right)$ was considered. We are going to discuss this operator in more detail in later. In the rest of this chapter we concentrate our attention on rank one singular perturbations.

## Scale of Hilbert spaces

The scale of Hilbert spaces associated with the self-adjoint operator $A$ acting in the Hilbert space $H$ will be defined using the modulus $|A|$ of the operator $A$, where

$$
|A| \equiv\left(A^{*} A\right)^{1 / 2}
$$

The operator $|A|$ is positive and self-adjoint, its domain coincides with the domain of the operator $A$. For $s \geq 0, \mathcal{H}_{s}(A)$ is $\operatorname{Dom}\left(|A|^{s / 2}\right)$ with norm equal to the graph norm of the operator

$$
\begin{equation*}
\|\psi\|_{s}=\left\|(|A|+1)^{s / 2} \psi\right\|_{\mathcal{H}} \tag{15}
\end{equation*}
$$

The space $\mathcal{H}_{s}$ with norm $\|\cdot\|_{s}$ is complete. The adjoint spaces formed by the linear bounded functionals will be denoted by $\mathcal{H}_{-s}(A)=\mathcal{H}_{s}(A)^{*}$. The norm in the space $\mathcal{H}_{-s}(A)$ is defined by the formula

$$
\begin{equation*}
\|\psi\|_{-s}=\left\|\frac{1}{(|A|+1)^{s / 2}} \psi\right\|_{\mathcal{H}}, \tag{16}
\end{equation*}
$$

where the operator $1 /(|A|+1)^{s / 2}$ is defined in the generalized sense. Let $\psi \in \mathcal{H}_{-s}(A), \eta \in H=\mathcal{H}_{0}(A)$. Then

$$
\left\langle\frac{1}{(|A|+1)^{s / 2}} \psi, \eta\right\rangle=\left\langle\psi, \frac{1}{(|A|+1)^{s / 2}} \eta\right\rangle .
$$

It follows that $\left(1 /(|A|+1)^{s / 2}\right) \psi \in H$ and the norm of the functional is given by the formula (16).

The operator $(|A|+1)^{t / 2}$ defines an isometry from $\mathcal{H}_{s}(A)$ to $\mathcal{H}_{s-t}(A)$. Each space $\mathcal{H}_{-s}(A)$ is equal to the completion of the Hilbert space $H$ in the norm (16).

In what follows we are going to use the brackets $\langle\cdot, \cdot\rangle$ to denote not only the scalar product in the Hilbert space $H$, but the action of the functionals. Let $\psi \in \mathcal{H}_{-s}(A), \eta \in \mathcal{H}_{s}(A)$. Then we define

$$
\begin{equation*}
\langle\psi, \eta\rangle \equiv\left\langle\frac{1}{(|A|+1)^{s / 2}} \psi,(|A|+1)^{s / 2} \eta\right\rangle \tag{17}
\end{equation*}
$$

where the bracket on the right hand side denotes the scalar product.
The spaces $\mathcal{H}_{s}(A)$ form the following chain of triplets

$$
\ldots \subset \mathcal{H}_{2}(A) \subset \mathcal{H}_{1}(A) \subset H=\mathcal{H}_{0}(A) \subset \mathcal{H}_{-1}(A) \subset \mathcal{H}_{-2}(A) \subset \ldots
$$

The space $\mathcal{H}_{2}(A)$ coincides with the domain of the operator $A$ and $\mathcal{H}_{1}(A)$ is the domain of $|A|^{1 / 2}$. For every two $s, t ; s \leq t$, the space $\mathcal{H}_{t}(A)$ is dense in $\mathcal{H}_{s}(A)$ in the norm $\|\cdot\|_{s}$. The norm in the original Hilbert space $H$ will be denoted

$$
\begin{equation*}
\|\cdot\|_{0}=\|\cdot\|_{H} \tag{18}
\end{equation*}
$$

The norm in the space $\mathcal{H}_{1}(A)$ can be calculated as follows

$$
\begin{align*}
\|\psi\|_{1}^{2} & =\left\langle(|A|+1)^{1 / 2} \psi,(|A|+1)^{1 / 2} \psi\right\rangle \\
& =\langle\psi,(|A|+1) \psi\rangle  \tag{19}\\
& =\langle(\sqrt{|A|}+i) \psi,(\sqrt{|A|}+i) \psi\rangle .
\end{align*}
$$

Similarly we have

$$
\begin{equation*}
\|\psi\|_{-1}^{2}=\left\langle\frac{1}{\sqrt{|A|}+i} \psi, \frac{1}{\sqrt{|A|}+i} \psi\right\rangle \tag{20}
\end{equation*}
$$

One can also introduce the norms $\|\cdot\|_{2}^{*}$ and $\|\cdot\|_{-2}^{*}$ in the spaces $\mathcal{H}_{2}(A)$ and $\mathcal{H}_{-2}(A)$, which are equivalent to the standard norms in these spaces

$$
\begin{align*}
\|\psi\|_{2}^{* 2} & =\|(A-i) \psi\|^{2}
\end{aligned}=\left\langle\psi,\left(A^{2}+1\right) \psi\right\rangle ; 口 \begin{aligned}
& \|\psi\|_{-2}^{* 2}
\end{align*}=\left\|\frac{1}{A-i} \psi\right\|^{2}=\left\langle\psi, \frac{1}{A^{2}+1} \psi\right\rangle .
$$

In fact the spaces $\mathcal{H}_{s}(A)$ are Hilbert spaces with the scalar product associated with the standard norm

$$
\langle\psi, \varphi\rangle_{s}=\left\langle\psi,(|A|+1)^{s} \varphi\right\rangle .
$$

## Form bounded and form unbounded perturbations

It has been shown that rank one perturbations of a given operator $A$ can be defined only by the vectors $\varphi$ which are bounded linear functionals on the domain of the operator $A$, i.e. by the vectors $\varphi$ being elements from the space $\mathcal{H}_{-2}(A)$. If the operator $A$ is positive then every rank one perturbation defined by the vectors $\varphi$ from $\mathcal{H}_{-1}(A)$ can be defined using the form perturbation theory. Let us explain this in more detail. The linear operator

$$
\langle\varphi, \cdot\rangle \varphi: \mathcal{H}_{2}(A) \rightarrow \mathcal{H}_{-2}(A)
$$

defines naturally the following sesquilinear positive form

$$
\mathbf{V}_{\varphi}[\psi, \eta]=\langle\varphi, \psi\rangle\langle\eta, \varphi\rangle=\langle\varphi, \psi\rangle \overline{\langle\varphi, \eta\rangle}
$$

for $\psi, \eta \in \mathcal{H}_{2}(A)$. The sesquilinear positive form $V[\psi, \eta]$ will be called form bounded with respect to the operator $A$ if and only if the domain $\operatorname{Dom}(V)$ of the form is contained in the space $\mathcal{H}_{1}(A)$ and there exist two positive real constants $a$ and $b$ such that for any $\psi \in \operatorname{Dom}(V)$ the following estimate holds:

$$
\begin{equation*}
V[\psi, \psi] \leq a\|\psi\|_{1}^{2}+b\|\psi\|_{H}^{2} . \tag{22}
\end{equation*}
$$

If the constant $a$ can be chosen arbitrarily small, the form $V$ is said to be infinitesimally form bounded with respect to the operator $A$. Note that to define form bounded and infinitesimally form bounded perturbations we have actually used the quadratic form of the positive operator $|A|$, since the norms defined by

$$
a\|\psi\|_{1}^{2}+b\|\psi\|_{H}^{2}
$$

and

$$
a\langle\psi,| A|\psi\rangle+b\langle\psi, \psi\rangle
$$

are equivalent.
Lemma 2 Let $\varphi \in \mathcal{H}_{-1}(A)$. Then the sesquilinear from $\mathbf{V}_{\varphi}[\psi, \eta]=\langle\varphi, \psi\rangle\langle\varphi, \eta\rangle$ is infinitesimally form bounded with respect to the operator $A$.

Proof The Hilbert space $H$ is dense in $\mathcal{H}_{-1}(A)$ and for any $\epsilon>0$ there exists $\varphi_{0} \in H$ such that $\left\|\varphi-\varphi_{0}\right\|_{-1}^{2} \leq \epsilon / 2$. For any $\psi \in \mathcal{H}_{2}(A) \subset \mathcal{H}_{1}(A)$ the following estimate proves the lemma

$$
\begin{aligned}
\mathbf{V}_{\varphi}[\psi, \psi] & =|(\varphi, \psi)|^{2} \leq 2\left|\left(\varphi-\varphi_{0}, \psi\right)\right|^{2}+2\left|\left(\varphi_{0}, \psi\right)\right|^{2} \\
& \leq \epsilon\|\psi\|_{1}^{2}+2\left\|\varphi_{0}\right\|^{2}\|\psi\|_{H}^{2} .
\end{aligned}
$$

The lemma is proven.

The latter lemma is valid for any self-adjoint operator $A$. If the operator $A$ is a positive self-adjoint operator then the KLMN theorem [?] implies that for any real $\alpha$ the formal expression $A+\alpha\langle\varphi, \cdot\rangle \varphi$ defines a certain selfadjoint operator $A_{\alpha}$. One can prove that perturbations defined by vectors from $\mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$ are not form bounded.

Lemma 3 Let $\varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$. Then the sesquilinear from

$$
\mathbf{V}_{\varphi}[\psi, \eta]=\langle\varphi, \psi\rangle \overline{\langle\varphi, \eta\rangle}
$$

is not form bounded with respect to the operator $A$.
Proof Suppose that the bilinear form $\mathbf{V}_{\varphi}$ is form bounded i.e. there exist positive constants $a, b$ such that

$$
\mathbf{V}_{\varphi}[\psi, \psi] \leq a\|\psi\|_{1}^{2}+b\|\psi\|_{H}^{2}
$$

The latter estimate is valid for every $\psi \in \mathcal{H}_{2}(A)$, which is dense in $\mathcal{H}_{1}(A)$. It follows that $\varphi$ can be extended as a linear bounded functional to the whole of $\mathcal{H}_{1}(A)$. We get a contradiction which proves the lemma.

Thus we have proven that rank one perturbations of positive operators are uniquely defined if $\varphi \in \mathcal{H}_{-1}(A)$. The perturbations defined by vectors $\varphi \in$ $\mathcal{H}_{-2} \backslash \mathcal{H}_{-1}(A)$ cannot be defined using the form perturbation technique.

In what follows we are going to call form bounded all rank one perturbations determined by the vectors $\varphi$ from the space $\mathcal{H}_{-1}(A)$, even if the operator $A$ is not positive. Rank one perturbations defined by the vectors $\varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$ will be called form unbounded. Note that Lemma 2 implies that every form bounded rank one perturbation is in fact infinitesimally form bounded.

Rank one perturbations and the extension theory for symmetric operators

Consider the self-adjoint operator $A$ and its rank one perturbation $A_{\alpha}=$ $A+\alpha\langle\varphi, \cdot\rangle \varphi$ restricted to the set of functions $\operatorname{Dom}\left(A^{0}\right)=\{\psi \in \operatorname{Dom}(A):$ $\langle\varphi, \psi\rangle=0\}$. Let us denote the restricted operator by $A^{0}$. In the case of a bounded perturbation the operator $A^{0}$ is not densely defined. If the rank one perturbation is singular, then the operator $A^{0}$ is a densely defined symmetric operator with deficiency indices $(1,1)$.

Lemma 4 Let $A$ be a self-adjoint operator acting in the Hilbert space $H$ and let $\varphi \in \mathcal{H}_{-2}(A) \backslash H$. Then the restriction $A^{0}$ of the operator $A$ to the domain of functions $\operatorname{Dom}\left(A^{0}\right)=\{\psi \in \operatorname{Dom}(A):\langle\varphi, \psi\rangle=0\}$ is a densely defined symmetric operator with deficiency indices $(1,1)$.

Proof We prove first that the restricted operator is densely defined. The operator $A$ is densely defined and thus for every $f \in H$ there exists a sequence $f_{n} \in \operatorname{Dom}(A)=\mathcal{H}_{2}(A)$ converging to $f$ in the Hilbert space norm:

$$
\lim _{n \rightarrow \infty}\left\|f-f_{n}\right\|_{H}=0
$$

The functional $\varphi$ is not a bounded functional on the Hilbert space $H$. It follows that there exists a sequence $\psi_{n} \in \operatorname{Dom}(A)=\mathcal{H}_{2}(A)$ with the unit norm $\left\|\psi_{n}\right\|_{H}=1$ such that the corresponding sequence $\left\langle\varphi, \psi_{n}\right\rangle$ diverges to infinity. This sequence can be chosen in such a way that $\lim _{n \rightarrow \infty}\left\langle\varphi, f_{n}\right\rangle /\left\langle\varphi, \psi_{n}\right\rangle=0$. Then the sequence $f_{n}-\left(\left\langle\varphi, f_{n}\right\rangle /\left\langle\varphi, \psi_{n}\right\rangle\right) \psi_{n}$ belongs to the domain of the restricted operator

$$
\left\langle\varphi, f_{n}-\frac{\left\langle\varphi, f_{n}\right\rangle}{\left\langle\varphi, \psi_{n}\right\rangle} \psi_{n}\right\rangle=0
$$

and converges to the element $f$ in the Hilbert space norm

$$
\left\|f_{n}-\frac{\left(\varphi, f_{n}\right)}{\left(\varphi, \psi_{n}\right)} \psi_{n}-f\right\|_{H} \leq\left\|f_{n}-f\right\|_{H}+\left|\frac{\left(\varphi, f_{n}\right)}{\left(\varphi, \psi_{n}\right)}\right| \rightarrow_{n \rightarrow \infty} 0
$$

Thus the operator $A^{0}$ is densely defined.
The deficiency elements of the operator $A_{0}$ at the point $\lambda=i$ are equal to $g_{i}=(A-i)^{-1} \varphi$. The latter equality has to be understood in the generalized sense, i.e. $g_{\lambda}$ is the bounded linear functional which acts on every $\psi \in H$ in accordance with the formula

$$
\begin{aligned}
\left\langle g_{i}, \psi\right\rangle & =\left\langle(A-i)^{-1} \varphi, \psi\right\rangle \\
& =\left\langle\varphi,(A+i)^{-1} \psi\right\rangle \\
& \leq\|\varphi\|_{-2}\left(\left\|\frac{|A|}{A+i} \psi\right\|_{\mathcal{H}}+\left\|\frac{1}{A+i} \psi\right\|_{\mathcal{H}}\right) \\
& \leq 2\|\varphi\|_{-2}\|\psi\|_{0}
\end{aligned}
$$

Let $\psi \in \operatorname{Dom}\left(A^{0}\right)$. Then the following equalities hold

$$
\begin{aligned}
\left\langle\psi, A^{*} g_{i}\right\rangle & =\left\langle A \psi, g_{i}\right\rangle=\left\langle\frac{A}{A+i} \psi, \varphi\right\rangle \\
& =\left\langle\frac{-i}{A+i} \psi, \varphi\right\rangle
\end{aligned}=i\left\langle\psi, g_{i}\right\rangle .
$$

It follows that $g_{i}$ is the deficiency element for the restricted operator and corresponds to the complex number $i$. The deficiency element is unique (up to multiplication by complex numbers) and this finishes the proof of the lemma.

We see that the self-adjoint operator corresponding to the formal expression $A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi, \varphi \in \mathcal{H}_{-2}(A) \backslash H$, is one of the self-adjoint extensions of the symmetric operator $A^{0}$ having deficiency indices $(1,1)$. Let us discuss now the extension theory for such operators.

In what follows we are going to normalize the vector $\varphi \in \mathcal{H}_{-2}(A)$ using the norm $\|\cdot\|_{-2}^{*}$ defined by (21)

$$
\begin{equation*}
\left\|\frac{1}{A-i} \varphi\right\|=\|\varphi\|_{-2}^{*}=1 \tag{23}
\end{equation*}
$$

Then the deficiency elements $g_{ \pm i}=(1 /(A \mp i)) \varphi$ have unit norms in the Hilbert space $H$.

