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The index of a pair of
pure states and the QHE
w/ BACHMANN & TAUBER

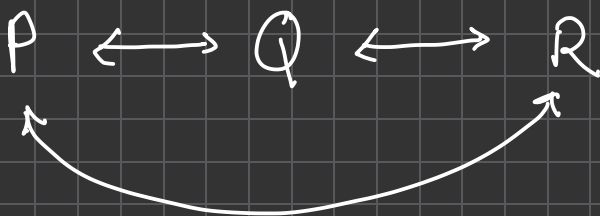
$\mathcal{H}$ P, Q $A = P - Q \in \mathcal{K}$



$$\text{index}(P, Q) \equiv \dim \mathcal{X}_{+1}(P-Q) - \dim \mathcal{X}_{-1}(P-Q) \in \mathbb{Z}.$$

$$\text{If } \|P-Q\| < 1 \Rightarrow \text{index}(P, Q) = 0.$$

$$\text{If } P-Q \in \mathcal{J}_1 \Rightarrow \text{index}(P, Q) = \text{tr}(P-Q) \\ (\text{tr}((P-Q)^{2n+1}))$$



$$P-Q \in K, \quad Q-R \in K$$

$$\text{index}(P, R) = \text{index}(P, Q) + \text{index}(Q, R).$$

U unitary

$$\text{index}(U^* P U, U^* Q U) = \text{index}(P, Q).$$

Fredholm: F

$$\dim \ker F < \infty$$

$$\dim \text{coker } F < \infty$$

$$F: \text{im } P \rightarrow \text{im } Q$$

$$\psi \mapsto Q\psi$$

P, Q non-triv.

$$\mathcal{H} = \text{im } P \oplus \text{ker } P$$

$$= \text{im } Q \oplus \text{ker } Q$$

$\exists$ unitary U

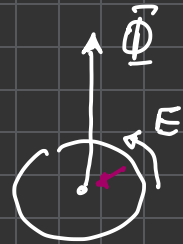
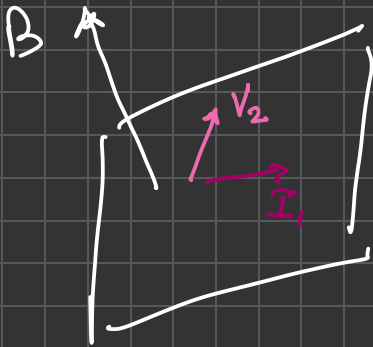
$$P = U^* Q U.$$

$$U_t = \exp(it \frac{1}{it} \log(U)).$$

$$P_t = U_t^* Q U_t$$

$$P_t - Q \in \mathcal{K}.$$

QHE



P on $\ell^2(\mathbb{Z}^2)$

$$P = \chi_{(-\infty, E_F)}(H)$$

$\exists C, \mu \in (0, \infty):$

P local

$$|P_{xy}| \leq C e^{-\mu|x-y|}.$$

$$L = \exp(i \arg(X_1 + iX_2))$$

P

vs.

$$L^* P L$$

$$P - L^* P L \in \mathcal{J}_3$$

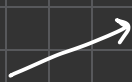
$$[P, L]_{xy} \sim e^{-\mu|x-y|} \frac{1}{|x|}$$

Thm. : $2\pi \underbrace{\sigma_{\text{Hall}}(H, E_F)}_{\text{Kubo}} = \text{index}(P, U^* P U).$

$$i \tilde{\text{tr}} P([x_1, P], [x_2, P])$$

Index of pair of pure states

$\mathcal{H}$



$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H}$$



~~$\mathcal{B}(\mathcal{F})$~~

$$A = \text{CAR}(\mathcal{H})$$

$$a: \mathcal{H} \rightarrow A$$

$$\{a(\varphi), a(\psi)^*\} = \langle \varphi, \psi \rangle 1_A$$

$$A \longleftrightarrow A^*$$



"States" $w: A \rightarrow \mathbb{C}$

$$w(a^*a) \geq 0$$

$$w(1_A) = 1$$

"Pure states"

extremal pts.

$$P \longrightarrow \omega \text{ pure state}$$

Quasi-free states

$$\omega_p(a(\varphi) * a(\psi)) = \langle \varphi, p\psi \rangle$$

$$\omega_1, \omega_2$$

$$\omega_1 - \omega_2 \in K?$$

$$\omega_1 - \omega_2 \in \mathcal{J}_1$$

$\mathcal{A}$ is $\mathcal{J}_1$, but not quadratic.