## The extension theory and Krein's formula

In this section we are going to study the self-adjoint extensions of symmetric operators with unit deficiency indices. Let $A^{0}$ be a certain densely defined symmetric operator acting in the Hilbert space $H$. Without loss of generality we suppose that the operator $A^{0}$ is closed. Suppose also that the deficiency indices are equal to $(1,1)$ and let $g_{i}$ and $g_{-i}$ be two normalized deficiency elements corresponding to $\lambda= \pm i$

$$
\begin{gathered}
A^{0 *} g_{ \pm i}= \pm i g_{ \pm i} \\
\left\|g_{ \pm i}\right\|_{H}=1
\end{gathered}
$$

Then the domain of the adjoint operator is equal to the following linear sum $\operatorname{Dom}\left(A^{0 *}\right)=\operatorname{Dom}\left(A^{0}\right) \dot{+} \mathcal{L}\left\{g_{i}, g_{-i}\right\}$, where $\dot{+}$ denotes the direct sum. Every element $\psi$ from the domain of the adjoint operator possesses the following representation

$$
\begin{equation*}
\psi=\hat{\psi}+a_{+}(\psi) g_{i}+a_{-}(\psi) g_{-i} \tag{24}
\end{equation*}
$$

where $\hat{\psi} \in \operatorname{Dom}\left(A^{0}\right), a_{ \pm}(\psi) \in \mathbf{C}$. The adjoint operator $A^{0 *}$ acts as follows on every $\psi \in \operatorname{Dom}\left(A^{0 *}\right)$

$$
\begin{equation*}
A^{0 *}\left(\hat{\psi}+a_{+}(\psi) g_{i}+a_{-}(\psi) g_{-i}\right)=A \hat{\psi}+i a_{+}(\psi) g_{i}-i a_{-}(\psi) g_{-i} \tag{25}
\end{equation*}
$$

All self-adjoint extensions of the operator $A^{0}$ can be parametrized by one unimodular parameter $v ;|v|=1$ using the von Neumann theory. Every self-adjoint extension $A(v)$ coincides with the restriction of the operator $A^{0 *}$ to the domain $\operatorname{Dom}(A(v))=\left\{\psi \in \operatorname{Dom}\left(A^{0 *}\right):-v a_{-}(\psi)=a_{+}(\psi)\right\}$.

Let us denote by $A$ the self-adjoint extension corresponding to $v=1$. Then the deficiency elements $g_{i}$ and $g_{-i}$ are related as follows

$$
\begin{equation*}
g_{-i}=\frac{A-i}{A+i} g_{i} . \tag{26}
\end{equation*}
$$

We are also going to use the following representation for the functions $\psi \in \operatorname{Dom}\left(A^{0 *}\right)$

$$
\begin{align*}
\psi & =\tilde{\psi}+\frac{b(\psi)}{2}\left(g_{i}+g_{-i}\right)  \tag{27}\\
& =\tilde{\psi}+b(\psi) \frac{A}{A+i} g_{i}
\end{align*}
$$

where $\tilde{\psi} \in \operatorname{Dom}(A)$. This representation is related to the representation (24) via the formulas

$$
\left\{\begin{aligned}
\tilde{\psi} & =\hat{\psi}+\frac{a_{+}(\psi)-a_{-}(\psi)}{2}\left(g_{i}-g_{-i}\right) \\
& =\hat{\psi}+\left(a_{+}(\psi)-a_{-}(\psi)\right) \frac{i}{A+i} g_{i} \\
b(\psi) & =a_{+}(\psi)+a_{-}(\psi)
\end{aligned}\right.
$$

where we have used (26). Using representation (27) the action of the adjoint operator $A^{0 *}$ is given by

$$
\begin{align*}
A^{0 *} \psi & =A^{0 *}\left(\tilde{\psi}+b(\psi) \frac{A}{A+i} g_{i}\right)  \tag{28}\\
& =A \tilde{\psi}-b(\psi) \frac{1}{A+i} g_{i}
\end{align*}
$$

The latter formula follows directly from (25). Let us calculate the boundary form $\left\langle A^{0 *} \psi, \eta\right\rangle-\left\langle\psi, A^{0 *} \eta\right\rangle$ of the adjoint operator for two functions $\psi, \eta \in$
$\operatorname{Dom}\left(A^{0 *}\right)$

$$
\left\langle A^{0 *} \psi, \eta\right\rangle-\left\langle\psi, A^{0 *} \eta\right\rangle
$$

$$
=\left\langle A \tilde{\psi}-b(\psi) \frac{1}{A+i} g_{i}, \tilde{\eta}+b(\eta) \frac{A}{A+i} g_{i}\right\rangle
$$

$$
-\left\langle\tilde{\psi}+b(\psi) \frac{A}{A+i} g_{i}, A \tilde{\eta}-b(\eta) \frac{1}{A+i} g_{i}\right\rangle
$$

$$
=\langle A \tilde{\psi}, \tilde{\eta}\rangle-\bar{b}(\psi)\left\langle\frac{1}{A+i} g_{i}, \tilde{\eta}\right\rangle
$$

$$
+b(\eta)\left\langle A \tilde{\psi}, \frac{A}{A+i} g_{i}\right\rangle-\bar{b}(\psi) b(\eta)\left\langle\frac{1}{A+i} g_{i}, \frac{A}{A+i} g_{i}\right\rangle
$$

$$
-\langle\tilde{\psi}, A \tilde{\eta}\rangle-\bar{b}(\psi)\left\langle\frac{A}{A+i} g_{i}, A \tilde{\eta}\right\rangle
$$

$$
+b(\eta)\left\langle\tilde{\psi}, \frac{1}{A+i} g_{i}\right\rangle+\bar{b}(\psi) b(\eta)\left\langle\frac{A}{A+i} g_{i}, \frac{1}{A+i} g_{i}\right\rangle
$$

$$
=\overline{\left\langle(A-i) g_{i}, \tilde{\psi}\right\rangle} b(\eta)-\overline{b(\psi)}\left\langle(A-i) g_{i}, \tilde{\eta}\right\rangle
$$

The deficiency element $g_{i}$ belongs to the Hilbert space. Therefore the vector $(A-i) g_{i}$ is a bounded linear functional on the domain of the operator $A$, i.e. belongs to $\mathcal{H}_{-2}(A)$ and the scalar products $\left\langle(A-i) g_{i}, \tilde{\psi}\right\rangle$ and $\left\langle(A-i) g_{i}, \tilde{\eta}\right\rangle$ appearing in the latter formula are well defined. Another way to parametrize the self-adjoint extensions is to use a real parameter $\gamma$ instead of the unitary parameter $v$ appeared in the von Neumann formulas. Let us denote by $A^{\gamma}$ the restriction of the operator $A^{0 *}$ to the domain

$$
\begin{equation*}
\operatorname{Dom}\left(A^{\gamma}\right)=\left\{\psi \in \operatorname{Dom}\left(A^{0 *}\right):\left\langle(A-i) g_{i}, \tilde{\psi}\right\rangle=\gamma b(\psi)\right\} \tag{29}
\end{equation*}
$$

We are even going to consider infinite values of the parameter $\gamma: \gamma \in$ $\mathbf{R} \cup\{\infty\}$. For $\gamma=\infty$ we put formally

$$
\operatorname{Dom}\left(A^{\infty}\right)=\left\{\psi \in \operatorname{Dom}\left(A^{0 *}\right): b(\psi)=0\right\} \equiv \operatorname{Dom}(A),
$$

i.e. the operator $A^{\infty}$ coincides with the operator $A$. The operators $A(v)$ and $A^{\gamma}$ coincide if the parameters $v$ and $\gamma$ are related as follows

$$
\begin{equation*}
v=\frac{\gamma+i}{\gamma-i} \Leftrightarrow \gamma=-i \frac{1+v}{1-v} \tag{30}
\end{equation*}
$$

When $\gamma$ runs over all real numbers including infinity, the parameter $v$ runs over all unimodular complex numbers. Therefore every self-adjoint extension of $A^{0}$ is described by a certain parameter $\gamma \in \mathbf{R} \cup\{\infty\}$. In what follows we are going to use both descriptions of self-adjoint extensions.

The resolvents of two self-adjoint extensions of one symmetric operator are related by Krein's formula [?, ?]:

Theorem 2 Let $A$ and $B$ be two self-adjoint extensions of a certain symmetric densely defined operator $A^{0}$ with unit deficiency indices. Then there exists a real number $\gamma \in \mathbf{R} \cup\{\infty\}$ such that the resolvents of the operators $A$ and $B$ are related as follows

$$
\begin{equation*}
\frac{1}{B-z}=\frac{1}{A-z}+\frac{1}{\gamma-\left\langle g_{i}, \frac{1+A z}{A-z} g_{i}\right\rangle}\left\langle\frac{A-i}{A-\bar{z}} g_{i},\right\rangle \frac{A-i}{A-z} g_{i}, \Im z \neq 0 \tag{31}
\end{equation*}
$$

where $g_{i}$ is the normalized deficiency element for $A^{0}$ at the point $\lambda=i$. If $A=B$, then $\gamma=\infty$ and the resolvents of the self-adjoint operators coincide:

$$
\begin{equation*}
\frac{1}{B-z}=\frac{1}{A-z}, \Im z \neq 0 \tag{32}
\end{equation*}
$$

Proof The operator $A$ is a self-adjoint extension of $A^{0}$. Therefore the function $((A-i) /(A+i)) g_{i}$ is a deficiency element for $A^{0}$ at the point $\lambda=-i$ and we can choose

$$
g_{-i}=\frac{A-i}{A+i} g_{i}
$$

Let us describe the self-adjoint extensions of $A^{0}$ using (29). Then the operator $A$ coincides with the operator $A^{\infty}$. The operator $B$ is also a self-adjoint extension of $A^{0}$, therefore there exists a certain real parameter $\gamma \in \mathbf{R} \cup\{\infty\}$ such that $B=A^{\gamma}$.

If $\gamma=\infty$, then the operators $A$ and $B$ coincide and formula (32) obviously holds.

To prove the theorem we have to calculate the resolvent of the operator $A^{\gamma}$, i.e. we have to solve the following equation

$$
\begin{equation*}
h=\left(A^{\gamma}-z\right) f, \tag{33}
\end{equation*}
$$

for a given $h \in H$. The function $f$ belongs to the domain of the operator $A^{\gamma}$, $f \in \operatorname{Dom}\left(A^{\gamma}\right)$. Let $\gamma \neq 0, \infty$. Then every function $f$ from the domain of the operator $A^{\gamma}$ possesses the representation

$$
f=\tilde{f}+\frac{1}{\gamma}\left\langle(A-i) g_{i}, \tilde{f}\right\rangle \frac{A}{A+i} g_{i}
$$

Equality (33) and formula (28) imply that

$$
h=A \tilde{f}-z \tilde{f}-\frac{1}{\gamma}\left\langle(A-i) g_{i}, \tilde{f}\right\rangle \frac{1+A z}{A+i} g_{i}
$$

Applying the resolvent of the original operator $A$ to the latter equation and then projecting to the element $(A-i) g_{i} \in \mathcal{H}_{-2}(A)$ we get

$$
\begin{equation*}
\left\langle(A-i) g_{i}, \tilde{f}\right\rangle=\frac{\left\langle\frac{A-i}{A-\bar{z}} g_{i}, h\right\rangle}{1-\frac{1}{\gamma}\left\langle g_{i}, \frac{1+A z}{A-z} g_{i}\right\rangle} \tag{34}
\end{equation*}
$$

It follows that the function $f$ is given by

$$
\begin{aligned}
f= & \frac{1}{A-z} h+\frac{1}{\gamma}\left\langle(A-i) g_{i}, \tilde{f}\right\rangle \frac{1+A z}{A-z} \frac{1}{A+i} g_{i} \\
& +\frac{1}{\gamma}\left\langle(A-i) g_{i}, \tilde{f}\right\rangle \frac{A}{A+i} g_{i} \\
= & \frac{1}{A-z} h+\frac{1}{\gamma}\left\langle(A-i) g_{i}, \tilde{f}\right\rangle \frac{A-i}{A-z} g_{i} \\
= & \frac{1}{A-z} h+\frac{\left\langle\frac{A-i}{A-\bar{z}} g_{i}, h\right\rangle}{\gamma-\left\langle g_{i}, \frac{1+A z}{A-z} g_{i}\right\rangle} \frac{A-i}{A-z} g_{i} .
\end{aligned}
$$

The latter formula implies (31).

In the exceptional case $\gamma=0$ the resolvent formula is obviously satisfied. The theorem is proven.

In fact we have proven that the resolvent of any self-adjoint extension $A^{\gamma}$ of $A^{0}$ is given by

$$
\begin{equation*}
\frac{1}{A^{\gamma}-z}=\frac{1}{A-z}+\frac{1}{\gamma-\left\langle g_{i}, \frac{1+A z}{A-z} g_{i}\right\rangle}\left\langle\frac{A-i}{A-\bar{z}} g_{i}, \cdot\right\rangle \frac{A-i}{A-z} g_{i}, \Im z \neq 0 \tag{35}
\end{equation*}
$$

The function

$$
\begin{equation*}
Q(z)=\left\langle g_{i}, \frac{1+A z}{A-z} g_{i}\right\rangle \tag{36}
\end{equation*}
$$

is called Krein's $Q$-function. It is a Nevanlinna function, since the deficiency element $g_{i}$ has finite norm. Usually the $Q$-function is defined by the following relation

$$
\begin{equation*}
\frac{Q(\lambda)-\bar{Q}(z)}{\lambda-\bar{z}}=\left\langle g_{\lambda}, g_{z}\right\rangle \equiv\left\langle\frac{A-i}{A-\lambda} g_{i}, \frac{A-i}{A-z} g_{i}\right\rangle . \tag{37}
\end{equation*}
$$

Obviously the latter relation defines the $Q$-function uniquely up to real constants. In general two self-adjoint extensions of one symmetric operator with deficiency indices $(1,1)$ define a one-parameter family of $Q$-functions which differ by real constants. In what follows we are going to use definition (36) for the Krein's $Q$-function, i.e. we are going to distinguish the $Q$-functions which differ by real constants. We hope that this convention will not cause any problem for the reader.

Comparing formula (36) and representation (7) for an arbitrary Nevanlinna function we see that the $Q$-functions corresponding to self-adjoint extensions of densely defined symmetric operators do not have a linear term in the asymptotics, i.e. the constants $b$ appeared in (7) are always equal to zero.

A similar analysis can be carried out in the case where the operator $A^{0}$ is not densely defined [?, ?]. The resolvents of the self-adjoint extensions are related by similar formulas. The difference is that Krein's formula for the resolvent describes in this case not only the resolvents of all extensions that are self-adjoint operators but also the resolvents of the self-adjoint relations which are extensions of the symmetric operator.

## $Q$-function for rank one perturbations

Let us continue discussion of the operator $A_{\alpha}$ formally given by $A_{\alpha}=$ $A+\alpha\langle\varphi, \cdot\rangle \varphi$ already started in Section . Lemma 4 implies that the deficiency element for the operator $A^{0}$ at the point $\lambda=i$ is given by $g_{i}=(1 /(A-i)) \varphi$. Therefore the $Q$-function corresponding to the operators $A^{0}$ and $A$ is given by

$$
\begin{equation*}
Q(z)=\left\langle g_{i}, \frac{1+A z}{A-z} g_{i}\right\rangle=\left\langle\varphi, \frac{1+A z}{A-z} \frac{1}{A^{2}+1} \varphi\right\rangle . \tag{38}
\end{equation*}
$$

If $\varphi \in \mathcal{H}_{-2}(A)$, then the scalar product appeared in the latter formula is well defined.