Def.: ω_1, ω_2 "comp."

$$\Leftrightarrow \exists u \in \mathcal{U}(\mathcal{A}):$$

$$\omega_2 = \omega_1(u^* \cdot u).$$

$$\omega_2 - \omega_1 = \omega_1 \left(\underbrace{\text{id} - \text{Ad}_u}_{\approx \delta_1} \right)$$

$$\mathcal{N}(\omega_1, \omega_2) := i \int \omega_1(u^* \delta u) \in \mathbb{Z}$$

$U(1)$ charge symmetry

generator: $\delta = i[Q, \cdot]$

$$\|\omega_1 - \omega_2\| < 2 \Rightarrow \mathcal{N}(\omega_1, \omega_2) = 0.$$

$\omega \rightarrow \omega \circ \ell$ ← Laughlin ansatz

$$\mathcal{N}(\omega, \omega \circ \ell) \in \mathbb{Z}$$

$$\mathcal{N}(\omega, \omega \circ \ell) = 2\pi q \sigma_{\text{Hall}}(\omega).$$

$$\ell a(\varphi) \equiv a(L\varphi)$$

RESEARCH ARTICLE

The index of a pair of pure states and the interacting integer quantum Hall effect

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Abstract

We introduce the index $\mathcal{N}(\omega_1, \omega_2)$ of a pair of pure states on a unital C^* -algebra, which is a generalization of the notion of the index of a pair of projections on a Hilbert space. We then show that the Hall conductance associated with an invertible state ω of a two-dimensional interacting electronic system which is symmetric under $U(1)$ charge transformation may be written as the index $\mathcal{N}(\omega, \omega_D)$, where ω_D is obtained from ω by inserting a unit of magnetic flux. This exhibits the integrality and continuity properties of the Hall conductance in the context of general topological features of $\mathcal{N}$.

1. Introduction

The index of a pair of projections introduced in [36, 21, 6] and explored further in [7] is an integer-valued index that can be associated with two projections P, Q on a Hilbert space whenever their difference is compact:

$$\text{index}(P, Q) = \dim(\text{im}(P) \cap \ker(Q)) - \dim(\text{im}(Q) \cap \ker(P)) \in \mathbb{Z}.$$

It is ‘topological’ in the sense that it is invariant under compact and norm-continuous deformations. In the case where one of the projections is a unitary conjugate of the other, $Q = U^*PU$, the index reduces to a Fredholm index of $PUP + P^\perp$.

Since its introduction, this notion has been intimately connected to the quantum Hall effect. It is indeed one of the possible expressions for the Hall conductance, when the currents are driven by the adiabatic increase of a magnetic flux through the two-dimensional electron gas. The physical meaning of P is that of a Fermi projection whenever the Fermi energy lies in a spectral gap or a mobility gap [1]. In particular, this picture is valid in a noninteracting setting where the many-body ground state reduces to a one-body projection, see [30] for an overview and further references. This concept was eventually generalized to yield the index of all entries of the Kitaev periodic table of topological insulators [37, 38].

There have been various attempts of generalizing the index of a pair of projections to an interacting setting where the state is not simply given by a projection in Hilbert space, most notably [9] in an arbitrarily large finite volume followed by [33] in the infinite volume setting, where quantization can be proved under the assumption of invertibility of the initial state. In both cases, the attention is on the Hall effect.

In this work, we introduce a generalization of the above indices which is defined in a completely abstract setting. It is associated with two pure states (ω_1, ω_2) of a C^* -algebra $\mathcal{A}$ that are related by an inner automorphism, in the presence of a $U(1)$ symmetry. Specifically, when $\omega_2 = \omega_1 \circ \text{Ad}_u$, where $u \in \mathcal{A}$ is unitary the index is given by

$$\mathcal{N}_\rho(\omega_1, \omega_2) := i \omega_1(u^* \delta^\rho(u)). \quad (1)$$

where δ^ρ is the generator of the $U(1)$ -symmetry. We prove, among other things, that the index is integer valued, operator-norm-continuous, and invariant under deformations by symmetric automorphisms. We also show that if the algebra is chosen to be the CAR algebra describing Fermions and if the two states are quasi-free, then our index reduces to the index of a pair of projections. Therefore, this new index of a pair of pure states generalizes [7], as well as the noncommutative geometric approach of [17].