Let $\varphi$ be an element from $\mathcal{H}_{-1}(A)$. One can write the following formula for the corresponding $Q$-function

$$
\begin{equation*}
Q(z)=\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle-\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle . \tag{39}
\end{equation*}
$$

It follows that $Q(z)$ possesses the integral representation

$$
\begin{equation*}
Q(z)=a+\int_{-\infty}^{+\infty} \frac{d \tau(\lambda)}{\lambda-z}, \Im z \neq 0 \tag{40}
\end{equation*}
$$

where $a \in \mathbf{R}$ and

$$
\int_{-\infty}^{+\infty} \frac{d \tau(\lambda)}{1+|\lambda|}<\infty
$$

Therefore the function $Q(z)$ belongs to the class $\mathcal{R}_{1}$ of Nevanlinna functions [?]. The class $\mathcal{R}_{1}$ is the subset of Nevanlinna functions $R$ with the following property

$$
\begin{equation*}
\int_{1}^{+\infty} \frac{\Im R(i y)}{y} d y<\infty \tag{41}
\end{equation*}
$$

Obviously the class $\mathcal{R}_{1}$ contains the class $\mathcal{R}_{0}$ of Nevanlinna functions.
Let us define another scale of Hilbert spaces associated with the operator $A$ and the vector $\varphi \in \mathcal{H}_{-2}(A) \backslash H$

$$
\begin{equation*}
\mathcal{H}_{2}(A)=\operatorname{Dom}(A) \subset H_{\varphi}(A) \subset H \subset H_{\varphi}(A)^{*} \subset \operatorname{Dom}(A)^{*}=\mathcal{H}_{-2}(A) \tag{42}
\end{equation*}
$$

Here $H_{\varphi}(A)$ denotes the domain of the adjoint operator $A^{0 *}$

$$
H_{\varphi}(A)=\operatorname{Dom}\left(A^{0 *}\right) .
$$

In the spaces $\mathcal{H}_{2}(A)$ and $\mathcal{H}_{-2}(A)$ we are going to use the norms $\|\cdot\|_{2}^{*}$ and $\|\cdot\|_{-2}^{*}$ defined by (21).

Consider the resolvent $1 /(A-i)$ of the operator $A$ acting in the generalized sense. Let $\varphi \in \operatorname{Dom}(A)^{*}$. Then $(1 /(A-i)) \varphi$ is the linear functional which acts on every $\psi \in H$ in accordance with the formula

$$
\left|\left\langle\frac{1}{A-i} \varphi, \psi\right\rangle\right|=\left|\left\langle\varphi, \frac{1}{A+i} \psi\right\rangle\right| \leq\|\varphi\|_{2}^{*}\left\|\frac{1}{A+i} \psi\right\|_{2} .
$$

It follows that $(1 /(A-i)) \varphi$ is a bounded functional on $H$ and thus is an element from the Hilbert space $H$.

The norm in the space $H_{\varphi}(A)$ will be defined using representation (27) as follows

$$
\begin{align*}
\|\psi\|_{H_{\varphi}(A)} & =\left\|\tilde{\psi}+b(\psi) \frac{A}{A^{2}+1} \varphi\right\|_{H_{\varphi}(A)}  \tag{43}\\
& =\|\tilde{\psi}\|_{2}^{*}+|b(\psi)|
\end{align*}
$$

since $H_{\varphi}(A)$ is a one dimensional extension of the space $\operatorname{Dom}(A)$. The space $H_{\varphi}(A)^{*}$ is adjoint to $H_{\varphi}(A)$. Let $\psi \in \operatorname{Dom}(A)$. Then

$$
\|\psi\|_{B_{\varphi}(A)}=\|\psi\|_{2}^{*}
$$

The inclusion (42) are now obvious.
The second scale of spaces is constructed using the functional $\varphi$, while the standard scale of Hilbert spaces is determined by the operator $A$ only. This determines the main difference between the two scales of Hilbert spaces.

## Singular rank one perturbations

## Form bounded rank one singular perturbations

Let us consider first form bounded rank one perturbations. We have seen that the operator

$$
\begin{equation*}
A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi, \quad \varphi \in \mathcal{H}_{-1}(A) \tag{44}
\end{equation*}
$$

can be defined using the form perturbation technique if the operator $A$ is positive. Another way to define the self-adjoint operator corresponding to the latter formal expression is to consider the linear operator defined by this expression. Consider the operator $A$ defined in the generalized sense. Then formula (44) determines a linear operator on $\mathcal{H}_{1}(A)$ with the range in the
space $\mathcal{H}_{-1}(A)$. The corresponding operator acting in the Hilbert space is defined by the restriction of the linear operator $A_{\alpha}$ to the following domain

$$
\begin{equation*}
\operatorname{Dom}\left(A_{\alpha}\right)=\left\{\psi \in \mathcal{H}_{1}(A) \subset H: A_{\alpha} \psi \in H\right\} \tag{45}
\end{equation*}
$$

The operator $A_{\alpha}$ restricted in this way is self-adjoint and will be considered as the unique self-adjoint operator corresponding to the heuristic expression (44).

Theorem 3 Let $\varphi \in \mathcal{H}_{-1}(A) \backslash H$. Then the domain of the self-adjoint operator $A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi$ coincides with the following set

$$
\begin{equation*}
\operatorname{Dom}\left(A_{\alpha}\right)=\left\{\psi \in H_{\varphi}(A):\langle\varphi, \tilde{\psi}\rangle=-\left(\frac{1}{\alpha}+\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle\right) b(\psi)\right\} \tag{46}
\end{equation*}
$$

$A_{\alpha}$ is a self-adjoint extension of $A^{0}$. For $\alpha=0$ we have $A_{0}=A$.
Proof The linear operator $A_{\alpha}$ maps the vector space $\mathcal{H}_{1}(A)$ to the space $\mathcal{H}_{-1}(A)$. Let $\psi$ be an element from $\mathcal{H}_{1}(A)$. Let us study the question: Under what conditions is the distribution $A_{\alpha} \psi$ an element from the Hilbert space $H$ ? Consider an arbitrary vector $\eta$ from the domain $\operatorname{Dom}\left(A^{0}\right) \subset H$. Then $\left\langle\eta, A_{\alpha} \psi\right\rangle$ is a bounded linear functional on $\eta$ only if $\psi \in \operatorname{Dom}\left(A^{0 *}\right)$, since the following equalities hold

$$
\begin{aligned}
\left\langle\eta, A_{\alpha} \psi\right\rangle & =\langle\eta, A+\alpha\langle\varphi, \psi\rangle \varphi\rangle \\
& =\langle\eta, A \psi\rangle+\alpha\langle\varphi, \psi\rangle\langle\eta, \varphi\rangle \\
& =\langle A \eta, \psi\rangle .
\end{aligned}
$$

We have taken into account that $\langle\eta, \varphi\rangle=0$ (as an element from $\operatorname{Dom}\left(A^{0}\right)$ ) and the operator $A$ is defined in the generalized sense on the vectors from $\mathcal{H}_{1}(A)$.

Let $\psi \in \operatorname{Dom}\left(A^{0 *}\right)=H_{\varphi}(A)$. Then the representation (27) is valid and
the linear operator acts as follows

$$
\begin{align*}
A_{\alpha} \psi= & (A+\alpha\langle\varphi, \cdot\rangle \varphi)\left(\tilde{\psi}+b(\psi) \frac{A}{A^{2}+1} \varphi\right) \\
= & A \tilde{\psi}+\alpha\langle\varphi, \tilde{\psi}\rangle \varphi+b(\psi) \frac{A^{2}}{A^{2}+1} \varphi \\
& +\alpha b(\psi)\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle \varphi  \tag{47}\\
= & \left\{A \tilde{\psi}-b(\psi) \frac{1}{A^{2}+1} \varphi\right\} \\
& +\left[\alpha\langle\varphi, \tilde{\psi}\rangle b(\psi)+\alpha b(\psi)\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle\right] \varphi
\end{align*}
$$

The expression in the braces $\}$ belongs to the original Hilbert space $H$. Therefore the vector element $A_{\alpha} \psi$ belongs to $H$ if and only if the expression in the square brackets [ ] is equal to zero, i.e. if the following equality holds

$$
\begin{equation*}
\langle\varphi, \tilde{\psi}\rangle=-\left(\frac{1}{\alpha}+\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle\right) b(\psi) \tag{48}
\end{equation*}
$$

The parameter

$$
\begin{equation*}
\gamma=-\frac{1}{\alpha}-\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle \tag{49}
\end{equation*}
$$

is real and the adjoint operator $A^{0 *}$ restricted to the domain of functions from $H_{\varphi}(A)$ satisfying the boundary condition (48) is self-adjoint. The restrictions of the operators $A_{\alpha}$ and $A^{0 *}$ to this domain are identical since the expression in the square brackets [ ] in formula (47) vanishes for the elements satisfying the boundary conditions (48). Thus we have proven that the self-adjoint operator defined by the formal expression (44) is a self-adjoint extension of the operator $A^{0}$ described by the parameter $\gamma$ given by (49).

If $\alpha=0$ then the parameter $\gamma=\infty$ and corresponding operator coincides with the original operator $A$. The theorem is thus proven.

The latter theorem describes the rank one perturbations of the operator $A$
using the real parameter $\gamma$. The unitary parameter $v$ describing the $A_{\alpha}$ is given by

$$
\begin{equation*}
v=\frac{1+\alpha\left\langle\varphi, \frac{1}{A+i} \varphi\right\rangle}{1+\alpha\left\langle\varphi, \frac{1}{A-i} \varphi\right\rangle} \tag{50}
\end{equation*}
$$

Considering different $\alpha \in \mathbf{R} \cup\{\infty\}$ all self-adjoint extensions of the symmetric operator $A^{0}$ can be obtained. The formula (48) establishes a one-to-one correspondence between the parameters $\alpha$ and $\gamma, \alpha, \gamma \in \mathbf{R} \cup\{\infty\}$. The parameter $\alpha$ describes all self-adjoint extensions of the symmetric operator $A^{0}$ in an additive manner, while the real parameter $\gamma$ appeared in Krein's formula and the unitary parameter $v$ from the von Neumann formula are not additive.

We have proven once more that the self-adjoint operator corresponding to a singular rank one perturbation is a self-adjoint extension of the symmetric operator $A^{0}$, which is a restriction of the original operator $A$. In the case of form bounded perturbations the self-adjoint operator is uniquely defined even for operators that are not semibounded. Form unbounded perturbations will be studied in the following section.

## Family of rank one form unbounded perturbations

Consider a form unbounded rank one perturbation defined by the same formal expression

$$
\begin{equation*}
A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi, \quad \varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A) \tag{51}
\end{equation*}
$$

We have shown that any self-adjoint operator corresponding to this formal expression is an extension of the symmetric operator $A^{0}$. Considering rank one form bounded perturbations we have determined the unique self-adjoint extension of the operator $A^{0}$ which coincides with the linear operator $A_{\alpha}$ defined in the generalized sense. In the case under consideration the linear operator $A_{\alpha}$ is not in general defined on the space $H_{\varphi}(A)=\operatorname{Dom}\left(A^{0 *}\right)$. The reason is that the linear functional $\varphi$ is not defined on this domain. It is defined on the domain $\operatorname{Dom}(A)=\mathcal{H}_{-2}(A)$. Thus to define the linear operator on the domain $\operatorname{Dom}\left(A^{0 *}\right)$ one has to extend the functional $\varphi$. The extension has to be chosen in such a way that the corresponding sesquilinear form is real. The following lemma describes all possible real extensions.

Lemma 5 Every extension of the functional $\varphi$ to the domain $H_{\varphi}(A)=$ $\operatorname{Dom}\left(A^{0 *}\right)$ is defined by one parameter $c \in \mathbf{C}$. Let

$$
\psi=\tilde{\psi}+b(\psi) \frac{A}{A^{2}+1} \in H_{\varphi}(A)
$$

Then the extended functional $\varphi_{c}$ acts as follows

$$
\begin{equation*}
\left\langle\varphi_{c}, \psi\right\rangle=\langle\varphi, \tilde{\psi}\rangle+\bar{c} b(\psi) \tag{52}
\end{equation*}
$$

This extension defines a real quadratic form $Q[\psi, \psi]=\left\langle\psi,\left(A /\left(A^{2}+1\right)\right) \psi\right\rangle$ with domain $\operatorname{Dom}(Q)=H \dot{+} \mathcal{L}\{\varphi\}$ if and only if the parameter $c$ is real.

Proof The linear functional $\varphi_{c}$ defined by formula (52) is bounded and defined on any element $\psi$ from the domain of the adjoint operator. The norm in the space $H_{\varphi}(A)$ has been defined by (43). The quadratic form corresponding to this extension is real if the parameter $c$ is real.

Consider now an arbitrary bounded linear extension $\hat{\varphi}$ of the functional $\varphi$ to the domain of the adjoint operator. Let $\psi=\tilde{\psi}+b(\psi)\left(A /\left(A^{2}+1\right)\right) \varphi$ be an element from the domain $\operatorname{Dom}\left(A^{0 *}\right)$ of the adjoint operator. Since the functional $\hat{\varphi}$ is a linear extension of $\varphi$, the following equality holds

$$
\langle\hat{\varphi}, \psi\rangle=\langle\varphi, \tilde{\psi}\rangle+b(\psi)\left\langle\hat{\varphi}, \frac{A}{A^{2}+1} \varphi\right\rangle .
$$

Thus every bounded linear extension of the functional $\varphi$ is defined by one parameter

$$
\bar{c}=\left\langle\hat{\varphi}, \frac{A}{A^{2}+1} \varphi\right\rangle .
$$

Consider an arbitrary element $\psi=\tilde{\psi}+q(\psi) \varphi \in \operatorname{Dom}(Q)$, where $\tilde{\psi} \in$ $H, q(\psi) \in \mathbf{C}$. Then the quadratic form can be calculated as follows

$$
\begin{aligned}
Q[\psi, \psi]= & \left\langle\tilde{\psi}+q(\psi) \hat{\varphi}, \frac{A}{A^{2}+1}(\tilde{\psi}+q(\psi) \hat{\varphi})\right\rangle \\
= & \left\langle\tilde{\psi}, \frac{A}{A^{2}+1} \tilde{\psi}\right\rangle+\bar{q}(\psi)\left\langle\varphi, \frac{A}{A^{2}+1} \tilde{\psi}\right\rangle \\
& +q(\psi)\left\langle\tilde{\psi}, \frac{A}{A^{2}+1} \varphi\right\rangle+|q(\psi)|^{2}\left\langle\hat{\varphi}, \frac{A}{A^{2}+1} \varphi\right\rangle \\
= & \Re\left(\left\langle\tilde{\psi}, \frac{A}{A^{2}+1} \tilde{\psi}\right\rangle+2 q(\psi)\left\langle\varphi, \frac{A}{A^{2}+1} \tilde{\psi}\right\rangle\right)+|q(\psi)|^{2} \bar{c}
\end{aligned}
$$

The latter formula shows that the quadratic form is real if and only if the parameter $c$ is real. The lemma is proven.

The following definition will be used below.
Definition 1 Let $\varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$. Then the functional $\varphi_{c}$ is the linear bounded extension of the functional $\varphi$ to the domain $H_{\varphi}(A)$ defined by the condition

$$
\begin{equation*}
\left\langle\varphi_{c}, \frac{A}{A^{2}+1} \varphi\right\rangle=c, \tag{53}
\end{equation*}
$$

where $c \in \mathbf{R}$.
The following theorem describes the domain of the self-adjoint operator corresponding to the formal expression (51) and extension (53).

Theorem 4 Let $\varphi_{c}$ be a linear bounded extension of the functional $\varphi \in$ $\mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$. Then the domain of the self-adjoint operator

$$
A_{\alpha}=A+\alpha\left\langle\varphi_{c}, \cdot\right\rangle \varphi
$$

being a rank one form unbounded perturbation of $A$, coincides with the following set

$$
\begin{gathered}
\operatorname{Dom}\left(A_{\alpha}\right)=\left\{\psi=\tilde{\psi}+b(\psi) \frac{A}{A^{2}+1} \varphi \in \operatorname{Dom}\left(A^{0 *}\right):\right. \\
\left.\langle\varphi, \tilde{\psi}\rangle=-\left(\frac{1}{\alpha}+c\right) b(\psi)\right\}
\end{gathered}
$$

$A_{\alpha}$ is a self-adjoint extension of $A^{0}$ if $c \in \mathbf{R}$. For $\alpha=0$ we have $A_{0}=A$.
Proof The proof is similar to that of theorem 3. The linear operator $A_{\alpha}$
acts as follows on the domain $\operatorname{Dom}\left(A^{0 *}\right) \ni \psi$

$$
\begin{aligned}
A_{\alpha} \psi= & \left(A+\alpha\left\langle\varphi_{c}, \cdot\right\rangle \varphi\right)\left(\tilde{\psi}+b(\psi) \frac{A}{A^{2}+1} \varphi\right) \\
= & A \tilde{\psi}+\alpha\langle\varphi, \tilde{\psi}\rangle \varphi+b(\psi) \frac{A^{2}}{A^{2}+1} \varphi \\
& +\alpha b(\psi)\left\langle\varphi_{c}, \frac{A}{A^{2}+1} \varphi\right\rangle \\
= & \left\{A \hat{\psi}-b(\psi) \frac{1}{A^{2}+1} \varphi\right\} \\
& +[\alpha\langle\varphi, \tilde{\psi}\rangle+b(\psi)+\alpha c b(\psi)] \varphi .
\end{aligned}
$$

The range of the linear operator $A_{\alpha}$ does not belong to the Hilbert space. The domain of the self-adjoint operator $A_{\alpha}$ is equal to the following set

$$
\operatorname{Dom}\left(A_{\alpha}\right)=\left\{\psi \in H_{\varphi}(A): A_{\alpha} \psi \in H\right\} .
$$

The element $A_{\alpha} \psi$ belongs to $H$ if and only if the following condition is satisfied

$$
\langle\varphi, \tilde{\psi}\rangle=-\left(\frac{1}{\alpha}+c\right) b(\psi)
$$

The parameter

$$
\begin{equation*}
\gamma=-\left(\frac{1}{\alpha}+c\right) \tag{54}
\end{equation*}
$$

is real. The operator $A^{0 *}$ restricted to the domain of functions satisfying the latter condition is self-adjoint and coincides with the operator $A_{\alpha}$ restricted to the same domain. Thus the theorem is proven.

The latter theorem describes rank one perturbations using the real parameter $\gamma$ appearing in the boundary condition. The unitary parameter $v$ describing the same self-adjoint extension of the operator $A^{0}$ is given by

$$
v=\frac{1+\alpha(c+i)}{1+\alpha(c-i)} .
$$

Considering different $\alpha \in \mathbf{R} \cup\{\infty\}$ we get all self-adjoint extensions of the operator $A^{0}$. In general the extension depends on the parameter $c$ which describes the extension of the functional $\varphi$.

We have considered only extensions of the functional $\varphi$ determined by the real parameters $c$. One can see that unreal values of this parameter lead to the boundary conditions defining non-self-adjoint operators (the corresponding parameter $\gamma$ is not real). Considering different extensions of the functional (different values of the constant $c \in \mathbf{R}$ ) and one particular $\alpha \neq 0$ we also get all except one self-adjoint extensions of the operator $A^{0}$. The exceptional extension $A^{\infty}$ coincides with the original operator $A$ (see Section).

Singular rank one form unbounded perturbations of homogeneous operators

This section is devoted to the investigation of form unbounded rank one perturbations in the case where the original operator and the element $\varphi$ are homogeneous with respect to a certain group of unitary transformations of the Hilbert space $H$. The extension of the functional $\varphi$ in general can be uniquely defined using the homogeneity properties of the operator and its perturbation.

Lemma 6 Let the self-adjoint operator $A$ and the vector $\varphi \in \operatorname{Dom}(A)^{*}$ be homogeneous with respect to a certain unitary group $G(t)$, i.e. there exist real constants $\beta, \theta \in \mathbf{R}$ such that

$$
\begin{gather*}
G(t) A=t^{-\beta} A G(t) ;  \tag{55}\\
\langle G(t) \varphi, \psi\rangle=\langle\varphi, G(1 / t) \psi\rangle=t^{\theta}\langle\varphi, \psi\rangle \tag{56}
\end{gather*}
$$

for every $\psi \in \operatorname{Dom}(A)$. Then $\varphi$ can be extended as a homogeneous linear bounded functional to the domain $H_{\varphi}(A)$ if and only if

$$
\begin{equation*}
f(t)=i \frac{1-t^{\beta}}{1-t^{-\beta-2 \theta}}\left\langle\varphi, \frac{1}{(A-i)\left(A-t^{\beta} i\right)} \varphi\right\rangle \tag{57}
\end{equation*}
$$

does not depend on $t \neq 1$.
Proof Consider an arbitrary linear bounded extension $\varphi_{c}$ of the functional $\varphi$ which is defined by one parameter (see Lemma 5)

$$
c=\left\langle\varphi_{c}, \frac{A}{A^{2}+1} \varphi\right\rangle .
$$

Suppose that this extension is homogeneous and thus satisfies equation (56). Then the function $f(t)$ can be calculated

$$
\begin{aligned}
f(t) & =i \frac{1-t^{\beta}}{1-t^{-\beta-2 \theta}}\left\langle\varphi, \frac{1}{(A-i)\left(A-t^{\beta} i\right)} \varphi\right\rangle \\
& =\frac{1}{1-t^{-\beta-2 \theta}}\left\langle\varphi_{c},\left(\frac{1}{A-i}-\frac{1}{A-t^{\beta} i}\right) \varphi\right\rangle \\
& =\frac{1}{1-t^{-\beta-2 \theta}}\left\{\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle-\left\langle\varphi_{c}, \frac{1}{A-t^{\beta} i} \varphi\right\rangle\right\} \\
& =\frac{1}{1-t^{-\beta-2 \theta}}\left\{\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle-t^{-\theta}\left\langle\varphi_{c}, \frac{1}{A-t^{\beta} i} G(t) \varphi\right\rangle\right\} \\
& =\frac{1}{1-t^{-\beta-2 \theta}}\left\{\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle-t^{-\beta-\theta}\left\langle\varphi_{c}, G(t) \frac{1}{A-i} \varphi\right\rangle\right\} \\
& =\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle \\
& =c+i\left\langle\varphi, \frac{1}{A^{2}+1} \varphi\right\rangle .
\end{aligned}
$$

It follows that for any homogeneous extension $\varphi_{c}$ the function $f(t)$ is equal to a certain constant determined by the extension. The imaginary part of $f(t)$ is always equal to 1 if the parameter $c$ is real.

Suppose conversely that the function $f(t)$ is equal to a given constant. Let us define the extension of the functional $\varphi$ by the following condition

$$
\begin{equation*}
\left\langle\varphi_{c}, \frac{A}{A^{2}+1} \varphi\right\rangle=c=f(t)-i \tag{58}
\end{equation*}
$$

The imaginary part of $f(t)$ is always equal to 1 :

$$
\begin{aligned}
\Im f(t) & =\frac{1-t^{\beta}}{1-t^{-\beta-2 \theta}}\left\langle\varphi, \frac{A^{2}-t^{\beta}}{\left(A^{2}+1\right)\left(A^{2}+t^{2 \beta}\right)} \varphi\right\rangle \\
& =\frac{1}{1-t^{-\beta-2 \theta}}\left(\left\langle\varphi, \frac{1}{A^{2}+1} \varphi\right\rangle-t^{\beta}\left\langle\varphi, \frac{1}{A^{2}+t^{2 \beta}} \varphi\right\rangle\right) \\
& =\frac{1}{1-t^{-\beta-2 \theta}}\left(1-t^{\beta} t^{-2 \beta-2 \theta}\right)\left\langle\varphi, \frac{1}{A^{2}+1} \varphi\right\rangle \\
& =\left\langle\varphi, \frac{1}{A^{2}+1} \varphi\right\rangle .
\end{aligned}
$$

Therefore the constant $c$ determined by (58) is always real. It is necessary to show that the extension of the functional is homogeneous in this case. In fact it is enough to prove this property only for the elements $(1 /(A-i)) \varphi$ and $(1 /(A+i)) \varphi$. We have:

$$
\begin{aligned}
\langle G(1 / t) & \left.\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle \\
& =\left\langle\varphi_{c}, G(t) \frac{1}{A-i} \varphi\right\rangle \\
& =t^{\theta+\beta}\left\langle\varphi_{c}, \frac{1}{A-t^{\beta} i} \varphi\right\rangle \\
& =t^{\theta+\beta}\left(\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle+\left(t^{\beta} i-i\right)\left\langle\varphi, \frac{1}{(A-i)\left(A-t^{\beta} i\right)} \varphi\right\rangle\right) \\
& =t^{\theta+\beta}\left(\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle-\left(1-t^{-\beta-2 \theta}\right)\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle\right) \\
& =t^{-\theta}\left\langle\varphi_{c}, \frac{1}{A-i} \varphi\right\rangle
\end{aligned}
$$

Similarly one can prove that

$$
\left\langle G(1 / t) \varphi_{c}, \frac{1}{A+i} \varphi\right\rangle=t^{-\theta}\left\langle\varphi_{c}, \frac{1}{A+i} \varphi\right\rangle
$$

and the lemma is proven.

It has been shown during the proof of the latter theorem that every homogeneous extension of the functional $\varphi$ is defined by the real parameter $c$. It follows that every homogeneous extension necessarily defines a real extension of the quadratic form $Q[\psi, \psi]=\left\langle\psi,\left(A /\left(A^{2}+1\right)\right) \psi\right\rangle$.

If the unitary group $G$ consists of only two elements

$$
G=\{G(1), G(-1)\}
$$

then the homogeneous extension can always be constructed and it is unique. This condition is true for example for the first derivative operator and Dirac operators in one dimension with the delta potential. The group of the unitary transformations coincides with the symmetry group with respect to the origin. These operators are studied at the end of this chapter.

Lemma 6 implies that if the original operator $A$ and the vector $\varphi \in$ $\mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$ are homogeneous and if the corresponding function $f(t)$ is constant, then there exists a unique self-adjoint operator corresponding to the formal rank one perturbation and possessing the same symmetry properties. Therefore in this case the unique self-adjoint operator can be determined even if the rank one perturbation is not form bounded, but we have to use extra assumptions to determine this operator. The function $f(t)$ is not always constant. For example consider the following operator with rank one singular perturbation

$$
-\Delta+\alpha\langle\delta, \cdot\rangle \delta
$$

where $\Delta$ is the Laplace operator in $L_{2}\left(\mathbf{R}^{n}\right)$ and $\delta$ is the delta function with the support at the origin. If $n=1,3$ then $f(t)$ is equal to a constant and the homogeneous extension can be constructed (see Section ). If $n=2$ then the function $f(t)$ is not constant and no homogeneous extension exists.

## Resolvent formulas

The resolvent of the perturbed operator can be calculated explicitly using the general Krein formula (31) and taking into account (54)

$$
\begin{equation*}
\frac{1}{A_{\alpha}-z}=\frac{1}{A-z}-\frac{1}{1 / \alpha+c+\left\langle\varphi, \frac{1+A z}{A-z} \frac{1}{A^{2}+1} \varphi\right\rangle}\left\langle\frac{1}{A-\bar{z}} \varphi, \cdot\right\rangle \frac{1}{A-z} \varphi \tag{59}
\end{equation*}
$$

The parameter $c$ which appears in the latter formula can be chosen arbitrary for $\mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$ perturbations. For $\mathcal{H}_{-1}(A)$ perturbations instead $c$ is determined according to

$$
c=\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle .
$$

Let us introduce the following Nevanlinna function

$$
\begin{align*}
F(z) & =c+\left\langle\varphi, \frac{1+A z}{A-z} \frac{1}{A^{2}+1} \varphi\right\rangle  \tag{60}\\
& =c+Q(z)
\end{align*}
$$

where $Q(z)$ is the $Q$-function associated with the operator $A$ and the vector $\varphi \in \mathcal{H}_{-2}(A)$. Using this notation formula (59) just coincides with the formula for the resolvent of rank one bounded perturbation (2).

If $\varphi \in \mathcal{H}_{-1}(A)$ then the function $F(z)$ is given by

$$
F(z)=c+Q(z)=\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle+\left\langle\varphi, \frac{1+A z}{A-z} \frac{1}{A^{2}+1} \varphi\right\rangle=\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle
$$

which is again formula (3). For $\varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$ the function $F(z)$ can be calculated using the extension $\varphi_{c}$ of the functional $\varphi$ as follows

$$
F(z)=\left\langle\varphi_{c}, \frac{1}{A-z} \varphi\right\rangle
$$

Let us introduce the function

$$
F_{\alpha}(z)=\left\langle\varphi_{c}, \frac{1}{A_{\alpha}-z} \varphi_{c}\right\rangle
$$

describing the rank one perturbations of the operator $A_{\alpha}$.
All five critical formulas for the rank one perturbation [?] can be written in the same form for bounded, form bounded and form unbounded perturbations

$$
\begin{gather*}
F_{\alpha}(z)=\frac{F(z)}{1+\alpha F(z)}  \tag{61}\\
\frac{1}{A_{\alpha}-z} \varphi=\frac{1}{1-\alpha F(z)} \frac{1}{A-z} \varphi \tag{62}
\end{gather*}
$$

$$
\begin{gather*}
\frac{1}{A_{\alpha}-z}=\frac{1}{A-z}-\frac{\alpha}{1+\alpha F(z)}\left(\frac{1}{A-\bar{z}} \varphi, \cdot\right) \frac{1}{A-z} \varphi ;  \tag{63}\\
\operatorname{Tr}\left[\frac{1}{A-z}-\frac{1}{A_{\alpha}-z}\right]=\frac{d}{d z} \ln (1+\alpha F(z)) .  \tag{64}\\
\int_{\mathbf{R}}\left[d \mu_{\alpha}(E)\right] d \alpha=d E, \tag{65}
\end{gather*}
$$

where $\mu_{\alpha}$ is the spectral measure corresponding to the operator $A_{\alpha}$. The formulas can be proven following the main lines of Section. We note that the result does not depend on the parameter $c$, which can be chosen arbitrary for form unbounded perturbations.

We are now going to prove that if the operators $A$ and $B$ are two selfadjoint extensions of one symmetric densely defined operator $A^{0}$ having deficiency indices $(1,1)$, then the operator $B$ is a rank one singular perturbation of $A$.

Theorem 5 Let $A^{0}$ be a densely defined symmetric operator with the deficiency indices $(1,1)$. Let $A$ and $B$ be two self-adjoint extensions of the operator $A^{0}$. Then the operator $B$ is a rank one singular perturbation of the operator $A$.

Proof Theorem 2 implies that the resolvents of the operators $A$ and $B$ are related by formula (31), where $g_{i}$ is the deficiency element for $A^{0}$ at the point $i$. Consider the vector $\varphi$ given by

$$
\varphi=(A-i) g_{i} .
$$

Since $g_{i}$ belongs to the Hilbert space, the vector $\varphi$ is an element of the Hilbert space $\mathcal{H}_{-2}(A)$. The closure of the operator $A^{0}$ then coincides with the restriction of the operator $A$ to the domain of functions orthogonal to the vector $\varphi$. Obviously the vector $\varphi$ does not belong to the Hilbert space, since the operator $A^{0}$ is densely defined.

To prove the theorem it is enough to show that there exists a real constant $\alpha$ such that

$$
B=A+\alpha\langle\varphi, \cdot\rangle \varphi .
$$

If the parameter $\gamma$ appearing in Krein's formula is infinite, then the operators $A$ and $B$ coincide. It follows that the operator $B$ is a rank one perturbation of the operator $A$ with the coupling constant $\alpha$ equal to zero.