We then turn to the quantum Hall effect. Echoing [33], we prove that the piercing of a magnetic flux from 0 to 2π starting from an *invertible* state corresponds to a situation where the many-body index is well-defined. The technical part here uses adiabatic flux insertion à la Laughlin [40] extended to the many-body setting in [10]. Unlike there, we focus on charge conservation rather than time-reversal symmetry. In this context, the many-body index equals the charge deficiency of the final state with respect to the initial state, and so to the Hall conductance by the Laughlin argument. This places the Hall index into a very general C^* -algebraic framework which is valid for both interacting and noninteracting fermionic systems, generalizing previous expressions [9, 33] and placing them in a functional analytic framework which complements other approaches using algebraic topology [34, 3].

We point out immediately that the invertibility assumption appears in two roles that are very distinct from each other. It first plays a role as a tool that allows us to ensure that the state is the gapped ground state of a so-called parent Hamiltonian. Secondly, it is crucial in concluding that the defect state obtained after flux insertion is locally comparable to the initial state, namely that the two differ only in the vicinity of the puncture and not at infinity. While the first role could be bypassed in a physical setting where the state is given as a gapped ground state, the second one appears fundamental. Indeed, it is that very assumption that ensures that the state has integer quantum Hall conductance as opposed to fractional conductance. In the latter case, the initial state is expected to have nontrivial superselection sectors corresponding to anyonic excitations (see [43] for one possible meaning of these terms and [25] for an overview of anyons in the fractional quantum Hall effect) and these cannot be realized upon an invertible state [39, 15].

Let us briefly discuss other approaches to (integer) quantization in the interacting quantum Hall effect. In the single particle picture, charge deficiency, charge transport and linear response coincide [40, 6, 30]. This equivalence continues to hold in the interacting picture [9, 33], where quantization of the Hall conductance was proved in [31, 8], see also [12, 42] for the validity of linear response. While the above assume the presence of a gap, this assumption is proved to hold in a perturbative setting in [28], while it is not needed if one considers nonequilibrium almost steady states [49, 46].

The paper is organized as follows. In Section 2, we describe the general algebraic setting, introduce the notion of two pure states being ρ -locally comparable, (see Definition 2.3) and define the index $\mathcal{N}_\rho(\omega_1, \omega_2)$. If ρ_t is a $U(1)$ -symmetry, then

$$\mathcal{N}_\rho(\omega_1, \omega_2) \in \mathbb{Z}.$$

In Section 2.2, we consider the Fermionic algebra $\mathcal{A} = \text{CAR}(\mathcal{H})$ on a one-particle Hilbert space $\mathcal{H}$. For P, Q two orthogonal projections on $\mathcal{H}$ and ω_P, ω_Q their corresponding quasi-free states on $\mathcal{A}$, we show that if $P - Q$ is trace class then

$$\mathcal{N}_\rho(\omega_P, \omega_Q) = \text{index}(P, Q),$$

the index of a pair of projection. Section 3 focusses on the quantum Hall effect in an interacting setting where the two states are an initial invertible state and a defect state obtained from the initial one by

inserting a unit of flux at the origin. Then the charge deficiency, and therefore the Hall conductance, is given by

$$\mathcal{N}_{\rho \otimes \text{id}}(\hat{\omega}, \hat{\omega}^D) \in \mathbb{Z}$$

where $\hat{\cdot}$ corresponds to a stacking operation associated to invertible states. The fact that we pick $\rho \otimes \text{id}$ rather than the stacked $\hat{\rho}$ means that we measure only the charge transported in the original system, rather than in the full, stacked system. We then show in Section 4 that this index covers gapped ground states of interacting Hamiltonian as well, and that in the quasi-free case it coincides with a single-particle spectral flow.

2. Abstract index theory

2.1. The index of a pair of pure states

Let $\mathcal{A}$ be a unital C^* -algebra. In later sections we shall apply the theory to $\mathcal{A} = \text{CAR}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$ but in fact throughout this section no assumptions on $\mathcal{A}$ will be made. We denote by $\mathcal{U}(\mathcal{A})$ the set of unitary elements of $\mathcal{A}$. A state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is a positive (i.e., $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$) linear functional on $\mathcal{A}$ with $\omega(\mathbb{1}) = 1$; the space of all states shall be denoted $\mathcal{S}(\mathcal{A})$. The condition $\omega(\mathbb{1}) = 1$ is equivalent to the normalization $\|\omega\| = 1$, where the norm is given by

$$\|\omega\| := \sup(\{|\omega(a)| : a \in \mathcal{A} : \|a\| \leq 1\}).$$

In other words, all states are on the unit sphere of $\mathcal{A}^*$ and so $\mathcal{S}(\mathcal{A})$ is a weakly*-compact and convex subset of $\mathcal{A}^*$. The extreme points of $\mathcal{S}(\mathcal{A})$ are called pure states and their collection is denoted by $\mathcal{P}(\mathcal{A})$.

Definition 2.1 (locally comparable pair of pure states). Let $\omega_1, \omega_2 \in \mathcal{P}(\mathcal{A})$ be a pair of pure states. We say that the pair (ω_1, ω_2) is *locally comparable* iff there exists some $u \in \mathcal{U}(\mathcal{A})$ such that

$$\omega_2 = \omega_1 \circ \text{Ad}_u. \quad (2)$$

Here and in the sequel, Ad_u denotes the automorphism $\text{Ad}_u(a) = u^*au$. In other words, ω_1 and ω_2 are locally comparable iff they are inner-automorphism equivalent.

Remark 2.2. The intuition we have in mind for two pure states to be locally comparable is that their difference is “compact” in a vague sense. Since

$$\omega_2 - \omega_1 = \omega_1 \circ (\text{Ad}_u - \text{id}),$$

if $\mathcal{A} = \text{CAR}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then $\mathcal{A}$ may be considered as the operator norm limit of finite-rank (but not necessarily quadratic) observables, and as such, since $u \in \mathcal{U}(\mathcal{A})$, we should consider $u - \mathbb{1}$ and hence also $\text{Ad}_u - \text{id}$ to be (the many-body analog of) compact. Then by the ideal property, the whole expression $\omega_1 \circ (\text{Ad}_u - \text{id})$ is.

Back to the case of a general C^* -algebra $\mathcal{A}$, when two pure states ω_1, ω_2 are locally comparable, we want to measure how different they are, with the expectation that if they are path-connected (in $\mathcal{P}(\mathcal{A})$) they should not be different at all. This is the case when $u \in \mathcal{U}_0(\mathcal{A})$, that is, when u may be continuously deformed to $\mathbb{1}$, whence we get a continuous deformation of ω_2 to ω_1 . However, there are C^* -algebras where $\pi_0(\mathcal{U}(\mathcal{A})) \neq \{0\}$ (for example $C(\mathbb{S}^1)$). It will turn out that we are not quite interested in such obstructions. Rather, in many interesting cases (such as UHF algebras), $\pi_0(\mathcal{U}(\mathcal{A})) = \{0\}$. Then, to gauge a topological obstruction we impose a further symmetry constraint.

Let $\{\rho_t\}_{t \in \mathbb{R}} \subseteq \text{Aut}(\mathcal{A})$ be a strongly continuous one-parameter group of $*$ -automorphisms. Its generator δ^ρ , given by

$$\delta^\rho := \text{s-lim}_{t \rightarrow 0^+} \frac{1}{t}(\rho_t - \text{id})$$

is a $*$ -derivation. In general ρ_t is not expected to be inner and δ^ρ is not expected to be bounded. As such, we must consider δ^ρ together with its domain

$$\mathcal{D}(\delta^\rho) \equiv \{a \in \mathcal{A} \mid \lim_{t \rightarrow 0^+} \frac{1}{t}(\rho_t(a) - a) \text{ exists in } \mathcal{A}\}.$$

With this, we refine the notion of locally comparable pure states as follows.

Definition 2.3 (ρ -locally comparable pair of pure states). Let ρ_t be as above and (ω_1, ω_2) be a pair of pure states. We say that this pair is ρ -locally comparable iff there exists some $u \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\delta^\rho)$ such that

$$\omega_2 = \omega_1 \circ \text{Ad}_u.$$

We now have all the definitions to set up our index.

Definition 2.4 (The index of a locally comparable pair of ρ -invariant pure states). Let ρ_t be as above and (ω_1, ω_2) be a pair of ρ -locally comparable pure states. Assume moreover that both ω_1 and ω_2 are ρ -invariant, that is,

$$\omega_j \circ \rho_t = \omega_j \quad (t \in \mathbb{R}, j = 1, 2).$$

Then the index of the pair (ω_1, ω_2) is defined as

$$\mathcal{N}_\rho(\omega_1, \omega_2) := i \omega_1(u^* \delta^\rho(u)) \quad (3)$$

where $u \in \mathcal{D}(\delta^\rho)$ is any unitary which obeys $\omega_2 = \omega_1 \circ \text{Ad}_u$.