If the parameter $\gamma$ is finite, $\gamma \neq \infty$, then we have to distinguish between form bounded and form unbounded perturbations. Suppose that $\varphi \in \mathcal{H}_{-1}(A)$. It follows from Theorem 3 that the operator $A_{\alpha}$ coincides with the operator $B$ if

$$
\alpha=\frac{-1}{\gamma+\left\langle\varphi, \frac{A}{A^{2}+1} \varphi\right\rangle} .
$$

Suppose now that $\varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$. To define the corresponding rank one perturbations we fix the real parameter $c=\left\langle\varphi_{c},\left(A /\left(A^{2}+1\right)\right) \varphi\right\rangle$. Then the operator $A_{\alpha}=A+\alpha\left\langle\varphi_{c}, \cdot\right\rangle \varphi$ coincides with $B$ if the coupling constant is chosen equal to

$$
\alpha=\frac{-1}{\gamma+c} .
$$

The theorem is proven.

The same result holds in the case when the symmetric operator $A^{0}$ is not densely defined. The vector $\varphi$ belongs to the original Hilbert space in this case and all except one self-adjoint extension of the restricted operator $A^{0}$ are defined on the same domain. The exceptional extension is not an operator but an operator relation. It corresponds to the infinite value of the coupling constant $\alpha$ (see Section).

The latter theorem implies that the self-adjoint extensions of any symmetric operator with unit deficiency indices can be considered as rank one perturbations of a self-adjoint operator and can therefore be parametrized by the additive parameter $\alpha$ instead of the nonadditive parameters $\gamma$ and $v$ appearing in the boundary conditions.

## Approximations of singular rank one perturbations

## Norm convergence of the approximations

We are going to discuss how to approximate singular rank one perturbations by bounded ones. More precisely we consider operators

$$
\begin{equation*}
A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi \tag{66}
\end{equation*}
$$

given by a singular perturbation, i.e. $\varphi \in \mathcal{H}_{-2}(A) \backslash H$. Let $\varphi_{n} \in H$ be a sequence of functions from the Hilbert space. Consider the sequence of
operators with bounded rank one perturbations

$$
\begin{equation*}
A_{\alpha}^{n}=A+\alpha\left\langle\varphi_{n}, \cdot\right\rangle \varphi_{n} \tag{67}
\end{equation*}
$$

The self-adjoint operators $A_{\alpha}$ and $A_{\alpha}^{n}$ have in general different domains, since $\operatorname{Dom}\left(A_{\alpha}^{n}\right)=\operatorname{Dom}(A)$. But one can consider linear operators defined by (66) and (67) in the generalized sense. Two different types of convergence will be studied.

Considering $A_{\alpha}$ and $A_{\alpha}^{n}$ only as self-adjoint operators in the Hilbert space we can study the corresponding resolvent operators, which are bounded operators and therefore have common domain $H$. We say that the operators $A_{\alpha}^{n}$ converge to $A_{\alpha}$ in the strong resolvent sense if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|\frac{1}{A_{\alpha}^{n}-z}-\frac{1}{A_{\alpha}-z}\right\|=0 \tag{68}
\end{equation*}
$$

for some $z, \Im z \neq 0$.
Considering the linear operators defined by formal expressions (66) and (67) in the generalized sense suppose that these operators can be defined as bounded linear operators on a certain normed space D with the range in perhaps a different normed space $\mathrm{D}^{\prime}$. We say that the operators $A_{\alpha}^{n}$ converge to the operator $A_{\alpha}$ in the sense of linear operators if and only if

$$
\begin{align*}
& \mathrm{D} \quad \operatorname{Dom}\left(A_{\alpha}\right),  \tag{69}\\
& \mathrm{D} \quad \operatorname{Dom}(A)=\operatorname{Dom}\left(A_{\alpha}^{n}\right),
\end{align*}
$$

and the following limit holds

$$
\begin{equation*}
\left\|A_{\alpha}^{n}-A_{\alpha}\right\|_{B\left(\mathrm{D} \rightarrow \mathrm{D}^{\prime}\right)} \rightarrow_{n \rightarrow \infty} 0 \tag{70}
\end{equation*}
$$

where $\|\cdot\|_{B\left(\mathrm{D} \rightarrow \mathrm{D}^{\prime}\right)}$ denotes the norm of the linear operator acting on D with the range in $\mathrm{D}^{\prime}$. Note that the operators $A_{\alpha}^{n}, A_{\alpha}$ defined in the original Hilbert space are not necessarily bounded, but these operators could be bounded as operators mapping D to $\mathrm{D}^{\prime}$.

Consider first the approximations in the sense of linear operators. We start by investigating the question of how to approximate arbitrary $\varphi \in$ $\mathcal{H}_{-2}(A) \backslash H$ by vectors from the Hilbert space.

Lemma 7 Let $f$ be an element from $H \backslash \mathcal{H}_{2}(A)$ and $\varphi$ be an element from $\mathcal{H}_{-2}(A)$; then for any $c$ there exists a sequence $\varphi_{n}$ of elements from $H$ converging to $\varphi$ in the $\mathcal{H}_{-2}(A)$ norm such that $\left\langle f, \varphi_{n}\right\rangle$ converges to $c$.

Proof The original Hilbert space $H=\mathcal{H}_{0}(A)$ is dense in $\mathcal{H}_{-2}(A)$. It follows that there exists a sequence $\tilde{\varphi}_{n}$ of elements from $H$ converging in the $\mathcal{H}_{-2}$ norm to $\varphi$. If the sequence $\left\langle f, \tilde{\varphi}_{n}\right\rangle=a_{n}$ converges to $c$, then the lemma is proven. If it does not then let us consider a sequence $\psi_{n} \in \mathcal{H}_{0}(A)$ with unit $\mathcal{H}_{-2}$ norm $\left\|\psi_{n}\right\|_{\mathcal{H}_{-2}}=1$ such that $\left|\left\langle f, \psi_{n}\right\rangle\right|$ diverges to $\infty$. Such a sequence exists because $f \notin \mathcal{H}_{2}(A)$. We can then choose a subsequence such that $c-a_{n} /\left\langle f, \psi_{n}\right\rangle \rightarrow 0$. We keep the same notation for the chosen subsequence. Consider then the sequence

$$
\varphi_{n}=\tilde{\varphi}_{n}+\frac{c-a_{n}}{\left\langle f, \psi_{n}\right\rangle} \psi_{n}
$$

The following estimates are valid

$$
\left\|\varphi_{n}-\varphi\right\|_{\mathcal{H}_{-2}} \leq\left\|\tilde{\varphi}_{n}-\varphi\right\|_{\mathcal{H}_{-2}}+\left|\frac{c-a_{n}}{\left\langle f, \psi_{n}\right\rangle}\right|
$$

It follows that $\varphi_{n}$ converge to $\varphi$ in the $\mathcal{H}_{-2}$ norm. At the same time the sequence $\left\langle f, \tilde{\varphi}_{n}\right\rangle=a_{n}+c-a_{n}=c$ obviously converges to $c$, hence the lemma is proven.

The convergence in $\mathcal{H}_{-2}(A)$ was crucial for the proof of the lemma. For example the following lemma is valid.

Lemma 8 Let $f$ be an element from $\mathcal{H}_{1}(A)$ and $\varphi$ be an element from $\mathcal{H}_{-1}(A)$. Then for every sequence $\varphi_{n} \in \mathcal{H}_{0}(A)$ converging to $\varphi$ in the norm of $\mathcal{H}_{-1}(A)$ the sequence $\left\langle f, \varphi_{n}\right\rangle$ converges to $\langle f, \varphi\rangle$.

Proof The statement of the lemma follows from the fact that strong convergence of bounded functionals implies weak convergence.

The operators $A_{\alpha}$ and $A_{\alpha}^{n}$ are defined on the common domain $H_{\varphi}(A)=$ $\operatorname{Dom}\left(A^{0 *}\right)$ :

$$
\begin{aligned}
\operatorname{Dom}\left(A^{0 *}\right) & \supset \operatorname{Dom}\left(A_{\alpha}\right) \\
\operatorname{Dom}\left(A^{0 *}\right) & \supset \operatorname{Dom}(A)=\operatorname{Dom}\left(A_{\alpha}^{n}\right) .
\end{aligned}
$$

The range of the linear operators $A_{\alpha}^{n}, A_{\alpha}$ belongs to the space $\mathcal{H}_{-2}(A)$ with the standard norm.

Theorem 6 Let the sequence $\varphi_{n} \in H$ converge to $\varphi$ in $\mathcal{H}_{-2}(A)$ and $\left\langle\varphi_{n},\left(A /\left(A^{2}+1\right)\right) \varphi\right\rangle$ converge to $c$. Then the sequence of linear operators $A_{\alpha}^{n}=A+\alpha\left\langle\varphi_{n}, \cdot\right\rangle \varphi_{n} d e$ fined on the domain $H_{\varphi}(A)$ converges in the operator norm to the operator $A_{\alpha}$.

Proof Consider an arbitrary element $g=\tilde{g}+b(g)\left(A /\left(A^{2}+1\right)\right) \varphi \in H_{\varphi}(A)$. Then the following estimates are valid

$$
\begin{aligned}
\| & \left(A_{\alpha}^{n}-A_{\alpha}\right) g \|_{-2} \\
= & |\alpha|\left\|\left\langle\varphi_{n}, g\right\rangle \varphi_{n}-\left\langle\varphi_{c}, g\right\rangle \varphi\right\|_{-2} \\
= & |\alpha| \|\left\langle\varphi_{n}, \tilde{g}\right\rangle \varphi_{n}+b(g)\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi\right\rangle \varphi_{n} \\
& -\langle\varphi, \tilde{g}\rangle \varphi-b(g)\left\langle\varphi_{c}, \frac{A}{A^{2}+1} \varphi\right\rangle \varphi \|_{-2} \\
\leq & |\alpha|\left\{\left|\left\langle\varphi_{n}, \tilde{g}\right\rangle-\langle\varphi, \tilde{g}\rangle\right|\left\|\varphi_{n}\right\|_{-2}+|\langle\varphi, \tilde{g}\rangle|\left\|\varphi_{n}-\varphi\right\|_{-2}\right. \\
& \left.+|b(g)|\left|\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi\right\rangle-c\right|\left\|\varphi_{n}\right\|_{-2}+|b(g)||c|\left\|\varphi_{n}-\varphi\right\|_{-2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\leq & |\alpha|\left\{\left\|\varphi_{n}\right\|_{-2}\left\|\varphi_{n}-\varphi\right\|_{-2}\|\tilde{g}\|_{2}+\left\|\varphi_{n}-\varphi\right\|_{-2}\|\varphi\|_{-2}\|\tilde{g}\|_{2}\right. \\
& \left.+\left|\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi\right\rangle-c\right|\left\|\varphi_{n}\right\|_{-2}|b(g)|+|c|\left\|\varphi_{n}-\varphi\right\|_{-2}|b(g)|\right\} \\
\leq & |\alpha|\left\{\left(\left\|\varphi_{n}\right\|_{-2}+\|\varphi\|_{-2}+|c|\right)\left\|\varphi_{n}-\varphi\right\|_{-2}\right. \\
& \left.+\left\|\varphi_{n}\right\|_{-2}\left|\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi\right\rangle-2 c\right|\right\}\|g\|_{H_{\varphi}(A)} .
\end{aligned}
$$

The sequence $\varphi_{n}$ converges to $\varphi$ in the $\mathcal{H}_{-2}(A)$ norm, the sequence \| $\varphi_{n} \|_{-2}$ is bounded and the sequence $\left\langle\varphi_{n},\left(A /\left(A^{2}+1\right)\right) \varphi\right\rangle$ converges to $c$. It follows that the linear operators converge in the operator norm.

Theorem 7 Let $\varphi \in \mathcal{H}_{-2}(A) \backslash H$. Then there exists a sequence $\varphi_{n} \in H$ converging to $\varphi$ in the $\mathcal{H}_{-2}(A)$ norm such that the sequence of linear operators $A_{\alpha}^{n}=A+\alpha\left\langle\varphi_{n}, \cdot\right\rangle \varphi_{n}$ defined on the domain $H_{\varphi}(A)$ converges in the operator norm to the operator $A_{\alpha}=A+\alpha\left\langle\varphi_{c}, \cdot\right\rangle \varphi$.

Proof The element $\left(A /\left(A^{2}+1\right)\right) \varphi$ belongs to the Hilbert space but does not belong to the domain of the operator. It follows from Lemma 7 that there exists a sequence $\varphi_{n}$ converging to $\varphi$ in the $\mathcal{H}_{-2}(A)$ norm and such that $\left\langle\varphi_{n},\left(A /\left(A^{2}+1\right)\right) \varphi\right\rangle$ converge to $c$. It follows from Theorem 6 that the operators $A_{\alpha}^{n}$ converge to $A_{\alpha}$ in the operator norm.

The approximating sequence $\varphi_{n}$ can be constructed using the spectral representation of the original operator $A$. If the element $\varphi \in \mathcal{H}_{-2}(A)$ then there exists a certain measure $d \mu(\lambda)$ such that

$$
\left\langle\frac{1}{A-i} \varphi, \frac{1+z A}{A-z} \frac{1}{A-i} \varphi\right\rangle=\int_{-\infty}^{+\infty} \frac{1+z \lambda}{\lambda-z} \frac{1}{\lambda^{2}+1} d \mu(\lambda)
$$

and $\int_{-\infty}^{+\infty} \frac{d \mu(\lambda)}{\lambda^{2}+1}<\infty$. Consider the spaces $\mathcal{H}_{-1,-2}(A)$ and $\mathcal{H}_{-2,-1}(A)$ formed by the elements from $\mathcal{H}_{-2}(A)$ satisfying the following additional conditions

$$
\int_{-\infty}^{0} \frac{|\lambda| d \mu(\lambda)}{\lambda^{2}+1}<\infty \text { and } \int_{0}^{\infty} \frac{|\lambda| d \mu(\lambda)}{\lambda^{2}+1}<\infty
$$

respectively. The following lemma can be proven.
Lemma 9 Let $\varphi \in \mathcal{H}_{-2}(A) \backslash\left(\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A)\right)$ then there exist two sequences $c_{n}, d_{n} \rightarrow \infty$ such that

$$
\lim _{n \rightarrow \infty} \int_{-c_{n}}^{d_{n}} \frac{\lambda}{\lambda^{2}+1} d \mu(\lambda)=c
$$

Proof Convergence of the integral $\int_{\infty}^{+\infty}\left(1 /\left(\lambda^{2}+1\right)\right) d \mu(\lambda)$ implies that the two sequences

$$
(n+1) \int_{n}^{n+1} \frac{1}{\lambda^{2}+1} d \mu(\lambda)
$$

and

$$
(n+1) \int_{-n-1}^{-n} \frac{\lambda}{\lambda^{2}+1} d \mu(\lambda)
$$

$n=1,2, \ldots$, have zero limits when $n \rightarrow \infty$. The sums of both sequences are diverging, since

$$
\begin{aligned}
& \int_{n}^{n+1} \frac{\lambda}{\lambda^{2}+1} d \mu(\lambda) \leq(n+1) \int_{n}^{n+1} \frac{1}{\lambda^{2}+1} d \mu(\lambda) \\
& \int_{-n}^{-n-1} \frac{|\lambda|}{\lambda^{2}+1} d \mu(\lambda) \leq(n+1) \int_{-n}^{-n-1} \frac{1}{\lambda^{2}+1} d \mu(\lambda)
\end{aligned}
$$

The sequences have different signs. It follows that the sequence

$$
\int_{-n}^{m} \frac{\lambda}{\lambda^{2}+1} d \mu(\lambda)
$$

can converge to any real number when $n, m \rightarrow \infty$. The lemma is proven.