We note that in general there is no reason for the unitary u above to be ρ_t -invariant; see Remark 3.22(iii) for an explicit example. Indeed, if this happens, then $\delta^\rho(u) = 0$ and so $\mathcal{N}_\rho(\omega_1, \omega_2) = 0$.

The condition that the pair (ω_1, ω_2) must be compatible with ρ in the above sense, namely that they are ρ -locally comparable and invariant, means they are generically *not* path connected, which is what the index measures.

Theorem 2.5 (Properties of the index). *The index defined above has the following properties:*

- (i) $\mathcal{N}_\rho(\omega_1, \omega_2)$ does not depend on the choice of $u \in \mathcal{D}(\delta^\rho) \cap \mathcal{U}(\mathcal{A})$ such that $\omega_2 = \omega_1 \circ \text{Ad}_u$.
- (ii) If $\rho_{2\pi} = \text{id}$, then $\mathcal{N}_\rho(\omega_1, \omega_2) \in \mathbb{Z}$.
- (iii) $\mathcal{N}_\rho(\omega_1, \omega_2)$ has the following continuity property:

$$\|\omega_1 - \omega_2\| < 2 \implies \mathcal{N}_\rho(\omega_1, \omega_2) = 0. \quad (4)$$

- (iv) For any automorphism α of $\mathcal{A}$ such that $\alpha \circ \delta^\rho = \delta^\rho \circ \alpha$,

$$\mathcal{N}_\rho(\omega_1 \circ \alpha, \omega_2 \circ \alpha) = \mathcal{N}_\rho(\omega_1, \omega_2). \quad (5)$$

- (v) $\mathcal{N}_\rho(\omega, \omega) = 0$ for any ρ -invariant pure state $\omega \in \mathcal{P}(\mathcal{A})$.

- (vi) If $\omega_1, \omega_2, \omega_3$ are three pairwise ρ -locally comparable pure states, all of which are ρ -invariant, then

$$\mathcal{N}_\rho(\omega_1, \omega_2) + \mathcal{N}_\rho(\omega_2, \omega_3) = \mathcal{N}_\rho(\omega_1, \omega_3). \quad (6)$$

(vii) *The index is anti-symmetric*

$$\mathcal{N}_\rho(\omega_1, \omega_2) = -\mathcal{N}_\rho(\omega_2, \omega_1). \quad (7)$$

(viii) *We have the following additivity with respect to tensor products. Let $\tilde{\mathcal{A}}$ be another C^* -algebra and $\tilde{\rho}$ as above; Let further $\tilde{\omega}_1, \tilde{\omega}_2$ be two $\tilde{\rho}$ -locally comparable pure states on $\tilde{\mathcal{A}}$ which are $\tilde{\rho}_t$ -invariant. Then*

$$\mathcal{N}_{\rho \otimes \tilde{\rho}}(\omega_1 \otimes \tilde{\omega}_1, \omega_2 \otimes \tilde{\omega}_2) = \mathcal{N}_\rho(\omega_1, \omega_2) + \mathcal{N}_{\tilde{\rho}}(\tilde{\omega}_1, \tilde{\omega}_2). \quad (8)$$

To clarify further, let us assume temporarily that $\mathcal{A}$ is a uniformly hyperfinite (UHF) algebra. In this case, it is well known (see, e.g., [19, Prop 3.2.52]) that one may find a sequence $\{q_n = q_n^*\}_n \subseteq \mathcal{A}$ such that

$$\delta^\rho = \text{s-lim}_{n \rightarrow \infty} i[q_n, \cdot].$$

Then it is clear that

$$\mathcal{N}_\rho(\omega_1, \omega_2) = \lim_n (\omega_1(q_n) - \omega_2(q_n)). \quad (9)$$

Since this expression does not involve the unitary u the index is well-defined. The remainder of this subsection does not assume that $\mathcal{A}$ is UHF.

Remark 2.6. If (ω_1, ω_2) are ρ -locally comparable and if $\tilde{\omega}_2 = \omega_2 \circ \text{Ad}_v$ for a $v \in \mathcal{U}(\mathcal{A})$ with $\delta^\rho(v) = 0$, then $(\omega_1, \tilde{\omega}_2)$ are also ρ -locally comparable and

$$\mathcal{N}_\rho(\omega_1, \tilde{\omega}_2) = \mathcal{N}_\rho(\omega_1, \omega_2) \quad (10)$$

by additivity (6) and $\mathcal{N}_\rho(\tilde{\omega}_2, \omega_2) = 0$.

Proof of Theorem 2.5. (i) If we have $\omega_2 = \omega_1 \circ \text{Ad}_u = \omega_1 \circ \text{Ad}_v$ for two a priori distinct unitaries $u, v \in \mathcal{U}(\mathcal{A})$ then this readily implies $\omega_1 = \omega_1 \circ \text{Ad}_{u^*v}$. Let us define $w := u^*v$ whence $\omega_1 = \omega_1 \circ \text{Ad}_w$. Since $v = uw$, we have

$$v^* \delta^\rho(v) = (uw)^* \delta^\rho(uw) = w^* u^* \delta^\rho(u) w + w^* \delta^\rho(w).$$

Applying ω_1 to both sides of this and using the w -invariance of ω_1 , it remains to show that $\omega_1(w^* \delta^\rho(w)) = 0$ to get the statement. Using Lemma 2.7 below we have that

$$|\omega_1(w)| = 1. \quad (11)$$

Defining $g := w - \omega_1(w)\mathbb{1}$, we calculate

$$\begin{aligned} \omega_1(|g|^2) &= \omega_1((w - \omega_1(w)\mathbb{1})^*(w - \omega_1(w)\mathbb{1})) \\ &= \omega_1(|w|^2) - |\omega_1(w)|^2 \\ &= 0 \end{aligned}$$

where we have used that $\omega_1(|w|^2) = \omega_1(w^* \mathbb{1} w) = \omega_1(\mathbb{1}) = 1$. Now by Cauchy-Schwarz,

$$|\omega_1(g^* a)|^2 \leq \omega_1(|g|^2) \omega_1(|a|^2) = 0 \quad (12)$$

for all $a \in \mathcal{A}$ which shows that $\ker \omega_1$ is a two-sided $*$ -ideal within $\mathcal{A}$. Hence

$$\begin{aligned}\omega_1(w^* \delta^\rho(w)) &= \omega_1((g + \omega_1(w)\mathbb{1})^* \delta^\rho(g + \omega_1(w)\mathbb{1})) \\ &= \omega_1(g^* \delta^\rho(g)) \\ &= 0\end{aligned}$$

where we used that $\omega_1 \circ \delta^\rho = 0$ since ω_1 is invariant under ρ_t in the second equality and (12) in the last equality.

(ii) Let $(\mathcal{H}_1, \pi_1, \Omega_1)$ be the GNS triplet associated with ω_1 . Let Q_1 be the self-adjoint operator on $\mathcal{H}_1$ such that e^{itQ_1} implements ρ_t ; see Lemma 2.8 right below. By construction, $Q_1 \Omega_1 = 0$.

We claim that $\Omega_2 := \pi_1(u)\Omega_1$ is also an eigenvector of Q_1 , namely $Q_1 \Omega_2 = n\Omega_2$ for some $n \in \mathbb{Z}$ since Q_1 has integer spectrum. To see this we proceed as follows. Since (ω_1, ω_2) is a locally comparable pair, ω_2 is a vector state in $\mathcal{H}_1$, namely the vector $\Omega_2 \equiv \pi_1(u)\Omega_1$ is such that

$$\langle \Omega_2, \pi_1(a)\Omega_2 \rangle = \omega_2(a) \quad (a \in \mathcal{A}).$$

Moreover, $\omega_2 \circ \rho_t = \omega_2$ implies, after going to the GNS representation and taking the derivative with respect to t at 0, that

$$\langle \Omega_2, [Q_1, A]\Omega_2 \rangle = 0 \quad (A \in \mathcal{B}(\mathcal{H}_1)),$$

since Ω_2 is in the domain of Q_1 . We may take A to be the orthogonal projection onto $\text{span}(\Psi)$ for some $\Psi \in \mathcal{H}_1$ to conclude that

$$\langle \Omega_2, Q_1 \Psi \rangle \langle \Psi, \Omega_2 \rangle \in \mathbb{R} \quad (\Psi \in \mathcal{H}_1). \quad (13)$$