Consider the approximating sequence of the elements from the Hilbert space
$\varphi_{n}=E_{\left(-c_{n}, d_{n}\right)}(A) \varphi$, where $E(A)$ denotes the spectral projector for the operator $A$. The following limit holds

$$
\lim _{n \rightarrow \infty}\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi_{n}\right\rangle=c
$$

The sequence $\varphi_{n}$ will be used in the following section to construct the approximations of rank one perturbations in the strong resolvent sense.

Strong resolvent convergence of the approximations

In this section we study the strong resolvent convergence of the operators. We have shown in fact that the difference of the resolvents of the original and perturbed operators has rank one. We prove first that every rank one $\mathcal{H}_{-1}$ perturbation can be approximated in the strong resolvent sense.

Theorem 8 Let A be a self-adjoint operator in the Hilbert space $\mathcal{H}$ and $\varphi$ be an element from $\mathcal{H}_{-1}(A)$. Let the sequence $\varphi_{n} \in \mathcal{H}$ converge to $\varphi$ in the norm $\mathcal{H}_{-1}(A)$. Then the sequence of operators $A_{\alpha}^{n}=A+\alpha\left\langle\varphi_{n}, \cdot\right\rangle \varphi_{n}$ converges to the operator $A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi$ in the strong resolvent sense for all $z, \Im z \neq 0$.

Proof Since the $\left\{1 /\left(A_{\alpha}^{n}-z\right)\right\}$ are uniformly bounded it is enough to prove the weak convergence of the resolvents. Consider two arbitrary vectors $\psi_{1}, \psi_{2}$ from the Hilbert space. The convergence in the space $\mathcal{H}_{-1}(A)$ implies

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left\langle\frac{1}{A-\bar{z}}\left(\varphi-\varphi_{n}\right), \psi_{1}\right\rangle=0  \tag{71}\\
& \lim _{n \rightarrow \infty}\left\langle\psi_{2}, \frac{1}{A-z}\left(\varphi_{n}-\varphi\right)\right\rangle=0 \tag{72}
\end{align*}
$$

Moreover the quadratic form of the resolvent converges at the point $z=i$ and similarly at all other points in the resolvent set of $A$ and we have

$$
\begin{aligned}
& \left|\left\langle\varphi_{n}, \frac{1}{A-z} \varphi_{n}\right\rangle-\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle\right| \\
\leq & 2\left\|\varphi_{n}-\varphi\right\|_{-1}\left(\left\|\varphi_{n}\right\|_{-1}+\|\varphi\|_{-1}\right) \rightarrow 0 .
\end{aligned}
$$

We have for the difference of the resolvents

$$
\begin{aligned}
\left\langle\psi_{2}, \frac{1}{A_{\alpha}^{n}-z} \psi_{1}\right\rangle & -\left\langle\psi_{2}, \frac{1}{A_{\alpha}-z} \psi_{1}\right\rangle \\
= & \frac{\alpha}{1+\alpha\left\langle\varphi, \frac{1}{A-z} \varphi\right\rangle}\left\langle\frac{1}{A-\bar{z}} \varphi, \psi_{1}\right\rangle\left\langle\psi_{2}, \frac{1}{A-z} \varphi\right\rangle \\
& -\frac{\alpha}{1+\alpha\left\langle\varphi_{n}, \frac{1}{A-z} \varphi_{n}\right\rangle}\left\langle\frac{1}{A-\bar{z}} \varphi_{n}, \psi_{1}\right\rangle\left\langle\psi_{2}, \frac{1}{A-z} \varphi_{n}\right\rangle .
\end{aligned}
$$

The weak resolvent convergence follows from the formulas (71),(72) and the convergence of the quadratic form of the resolvent. The denominator in the first quotient does not vanish because $\Im z \neq 0$.

Let us study rank one $\mathcal{H}_{-2}$ perturbations.
Theorem 9 Let $A$ be a self-adjoint operator and $\varphi$ be a functional from $\mathcal{H}_{-2}(A),\|(1 /(A-i)) \varphi\|=1$. Let $\varphi_{n}$ be any sequence from the Hilbert space converging to $\varphi$ in $\mathcal{H}_{-2}(A)$ and let $\lim _{n \rightarrow \infty}\left\langle\varphi_{n},\left(A /\left(A^{2}+1\right)\right) \varphi_{n}\right\rangle=c$. Then the sequence of self-adjoint operators

$$
A_{\alpha}^{n}=A+\alpha\left\langle\varphi_{n}, \cdot\right\rangle \varphi_{n}
$$

converges to $A_{\alpha}$ in the strong resolvent sense. If

$$
\lim _{n \rightarrow \infty}\left|\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi_{n}\right\rangle\right|=\infty
$$

the operators $A_{\alpha}^{n}$ converge to the original operator in the strong resolvent sense.

Proof The first part of the theorem can be proven using the fact that the convergence in $\mathcal{H}_{-2}$ implies weak convergence of the resolvents and formulas (71),(72) hold for every $\psi_{1}, \psi_{2} \in \mathcal{H}$. Calculations similar to thoes carried out during the proof of Theorem 8 lead to the result which has to be proven. One has to take into account only that

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left\langle\varphi_{n}\right. & \left.\frac{1}{A-z} \varphi_{n}\right\rangle \\
& =\lim _{n \rightarrow \infty}\left\langle\varphi_{n}, \frac{1+A z}{A-z} \frac{1}{A^{2}+1} \varphi_{n}\right\rangle+\lim _{n \rightarrow \infty}\left\langle\varphi_{n}, \frac{A}{A^{2}+1} \varphi_{n}\right\rangle \\
& =\left\langle\varphi, \frac{1+A z}{A-z} \frac{1}{A^{2}+1} \varphi\right\rangle+c \\
& =c+Q(z) \\
& =F(z)
\end{aligned}
$$

where $F(z)$ appeared in (60). Consider now the case where

$$
\lim _{n \rightarrow \infty}\left|\left(\varphi_{n}, \frac{1}{A-i} \varphi_{n}\right)\right|=\infty
$$

The difference of the resolvents of the original operator and its rank one perturbation is the rank one operator

$$
\frac{1}{A_{\alpha}^{n}-z}-\frac{1}{A-z}=-\frac{\alpha}{1+\alpha\left\langle\varphi_{n}, \frac{1}{A-z} \varphi_{n}\right\rangle}\left\langle\frac{1}{A-\bar{z}} \varphi_{n}, \cdot\right\rangle \frac{1}{A-z} \varphi_{n} .
$$

The first term on the right hand side of the last equality converges to zero. It follows that the difference of the resolvents converges weakly to zero since $(1 /(A-z)) \varphi_{n}$ and $(1 /(A-\bar{z})) \varphi_{n}$ converge to $(1 /(A-z)) \varphi$ and $(1 /(A-\bar{z})) \varphi$ respectively. Hence the theorem is proven.

Let $\varphi \in \mathcal{H}_{-2}(A) \backslash\left(\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A)\right)$. Then it is possible to construct a sequence of operators converging to $A_{\alpha}$ in the strong resolvent sense, according to the following

Theorem 10 Let $\varphi \in \mathcal{H}_{-2}(A) \backslash\left(\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A)\right)$. Then there exist two sequences $c_{n}, d_{n} \rightarrow \infty$ such that $\varphi_{n}=E_{\left(-c_{n}, d_{n}\right)}(A) \varphi$ determines the sequence of self-adjoint operators $A_{\alpha}^{n}=A+\alpha\left\langle\varphi_{n}, \cdot\right\rangle \varphi_{n}$ involving bounded perturbations of $A$ converging to the perturbed operator $A_{\alpha}=A+\alpha\langle\varphi, \cdot\rangle \varphi$ in the strong resolvent sense.

Proof The statement follows easily from Lemma 9 and Theorem 9.

The latter theorem shows how to construct the approximating sequence $\varphi_{n}$ leading to the approximations of the operator $A_{\alpha}$ in the strong resolvent sense.

If $\varphi \in \mathcal{H}_{-1}(A)$ then the sequence $\varphi_{n} \in H$ converging to $\varphi$ in the $\mathcal{H}_{-1}$ norm defines a sequence of self-adjoint operators converging to the perturbed operator $A_{\alpha}$ in the strong resolvent sense. If $\varphi \in \mathcal{H}_{-2}(A) \backslash\left(\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A)\right)$ then there exists a sequence $\varphi_{n}$ converging to $\varphi$ in the $\mathcal{H}_{-2}$ norm such that the sequence of the corresponding perturbed operators converges to $A_{\alpha}$ in the strong resolvent sense. If $\varphi \in \mathcal{H}_{-1,-2}(A) \backslash \mathcal{H}_{-1}(A)$ or $\varphi \in \mathcal{H}_{-2,-1}(A) \backslash \mathcal{H}_{-1}(A)$ then every sequence $\varphi_{n}$ converging to $\varphi$ in the $\mathcal{H}_{-2}$ norm defines a sequence of self-adjoint operators converging to the original operator in the strong resolvent sense. It follows that not every form unbounded rank one perturbation can be approximated in the strong resolvent sense by operators with bounded perturbations. For example if the original operator $A$ is semibounded, then the subspace $\mathcal{H}_{-2}(A) \backslash\left(\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A)\right)$ is trivial and no form unbounded perturbation of such an operator can be approximated in the strong resolvent sense without the renormalization of the coupling constant. See Section, where such an approximation with the renormalized coupling constant is constructed for the Laplace operator with the delta interaction in $\mathbf{R}^{3}$.

Approximations in the sense of linear operators can be constructed for every rank one perturbation. If the perturbation is form bounded then every sequence $\varphi_{n}$ converging to $\varphi$ in the $\mathcal{H}_{-1}$ norm determines a sequence of operators converging to the perturbed operator in the norm of linear operators. If $\varphi \in \mathcal{H}_{-2}(A) \backslash \mathcal{H}_{-1}(A)$ then one can prove only the existence of the approximating sequence.

## Differential operators with rank one

 singular perturbations
## Point interactions in dimension three

We consider now the Schrödinger operator in dimension three defined by the heuristic expression:

$$
\begin{equation*}
L_{\alpha}=-\Delta+\alpha \delta, \tag{73}
\end{equation*}
$$

where $\Delta$ is the Laplace operator, $\alpha$ is a real coupling constant and $\delta$ is a Dirac delta function in dimension three. This operator was studied for the first time from the mathematical point of view by F.A.Beresin and L.D.Faddeev [?]. The operator $L_{\alpha}$ to be defined in $L_{2}\left(\mathbf{R}^{3}\right)$ can be considered as a singular rank one perturbation of the Laplace operator because $\delta \varphi=\varphi(0) \delta=\langle\varphi, \delta\rangle \delta$ and the generalized function $\delta$ is an element from $\mathcal{H}_{-2}(-\Delta) \backslash \mathcal{H}_{-1}(-\Delta)$ in three dimensional space. Consider the group $S(t), t>0$, of scaling transformations of $L_{2}\left(\mathbf{R}^{3}\right)$ defined as follows: for every function $\psi \in D$ and distribution $f$

$$
\begin{aligned}
& (S(t) \psi)(x)=t^{3 / 2} \psi(t x) \\
& \langle S(t) f, \psi\rangle=\langle f, S(1 / t) \psi\rangle
\end{aligned}
$$

The Laplace operator and the delta function are homogeneous with respect to the group $S(t)$ :

$$
\begin{aligned}
S(t) \Delta & =t^{2} \Delta S(c) \\
S(t) \delta & =t^{-3 / 2} \delta
\end{aligned}
$$

The perturbed operator coincides with one of the self-adjoint extensions of the symmetric Laplace operator $-\Delta_{0}$ defined on functions from $W_{2}^{2}\left(\mathbf{R}^{3}\right)$ vanishing at the origin. The domain $\operatorname{Dom}\left(-\Delta_{0}^{*}\right)$ of the adjoint operator $-\Delta_{0}^{*}$ coincides with the space $W_{2}^{2}\left(\mathbf{R}^{3} \backslash\{0\}\right)$. The distribution $\delta$ possesses a unique extension to the set $W_{2}^{2}\left(\mathbf{R}^{3} \backslash\{0\}\right)$. To calculate the parameter $c$ defining the extension of the functional $\varphi=\delta$ one has to take into account that the vector $\delta$ does not fulfil the normalization condition $\|(1 /(A-i)) \varphi\|=1$. Formula (58) should be modified as follows

$$
\left\langle\varphi_{c}, \frac{A}{A^{2}+1} \varphi\right\rangle=c=f(t)-i\left\|\frac{1}{A-i} \varphi\right\|^{2}
$$

Therefore the parameter $c$ is equal to

$$
\begin{aligned}
c & =i \frac{1-t^{2}}{1-t}\left\langle\frac{1}{-\Delta+i} \delta, \frac{1}{-\Delta-t^{2} i} \delta\right\rangle-i\left\|\frac{1}{-\Delta-i} \delta\right\| \\
& =i(1+t)\left\langle\frac{e^{i \sqrt{-i}|x|}}{4 \pi|x|}, \frac{e^{i t \sqrt{i}|x|}}{4 \pi|x|}\right\rangle-i\left\langle\frac{e^{i \sqrt{i}|x|}}{4 \pi|x|}, \frac{e^{i \sqrt{i}|x|}}{4 \pi|x|}\right\rangle \\
& =-\frac{1}{4 \pi \sqrt{2}}
\end{aligned}
$$

Any function $\psi$ belongs to the domain of the adjoint operator $\psi \in \operatorname{Dom}\left(-\Delta_{0}^{*}\right)$ if and only if

$$
\psi(x)=\tilde{\psi}(x)+\frac{b(\psi)}{2}\left(\frac{e^{(-1 / \sqrt{2}+i / \sqrt{2})|x|}}{4 \pi|x|}+\frac{e^{(-1 / \sqrt{2}-i / \sqrt{2})|x|}}{4 \pi|x|}\right)
$$

where $\tilde{\psi} \in \operatorname{Dom}(-\Delta)=W_{2}^{2}\left(\mathbf{R}^{3}\right), b(\psi) \in \mathbf{C}$. Using the homogeneous extension of the delta functional we define the parameter $\gamma$ which describes the self-adjoint extension using (54)

$$
\gamma=-\frac{1}{\alpha}+\frac{1}{4 \pi \sqrt{2}}
$$

Therefore the self-adjoint operator corresponding to the formal expression (73) is the restriction of the adjoint operator to the domain of functions satisfying the boundary condition

$$
\begin{equation*}
\langle\delta, \tilde{\psi}\rangle=\left(-\frac{1}{\alpha}+\frac{1}{4 \pi \sqrt{2}}\right) b(\psi) \tag{74}
\end{equation*}
$$

Let us consider this extension of the linear functional $\delta$ in more detail to underline the main ideas of the calculations. Every function $\psi \in W_{2}^{2}\left(\mathbf{R}^{3} \backslash\{0\}\right)$ is continuous outside the origin and has the following asymptotics there

$$
\begin{equation*}
\psi(x)={ }_{x \rightarrow 0} \frac{\psi_{-}}{4 \pi|x|}+\psi_{0}+o(1) \tag{75}
\end{equation*}
$$

where the boundary values $\psi_{-}, \psi_{0}$ are equal to

$$
\left\{\begin{array}{l}
\psi_{-}=a(\psi) \\
\psi_{0}=-a(\psi) / 4 \pi \sqrt{2}+\tilde{\psi}(0)
\end{array}\right.
$$

The linear operator (73) is not defined on all such functions. The distribution $\varphi=\delta$ should be extended to the set of all functions having the asymptotics (75). We denote by $E$ the set of all $C^{\infty}\left(\mathbf{R}^{3} \backslash\{0\}\right)$ functions with compact support having the asymptotic behaviour (75) at the origin. Convergence in this space is defined using an arbitrary $C_{0}^{\infty}\left(\mathbf{R}^{3}\right)$ function $\chi$ equal to one in some neighbourhood of the origin.