Now if Ω_2 is *not* an eigenvector of Q_1 , then there is a nonzero $\Phi \in \text{span}(\Omega_2)^\perp$ such that

$$Q_1 \Omega_2 = \lambda \Omega_2 + \Phi$$

for some $\lambda \in \mathbb{C}$. Actually $\lambda \in \mathbb{R}$ because Q_1 is self-adjoint. Now if we invoke (13) with $\Psi = \Omega_2 + i\Phi$ we obtain

$$\begin{aligned}\mathbb{R} \ni \langle \Omega_2, Q_1(\Omega_2 + i\Phi) \rangle \langle \Omega_2 + i\Phi, \Omega_2 \rangle &= \langle \lambda \Omega_2 + \Phi, \Omega_2 + i\Phi \rangle \|\Omega_2\|^2 \\ &= \left(\lambda \|\Omega_2\|^2 + i\|\Phi\|^2 \right) \|\Omega_2\|^2,\end{aligned}$$

which is a contradiction unless $\Phi = 0$. Hence Ω_2 is an eigenvector of Q_1 , which has integer spectrum, so

$$\mathcal{N}_\rho(\omega_1, \omega_2) = \langle \Omega_1, Q_1 \Omega_1 \rangle - \langle \Omega_2, Q_1 \Omega_2 \rangle = -\langle \Omega_2, Q_1 \Omega_2 \rangle \in \mathbb{Z}. \quad (14)$$

(iii) Using the same GNS notation as above, if $\mathcal{N}_\rho(\omega_1, \omega_2) \neq 0$ then Ω_2 has a nonzero eigenvalue for Q_1 . Since Q_1 is a self-adjoint operator and Ω_1 is a zero-eigenvalue eigenvector for Q_1 , this implies that $\langle \Omega_2, \Omega_1 \rangle = 0$.

On the other hand, since both states ω_1, ω_2 are represented as vector states on the same Hilbert space, we have $\|\omega_1 - \omega_2\| = \sup_{a \in \mathcal{A}: \|a\| \leq 1} |\langle \Omega_1, \pi_1(a)\Omega_1 \rangle - \langle \Omega_2, \pi_1(a)\Omega_2 \rangle|$. Denoting P_j is the orthogonal projection onto the span of Ω_{ω_j} for $j = 1, 2$, we have

$$\|\omega_1 - \omega_2\| = \sup_{a \in \mathcal{A}: \|a\| \leq 1} |\text{Tr}(\pi_1(a)(P_1 - P_2))| = \|P_1 - P_2\|_1$$

The second equality is by duality since π_1 is irreducible, as ω_1 is pure. Since P_j are one-dimensional projections, the trace norm is easily computable and we conclude that

$$\|\omega_1 - \omega_2\| = 2\sqrt{1 - |\langle \Omega_2, \Omega_1 \rangle|^2}. \quad (15)$$

Hence,

$$|\mathcal{N}_\rho(\omega_1, \omega_2)| > 0 \implies \|\omega_1 - \omega_2\| = 2.$$

We note in passing that the reader may want to compare this with a related converse statement about unitary equivalence of the states when $\|\omega_1 - \omega_2\| < 2$, see [29, Corollary 9, Remark 10].

(iv) Since (ω_2, ω_1) are ρ -locally comparable, there is $u \in \mathcal{A} \cap \mathcal{D}(\delta^\rho)$ such that $\omega_2 = \omega_1 \circ \text{Ad}_u$. Therefore,

$$\omega_2 \circ \alpha = \omega_1 \circ \text{ad}_u \circ \alpha = (\omega_1 \circ \alpha) \circ \text{ad}_{\alpha^{-1}(u)} \quad (16)$$

so that $(\omega_2 \circ \alpha, \omega_1 \circ \alpha)$ are locally comparable. They are in fact ρ -locally comparable since α commutes with δ^ρ and so $\alpha^{-1}(\mathcal{D}(\delta^\rho)) \subset \mathcal{D}(\delta^\rho)$. The computation

$$-i\mathcal{N}_\rho(\omega_1 \circ \alpha, \omega_2 \circ \alpha) = (\omega_1 \circ \alpha) \left(\alpha^{-1}(u)^* \delta^\rho(\alpha^{-1}(u)) \right) = \omega_1(u^* \delta^\rho(u)) = -i\mathcal{N}_\rho(\omega_1, \omega_2)$$

yields the invariance.

(v) By (i), we can pick $u = 1$, for which $\delta^\rho(u) = 0$ and so the index vanishes.