Definition $2 A$ sequence $\left\{\psi_{n}\right\}$ of functions from $E$ is said to converge to a function $\psi \in E$ if and only if:

1. $\lim _{n \rightarrow \infty} \psi_{n-}=\psi_{-}$
2. There exists a bounded domain outside which all the functions $\psi_{n}$ vanish;
3. The sequence $\left\{\tilde{\psi}_{n}^{(k)}\right\}$ of the regularized derivatives of order $k$ :

$$
\tilde{\psi}_{n}^{(k)}(x)=\left(\psi_{n}(x)-\frac{\chi(x) \psi_{n-}}{4 \pi|x|}\right)^{(k)}
$$

converges uniformly to

$$
\tilde{\psi}^{(k)}(x)=\left(\psi(x)-\frac{\chi(x) \psi_{-}}{4 \pi|x|}\right)^{(k)}
$$

This definition does not depend on the choice of the function $\chi$. The derivative of any function from $E$ is defined pointwise everywhere outside the origin. We denote by $E^{\prime}$ the set of all bounded linear forms on $E$. The set $E$ contains the standard set of test functions $D=C_{0}^{\infty}\left(\mathbf{R}^{3}\right)$.

The following lemma follows easily from Lemma 6.
Lemma 10 Let the distribution $\tilde{\delta} \in E^{\prime}$

1. be equal to $\delta$ on the test functions from $D$;
2. be a homogeneous distribution;
then this distribution for every function $\psi \in E$ is equal to

$$
\begin{equation*}
\tilde{\delta}(\psi)=\psi_{0} \tag{76}
\end{equation*}
$$

This means that the distribution $\tilde{\delta}$ "does not feel" the singularity of the test function at the origin.

We are going to use the same notation $\delta$ for the delta distribution in $D^{\prime}$ and $E^{\prime}$ in what follows. This is justified because of the uniqueness of this extension under our assumptions.

Definition 3 The delta distribution $\delta$ in $E^{\prime}$ with support at the origin is the linear functional on $E$ defined by the formula (76).

Following Section we define the linear operator $L_{\alpha}$ on the whole Sobolev space $W_{2}^{2}\left(\mathbf{R}^{3} \backslash\{0\}\right)=\operatorname{Dom}\left(-\Delta_{0}^{*}\right)$ using the closure. The corresponding self-adjoint operator, also denoted by $L_{\alpha}$, is defined on the following domain

$$
\operatorname{Dom}\left(L_{\alpha}\right)=\left\{\psi \in W_{2}^{2}\left(\mathbf{R}^{3} \backslash\{0\}\right): L_{\alpha} \psi \in L^{2}\left(\mathbf{R}^{3}\right)\right\}
$$

The latter inclusion has to be understood in the distributional sense with $D$ as the set of test functions. It follows that every function $\psi$ from the domain $\operatorname{Dom}\left(L_{\alpha}\right)$ should satisfy the following boundary condition

$$
\begin{equation*}
\psi_{-}+\alpha \psi_{0}=0 . \tag{77}
\end{equation*}
$$

The latter condition implies (74).

## Relations with the Schrödinger operator on the half axis

Consider the subsets $E_{r} \subset E ; D_{r} \subset D$ consisting of functions $\psi$ in $E$, respectively $D$, which depend only on the absolute value of the coordinate, i.e. such that $\psi(x)=\psi(|x|)$. The corresponding distribution spaces will be denoted by $E_{r}^{\prime}$, respectively $D_{r}^{\prime}$. The space of square integrable functions on $\mathbf{R}^{3}$ depending on $|x|$ will be denoted by $L_{r}^{2}\left(\mathbf{R}^{3}\right)$. The transformation $T$ : $\psi(|x|) \rightarrow \sqrt{4 \pi} r \psi(r)$ acting from $L_{r}^{2}\left(\mathbf{R}^{3}\right)$ to $L^{2}\left(\mathbf{R}_{+}\right)$preserves the $L^{2}$ norm. This transformation transforms the set of test functions $D_{r}$ into the set $\mathbf{D}$ of $C^{\infty}\left(\mathbf{R}_{+}\right)$functions with compact support and equal to zero at the origin. The set of test functions $E_{r}$ is transformed into the set $\mathbf{E}$ of $C^{\infty}\left(\mathbf{R}_{+}\right)$functions with compact support having the following asymptotics at the origin:

$$
\psi(r)=\frac{\psi_{-}}{\sqrt{4 \pi}}+r \sqrt{4 \pi} \psi_{0}+o(r)
$$

The transformed linear operator $\mathbf{L}_{\alpha}=T L_{\alpha} T^{-1}$ is defined by the following formula:

$$
\begin{aligned}
\left\langle\varphi, \mathbf{L}_{\alpha} \psi\right\rangle_{L^{2}\left(\mathbf{R}_{+}\right)} & =\left\langle T^{-1} \varphi, L_{\alpha} T^{-1} \psi\right\rangle_{L^{2}\left(\mathbf{R}^{3}\right)} \\
& =\left\langle\varphi,-\frac{d^{2}}{d x^{2}} \psi\right\rangle_{L^{2}\left(\mathbf{R}_{+}\right)}+\alpha \bar{\varphi}_{0} \psi_{0} \\
& =\left\langle\varphi,\left(-\frac{d^{2}}{d x^{2}}+\frac{\alpha}{4 \pi} \delta^{(1)}\left\langle\delta^{(1)}, \cdot\right\rangle\right) \psi\right\rangle_{L^{2}\left(\mathbf{R}_{+}\right)} .
\end{aligned}
$$

Here $\delta^{(1)}$ denotes the derivative of the Dirac delta function, i.e. the functional defined on the functions from $\mathbf{E}$ as follows: $\left\langle\psi, \delta^{(1)}\right\rangle=-\psi^{\prime}(0)$.

Thus the three dimensional delta potential is quite similar to the pseudopotential on the half axis equal to the projector $P$ in $L^{2}\left(\mathbf{R}_{+}\right)$into the derivative of the delta function, i.e. $(P f)=\left\langle\delta^{(1)}, f\right\rangle \delta^{(1)}$. We remark that the element $\delta^{(1)}$ belongs to $\mathcal{H}_{-2}\left(-d^{2} / d x^{2}\right)$, where the operator $-d^{2} / d x^{2}$ is defined on the functions from $W_{2}^{2}\left(\mathbf{R}_{+}\right)$which satisfy the Dirichlet boundary condition at the origin. (See Section ?? where point interactions of the second derivative operator in $L_{2}(\mathbf{R})$ are studied in more detail.)

## Approximations of the delta potential

It follows from the previous consideration (see Section ) that it is possible to construct an approximation of the operator $L_{\alpha}$ by rank one perturbations from $\mathcal{H}_{0}(L)=L_{2}\left(\mathbf{R}^{3}\right)$. In the case of the Laplace operator in dimension three such an approximation can be constructed explicitly. The sequence of approximations can be chosen from the set of infinitely differentiable functions with compact support. We discuss first the approximation of the operator $\mathbf{L}_{\alpha}$. Let $\omega$ be a $C_{0}^{\infty}\left(\mathbf{R}_{+}\right)$real function with compact support and vanishing at the origin, normalized such that $\int_{0}^{\infty} \omega(x) d x=1$. An approximation of the delta function can be constructed with the help of scaling. We use the following definition $\omega_{\epsilon}(x)=(1 / \epsilon) \omega(x / \epsilon), x \in \mathbf{R}_{+}$. The following calculations show that the sequence $v_{\epsilon}(x)=d \omega_{\epsilon}(x) / d x$ converges when $\epsilon \rightarrow 0$ to the $\delta^{(1)}$ distribution for any function $\psi \in \mathbf{E}$ :

$$
\begin{aligned}
\left\langle v_{\epsilon}, \psi\right\rangle & =\int_{0}^{\infty} v_{\epsilon}(x) \psi(x) d x \\
& =\int_{0}^{\infty} \omega_{\epsilon}^{\prime}(x) \psi(x) d x \\
& =-\int_{0}^{\infty} \omega_{\epsilon}(x) \psi^{\prime}(x) d x+\left.\omega_{\epsilon} \psi\right|_{0} ^{\infty} .
\end{aligned}
$$

The integral in the latter formula converges as $\epsilon \searrow 0$ to the value of the function $\psi^{\prime}$ at the origin. The nonintegral terms are equal to zero because the function $\omega$ has, by assumption, zero limits at the origin and at infinity. By closure this result can be extended to all $\psi \in \mathbf{E}$. It follows that for any function $\psi \in \mathbf{E}$ and any test function $\varphi \in \mathbf{D}$ the following limit holds

$$
\lim _{\epsilon \rightarrow 0}\left\langle\varphi,\left(-\frac{d^{2}}{d x^{2}}+\frac{\alpha}{4 \pi} v_{\epsilon}\left\langle v_{\epsilon}, \cdot\right\rangle\right) \psi\right\rangle=\left\langle\varphi, \mathbf{L}_{\alpha} \psi\right\rangle .
$$

Thus the sequence of operators

$$
\mathbf{L}_{\alpha, \epsilon}=-\frac{d^{2}}{d x^{2}}+\frac{\alpha}{4 \pi} v_{\epsilon}\left\langle v_{\epsilon}, \cdot\right\rangle
$$

converges to the operator $\mathbf{L}_{\alpha}$ pointwise in the weak operator topology. An approximation of the operator $L_{\alpha}$ can be constructed using the same functional sequence $v_{\epsilon}$. We choose a special (but "standard") delta functional sequence equal to

$$
V_{\epsilon}(x)=\frac{-1}{4 \pi} \frac{v_{\epsilon}(|x|)}{|x|}
$$

$V_{\epsilon}$ has compact support and it is easily verified that

$$
\left.\begin{array}{rl}
\int_{\mathbf{R}^{3}} V_{1}(x) d^{3} x & =-\int_{0}^{\infty} r v_{1}(r) d r \\
& =-\left.r \omega(r)\right|_{0} ^{\infty}+\int_{0}^{\infty} \omega(r) d r
\end{array}\right)=1 .
$$

Moreover $V_{\epsilon}$ has the usual scaling properties:

$$
\begin{aligned}
& V_{\epsilon}(x)=\frac{-1}{4 \pi} \frac{v_{\epsilon}(|x|)}{|x|} \quad=\left.\frac{-1}{4 \pi r} \frac{\partial}{\partial r} \omega_{\epsilon}(r)\right|_{r=|x|} \\
& =\left.\frac{-1}{4 \pi r} \frac{\partial}{\partial r} \frac{1}{\epsilon} \omega\left(\frac{r}{\epsilon}\right)\right|_{r=|x|}=\frac{1}{\epsilon^{3}} V_{1}\left(\frac{x}{\epsilon}\right) .
\end{aligned}
$$

Finally, for any test function $\psi$ continuous in a neighbourhood of the origin the following limit holds

$$
\lim _{\epsilon \rightarrow 0} V_{\epsilon}(\psi)=\lim _{\epsilon \rightarrow 0} \int_{\mathbf{R}^{3}} V_{\epsilon}(x) \psi(x) d^{3} x=\psi(0) .
$$

Lemma 11 Let $\psi$ be any test function from E. Then the following limit holds

$$
\lim _{\epsilon \rightarrow 0}\left\langle V_{\epsilon}, \psi\right\rangle=\psi_{0}
$$

Proof Every function $\psi \in E$ possesses the following representation

$$
\psi(x)=\left(\frac{\psi_{-}}{4 \pi|x|}+\psi_{0}\right) \chi(x)+\tilde{\psi}(x)
$$

where $\chi$ has compact support and is equal to one in a neighbourhood of the origin and satisfies

$$
\tilde{\psi}(x)=o(1), x \rightarrow 0
$$

Then the following limits exist:

$$
\begin{gathered}
\lim _{\epsilon \rightarrow 0} \int_{\mathbf{R}^{3}} d^{3} x V_{\epsilon}(x) \tilde{\psi}(x)=0 \\
\lim _{\epsilon \rightarrow 0} \int_{\mathbf{R}^{3}} d^{3} x V_{\epsilon}(x) \chi(x) \psi_{0}=\psi_{0} \\
\lim _{\epsilon \rightarrow 0} \int_{\mathbf{R}^{3}} d^{3} x V_{\epsilon}(x) \frac{\chi(x)}{4 \pi|x|} \psi_{-}=0
\end{gathered}
$$

The last limit follows from the orthogonality of the functions $V_{\epsilon}(x)$ and $1 /|x|$ in $L^{2}\left(\mathbf{R}^{3}\right)$. The lemma is proven.

Consider now the sequence of linear operators defined in the generalized sense

$$
L_{\alpha, \epsilon}=-\Delta+\alpha V_{\epsilon}(x)\left\langle V_{\epsilon}(x), \cdot\right\rangle .
$$

This sequence of linear operators $L_{\alpha, \epsilon}$ converges as $\epsilon \searrow 0$ to the operator $L_{\alpha}$ in the weak operator topology. We prove now that the sequence of linear operators $L_{\alpha, \epsilon}$ converges to the operator $L_{\alpha}$ in the operator norm. All these operators are defined on the domain $\operatorname{Dom}\left(\Delta_{0}^{*}\right)$ and their ranges belong to $\mathcal{H}_{-2}(-\Delta)$. The norms are defined by equations (43) and (16) respectively.

Lemma 12 Let $\omega$ be an infinitely differentiable function with compact support on the positive half axis and assume $\omega(0)=0$ and $\int_{0}^{\infty} \omega(r) d r=1$. Then

$$
V_{\epsilon}(x)=\left.\left(\frac{-1}{4 \pi r} \frac{\partial}{\partial r} \frac{1}{\epsilon} \omega\left(\frac{r}{\epsilon}\right)\right)\right|_{r=|x|}, x \in \mathbf{R}^{3}
$$

converges to $\delta$ in $\mathcal{H}_{-2}(-\Delta)$ when $\epsilon \searrow 0$.
Proof We have to prove that

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0}\left\|\frac{1}{-\Delta+1}\left(\delta-V_{\epsilon}\right)\right\|_{L_{2}\left(\mathbf{R}^{3}\right)}=0 \tag{78}
\end{equation*}
$$

since the operator $-\Delta$ is positive. The Fourier transform $\hat{V}_{\epsilon}$ of the function $V_{\epsilon}$ can be calculated at any $p \in \mathbf{R}^{3}$ :

$$
\begin{aligned}
\hat{V}_{\epsilon}(p) & =\int_{0}^{\infty} d r r^{2} \int_{0}^{\pi} d \theta \sin \theta e^{i r p \cos \theta} \frac{-2 \pi}{4 \pi r} \frac{\partial}{\partial r} \frac{1}{\epsilon} \omega\left(\frac{r}{\epsilon}\right) \\
& =\int_{0}^{\infty} \cos r p \frac{1}{\epsilon} \omega\left(\frac{r}{\epsilon}\right) d r .
\end{aligned}
$$

The function

$$
\hat{V}_{\epsilon}(p)-1=\int_{0}^{\infty}(\cos r p-1) \frac{1}{\epsilon} \omega\left(\frac{r}{\epsilon}\right) d r
$$

is uniformly bounded and tends to zero uniformly on every compact domain $D \subset \mathbf{R}^{3}$. It follows that, with $g_{\epsilon}(p) \equiv\left(1 /\left(p^{2}+1\right)\right)\left(\hat{V}_{\epsilon}(p)-1\right)$,

$$
\left\|g_{\epsilon}\right\|_{L_{2}\left(\mathbf{R}^{3}\right)} \rightarrow_{\epsilon \rightarrow 0} 0
$$

and the limit (78) holds.

Theorem 11 The sequence of linear operators $L_{\alpha, \epsilon}$ converges in the operator norm to the linear operator $L_{\alpha}$ on $W_{2}^{2}\left(\mathbf{R}^{3} \backslash\{0\}\right)$.

Proof This follows easily from Lemma 12 and Theorem 6.
Approximations with the renormalized coupling constant
The operator $-\Delta$ is positive and the functional $\delta$ belongs to $\mathcal{H}_{-2}(-\Delta) \backslash$ $\mathcal{H}_{-1}(-\Delta)$. Therefore a point interaction in dimension three can be approximated in the strong resolvent sense or even norm resolvent sense only using a suitable renormalization of the coupling constant. Two approaches have been developed. In the first approach the operator with the point interaction is approximated by the sequence of operators

$$
L_{\alpha, \epsilon}=-\Delta+\alpha(\epsilon) V_{\epsilon}(x)
$$

The functions $V_{\epsilon}(x)$ are obtained from a certain function $V_{1}(x)$ by unitary scaling. In this approach the interaction term $\alpha\langle\delta, \cdot\rangle \delta=\alpha \delta$ is considered as a singular potential, not as a rank one operator. To get the norm resolvent convergence the coupling constant $\alpha$ should be chosen with a suitable dependence on the scaling parameter $\epsilon$. This approach is described in detail in the book by S.Albeverio, F.Gesztesy, R.Hoegh-Krohn, and H.Holden [?].

We are going to describe here in more detail the second approach where the approximating sequence of operators is constructed using the spectral representation of the Laplace operator. This approach was first developed by F.A.Beresin and L.D.Faddeev [?]. Consider the sequence of functions $u_{n}(x)$ converging to the $\delta(x)$. The sequence can be constructed using the Fourier transformation which is just the spectral representation for the Laplace operator

$$
\hat{u}_{n}(p)=\left\{\begin{array}{cl}
\frac{1}{(2 \pi)^{3 / 2}}, & p^{2}<n^{2} \\
0, & p^{2}>n^{2}
\end{array}\right.
$$

where $\hat{u}_{n}$ denotes the Fourier transform of the function $u_{n}$.
Obviously if $\psi \in C_{0}^{\infty}\left(\mathbf{R}^{3}\right)$ then

$$
\left\langle\psi, u_{n}\right\rangle=\int_{\mathbf{R}^{3}} \psi(x) u_{n}(x) d^{3} x \rightarrow \psi(0)=\langle\psi, \delta\rangle
$$

Consider the operator

$$
L^{n} \psi=-\Delta \psi+\alpha_{n} u_{n}(x) \int_{\mathbf{R}^{3}} \overline{u_{n}(y)} \psi(y) d^{3} y
$$

Let us calculate the resolvent of the operators in terms of the Fourier transform for some $z, \Im z \neq 0$

$$
\begin{gathered}
\left(L^{n}-z\right) \psi=f \\
\Rightarrow\left(p^{2}-z\right) \hat{\psi}(p)+\alpha_{n} \hat{u}_{n}(p) \int \hat{u}_{n}(q) \hat{\psi}(q) d^{3} q=\hat{f}(p)
\end{gathered}
$$

It follows that

$$
\begin{equation*}
\hat{\psi}(p)=\frac{\hat{f}(p)}{p^{2}-z}-\alpha_{n} \frac{\hat{u}_{n}(p)}{p^{2}-z} \int \hat{u}_{n}(q) \hat{\psi}(q) d q \tag{79}
\end{equation*}
$$

We multiply the latter equality by $\hat{u}_{n}(p)$ and integrate with respect to $p$ to get the following equation

$$
\int_{\mathbf{R}^{3}} \hat{u}_{n}(p) \hat{\psi}(p) d^{3} p=\frac{\int_{\mathbf{R}^{3}} \frac{\hat{f}(p)}{p^{2}-z} \hat{u}_{n}(p) d^{3} p}{1+\alpha_{n} \int_{\mathbf{R}^{3}} \frac{\left(\hat{u}_{n}(p)\right)^{2}}{p^{2}-z} d^{3} p}
$$

Finally we get the following formula for the resolvent

$$
\hat{\psi}(p)=\frac{\hat{f}(p)}{p^{2}-z}-\frac{\alpha_{n}}{1+\alpha_{n} \int_{\mathbf{R}^{3}} \frac{\hat{u}_{n}(q)^{2}}{q^{2}-z} d^{3} q}\left(\int_{\mathbf{R}^{3}} \frac{\hat{f}(q)}{q^{2}-z} \hat{u}_{n}(q) d^{3} q\right) \frac{\hat{u}_{n}(p)}{p^{2}-z}
$$

The resolvents of the operators $L^{n}$ have a nontrivial limit if and only if the fractions

$$
\frac{\alpha_{n}}{1+\alpha_{n} \int_{\mathbf{R}^{3}} \frac{\hat{u}_{n}(q)^{2}}{q^{2}-z} d^{3} q}
$$

converge to a nontrivial limit. The asymptotic of the integral can be computed explicitly using the spherical coordinates

$$
\int_{\mathbf{R}^{3}} \frac{\hat{u}_{n}(q)^{2}}{q^{2}-z} d^{3} q=\frac{4 \pi}{8 \pi^{3}} \int_{0}^{n} \frac{r^{2}}{r^{2}-z} d r=\frac{1}{2 \pi^{2}} n+o(n), \quad n \rightarrow \infty
$$

One can choose for example

$$
\frac{\alpha_{n}}{1+\alpha_{n} n / 2 \pi^{2}}=\alpha \Rightarrow \alpha_{n}=\frac{\alpha}{1-\alpha n / 2 \pi^{2}} .
$$

For this choice of the coupling constant the sequence of the self-adjoint operators $L^{n}$ converges to the operator $L_{\alpha}$ in the strong resolvent sense. We note that the sequence of coupling constants $\alpha_{n}$ is infinitesimal (in the sense that $\alpha_{n} \rightarrow 0$ as $n \rightarrow 0$ ). If the coupling constant does not depend on $n$ then Theorem 9 implies that the resolvents of the operators $L^{n}$ converge to the resolvent of the original operator.

Perturbations of the first derivative operator
Consider rank one perturbations defined by the formal expression

$$
\begin{equation*}
A_{\alpha}=\frac{1}{i} \frac{d}{d x}+\alpha \delta=\frac{1}{i} \frac{d}{d x}+\alpha\langle\delta, \cdot\rangle \delta \tag{80}
\end{equation*}
$$

The operator $A_{\alpha}$ can be considered as a rank one perturbation of the selfadjoint non-semibounded operator $A=1 / i d / d x$ with domain $\operatorname{Dom}(A)=$ $W_{2}^{1}(\mathbf{R})$. The $\delta$ measure defines a bounded linear functional on $W_{2}^{1}(\mathbf{R})$ due to the embedding theorem. But the element $(1 /(A-i)) \delta=i e^{-x} \Theta(x)$ does not belong to the domain of the operator $A$. $(\Theta(x)$ denotes here the Heaviside step function.) The restriction $A^{0}$ of the operator $A$ to the domain of functions $\operatorname{Dom}\left(A^{0}\right)=\left\{\psi \in W_{2}^{1}(\mathbf{R}): \psi(0)=0\right\}$ has deficiency indices $(1,1)$. The deficiency elements are given by

$$
\begin{aligned}
g_{i} & =\frac{1}{A-i} \delta=i e^{-x} \Theta(x) \\
g_{-i} & =\frac{1}{A+i} \delta=-i e^{x} \Theta(-x)
\end{aligned}
$$

Every function $\psi$ from the domain of the adjoint operator possesses the standard representation

$$
\psi(x)=\tilde{\psi}+\frac{b(\psi)}{2} i \operatorname{sign} x e^{-|x|}
$$

where $\hat{\psi} \in W_{2}^{1}(\mathbf{R})$. Consider the group of the central symmetries of the real line:

$$
\begin{gathered}
G(1)=I, G(-1)=J \\
G(-1)^{2}=G(1)
\end{gathered}
$$

where $I$ and $J$ are the identity and inversion operators respectively defined by the following formulas in the generalized sense

$$
\begin{gathered}
(I f)(x)=f(x) \\
(J f)(x)=f(-x)
\end{gathered}
$$

The original operator and the functional $\delta$ are homogeneous with respect to this group

$$
\begin{gathered}
A G(t)=t G(t) A \\
G(t) \delta=\delta
\end{gathered}
$$

The parameters $\beta$ and $\theta$ for this problem are equal to 1 and 0 respectively. The group consists of only two elements and the extension of the functional $\delta$ can be defined using the parameter $f(-1)$. The parameter $c$ defining the extension of the functional $\delta$ is given by
$c=f(-1)-i\left\|\frac{1}{A-i} \delta\right\|^{2}=i\left\langle\delta, \frac{1}{(A-i)(A+i)} \delta\right\rangle-i\left\langle\frac{1}{A-i} \delta, \frac{1}{A-i} \delta\right\rangle=0$.
The latter equality implies that the extension of the delta function, which is an even distribution, vanishes on every odd test function.

It follows from Theorem 4 that the self-adjoint operator $A_{\alpha}$ corresponding to the formal expression (80) is defined on the domain of functions satisfying the following conditions

$$
\tilde{\psi}(0)=-\frac{1}{\alpha} b(\psi)
$$

Thus the operator $A_{\alpha}$ is the self-adjoint operator $1 / i d / d x$ defined on the following domain

$$
\operatorname{Dom}\left(A_{\alpha}\right)=\left\{\psi \in W_{2}^{1}(\mathbf{R} \backslash\{0\}): \psi(-0)=\frac{1+i \frac{\alpha}{2}}{1-i \frac{\alpha}{2}} \psi(+0)\right\}
$$

The spectral analysis of the operator $A_{\alpha}$ can be easily carried out.
Dirac operator with a pseudopotential
A similar analysis to that made in the previous section can be carried out for the one dimensional Dirac operator with the delta potential

$$
\begin{gather*}
H_{\alpha}=\left(\begin{array}{cc}
m & -i \frac{d}{d x} \\
-i \frac{d}{d x} & -m
\end{array}\right)+V \vec{\delta}  \tag{81}\\
V=V^{*}=\left(\begin{array}{ll}
v_{11} & v_{12} \\
v_{21} & v_{22}
\end{array}\right)
\end{gather*}
$$

where $v_{11}, v_{22} \in \mathbf{R}, v_{12}=\overline{v_{21}} \in \mathbf{C}$. This family of Dirac operators with pseudopotentials is described by four real parameters. The original operator

$$
H=\left(\begin{array}{cc}
m & -i \frac{d}{d x} \\
-i \frac{d}{d x} & -m
\end{array}\right)
$$

is defined on the two component functions $f=\left(f_{1}, f_{2}\right) \in L_{2}(\mathbf{R}) \oplus L_{2}(\mathbf{R})$ from the domain $\operatorname{Dom}(H)=W_{2}^{1}(\mathbf{R}) \oplus W_{2}^{1}(\mathbf{R})$. Two delta functions $\delta_{1}, \delta_{2}$ defined as follows $\left\langle\delta_{i}, f\right\rangle=f_{i}(0), i=1,2$, are bounded linear functionals on the domain of the original operator. The delta function $\vec{\delta}$ is the linear map

$$
\begin{gathered}
\vec{\delta}: W_{2}^{1}(\mathbf{R}) \oplus W_{2}^{1}(\mathbf{R}) \rightarrow \mathbf{C}^{2}, \\
\langle\vec{\delta}, f\rangle=\binom{f_{1}(0)}{f_{2}(0)} .
\end{gathered}
$$

The product of the delta function and an arbitrary continuous function $f$ is equal to

$$
\begin{aligned}
\langle f \vec{\delta}, \psi\rangle & =\left\langle\vec{\delta},\binom{f_{1} \psi_{1}}{f_{2} \psi_{2}}\right\rangle \\
& =\binom{f_{1}(0) \psi(0)}{f_{2}(0) \psi(0)} \\
& =\left(\begin{array}{cc}
f_{1}(0) & 0 \\
0 & f_{2}(0)
\end{array}\right)\langle\vec{\delta}, \varphi\rangle,
\end{aligned}
$$

where $\psi$ is an arbitrary test function from $C_{0}^{\infty}(\mathbf{R}) \oplus C_{0}^{\infty}(\mathbf{R})$. The heuristic expression (81) can be written as

$$
H_{\alpha}=\left(\begin{array}{cc}
m & -i \frac{d}{d x}  \tag{82}\\
-i \frac{d}{d x} & -m
\end{array}\right)+V \operatorname{diag}\left\{\left\langle\delta_{i}, \cdot\right\rangle\right\} \vec{\delta}
$$

This operator can be considered as a rank two perturbation of the self-adjoint non-semibounded original operator $H$. In accordance with this approach we restrict the original operator $H$ to the domain of functions $\operatorname{Dom}\left(H^{0}\right)=$ $\{\psi \in \operatorname{Dom}(H):\langle\vec{\delta}, \psi\rangle=0\}$. The restricted operator $H^{0}$ has deficiency indices $(2,2)$. The adjoint operator $H^{0 *}$ is defined on the domain $W_{2}^{1}(\mathbf{R} \backslash$ $\{0\}) \oplus W_{2}^{1}(\mathbf{R} \backslash\{0\})$. To determine the perturbed operator, bounded linear functionals $\delta_{i}$ have to be extended to a set of functions which are discontinuous at the origin and continuous outside the origin. The delta functions are homogeneous with respect to the group of central symmetries of the real line and the extension is unique:

$$
\left\langle\delta_{i}, f\right\rangle=\frac{f_{i}(+0)+f_{i}(-0)}{2}, i=1,2
$$

This extension allows one to define the perturbed linear operator on the domain $W_{2}^{1}(\mathbf{R} \backslash\{0\}) \oplus W_{2}^{1}(\mathbf{R} \backslash\{0\})$ since the boundary values at the origin of the functions from this domain are well defined. The domain of the perturbed self-adjoint operator coincides with the set of all function $\psi \in L_{2}(\mathbf{R}) \oplus L_{2}(\mathbf{R})$, such that

$$
\left(\begin{array}{cc}
m & -i \frac{d}{d x} \\
-i \frac{d}{d x} & -m
\end{array}\right) \psi+V \vec{\delta}(x) \psi \in L_{2}(\mathbf{R}) \oplus L_{2}(\mathbf{R}) \text {. }
$$

Let us calculate the distribution

$$
f=\left(\begin{array}{cc}
m & -i \frac{d}{d x} \\
-i \frac{d}{d x} & -m
\end{array}\right) \psi+V \vec{\delta}(x) \psi
$$

for any function $\psi$ from the domain of the adjoint operator $H_{0}^{*}$. Every such distribution can be presented in the following form

$$
\begin{aligned}
f= & \tilde{f}-i\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right) \operatorname{diag}\left\{\psi_{1}(+0)-\psi_{1}(-0), \psi_{2}(+0)-\psi_{2}(-0)\right\} \vec{\delta} \\
& +\frac{1}{2} V \operatorname{diag}\left\{\psi_{1}(+0)+\psi_{1}(-0), \psi_{2}(+0)+\psi_{2}(-0)\right\} \vec{\delta}
\end{aligned}
$$

where $\tilde{f} \in L_{2}(\mathbf{R}) \oplus L_{2}(\mathbf{R})$. The vector $f$ belongs to the Hilbert space if and only if the coefficient in front of the delta function $\vec{\delta}$ is equal to zero. We get the following boundary conditions for the function $\psi$ at the origin:

$$
\left(\frac{1}{2} V-i\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\right) \psi(+0)=-\left(\frac{1}{2} V+i\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\right) \psi(-0)
$$

These boundary conditions can be written in the form:

$$
\begin{gather*}
\psi(+0)=\Lambda \psi(-0)  \tag{83}\\
\Lambda=-\left(\frac{1}{2} V-i\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)\right)\left(\frac{1}{2} V+i\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)\right) .
\end{gather*}
$$

One can show that

$$
\Lambda\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \Lambda^{*}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

and it follows that the operator $H^{0 *}$ restricted to the domain of functions satisfying the boundary conditions (83) is self-adjoint ([?, ?]).


[^0]:    ${ }^{1}$ Theorem (Baire)
    If a Banach space $X$ is the union of countable number of closed subsets $s_{n}$, at least one of the $s_{n}$ contains a ball.