(vi) To get the additivity of the index, assume that $\omega_2 = \omega_1 \circ \text{Ad}_u$ and $\omega_3 = \omega_2 \circ \text{ad}_v$. Hence $\omega_3 = \omega_1 \circ \text{Ad}_{vu}$ so that

$$\begin{aligned} \mathcal{N}_\rho(\omega_1, \omega_3) &= i\omega_1((vu)^* \delta^\rho(vu)) \\ &= i\omega_1(u^* v^* ((\delta^\rho(v))u + v\delta^\rho(u))) \\ &= i\omega_2(v^* \delta^\rho(v)) + i\omega_1(u^* \delta^\rho(u)) \\ &= \mathcal{N}_\rho(\omega_2, \omega_3) + \mathcal{N}_\rho(\omega_1, \omega_2). \end{aligned}$$

Finally, (vii) follows from (vi) with $\omega_3 = \omega_1$ and (v).

(viii) The result follows immediately since $\delta^{\rho \otimes \tilde{\rho}} = \delta^\rho \otimes \text{id} + \text{id} \otimes \tilde{\delta}^\rho$. □

Lemma 2.7. Let $\omega \in \mathcal{P}(\mathcal{A})$ and $u \in \mathcal{U}(\mathcal{A})$ be such that $\omega = \omega \circ \text{Ad}_u$. Then $\omega(u) \in U(1)$.

Proof. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS representation of ω . Then the invariance condition implies

$$\omega(a) = \langle \Omega, \pi(a)\Omega \rangle = \omega(u^* au) = \langle \Omega, \pi(u)^* \pi(a) \pi(u)\Omega \rangle.$$

We find that both Ω and $\pi(u)\Omega$ are cyclic vectors that generate the same state, where π is irreducible as ω is pure. This implies that $\pi(u)\Omega = e^{i\theta}\Omega$ for some $\theta \in \mathbb{R}$ and so $\omega(u) = \langle \Omega, \pi(u)\Omega \rangle = e^{i\theta}$ as claimed. □

Lemma 2.8. Let $\{\rho_t\}_{t \in \mathbb{R}} \subseteq \text{Aut}(\mathcal{A})$ be a one-parameter group of strongly continuous automorphisms such that $\rho_{2\pi} = \text{id}$ and let $\omega \in \mathcal{P}(\mathcal{A})$ be ρ -invariant. Then in the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of ω , there exists a self-adjoint operator $Q_\omega : \mathcal{H}_\omega \rightarrow \mathcal{H}_\omega$ such that

$$\pi_\omega(\rho_t(a)) = e^{-itQ_\omega} \pi_\omega(a) e^{itQ_\omega} \quad (t \in \mathbb{R}, \quad a \in \mathcal{A}).$$

One may choose Q_ω such that $Q_\omega \Omega_\omega = 0$, in which case $\sigma(Q_\omega) \subseteq \mathbb{Z}$.

Proof. For any $t \in \mathbb{R}$, the existence of a unitary implementer $U_{\omega,t}$ of ρ_t is a standard consequence of the uniqueness of the GNS representation. Strong continuity of $t \mapsto \rho_t$ implies strong continuity of $t \mapsto U_{\omega,t}$, and the existence of a self-adjoint generator Q_ω is ensured by Stone's theorem. Since ω is pure, π_ω is irreducible and hence Q_ω is defined up to an additive constant. The invariance of ω yields that Ω_ω is an eigenvector of Q_ω , which can be fixed by imposing $Q_\omega \Omega_\omega = 0$. With this, for any $a \in \mathcal{A}$,

$e^{2\pi i Q_\omega} \pi_\omega(a) \Omega_\omega = \pi_\omega(\rho_{2\pi}(a)) \Omega_\omega = \pi_\omega(a) \Omega_\omega$, which implies that $e^{2\pi i Q_\omega} = \mathbb{1}$ by cyclicity of GNS representation, so that $\sigma(Q_\omega) \subseteq \mathbb{Z}$. $\square$

Remark 2.9. The assumption of local-comparability and the standard Lemma 2.8 immediately imply that the index of just the difference of eigenvalues of a charge operator which has integer spectrum, see (14), and hence the name ‘charge deficiency’. The interest of phrasing our result in the abstract algebraic setting is two-fold. First of all, as we shall see in Section 3, establishing the ρ -local-comparability of two states is the essence of the problem in applications. Secondly, it allows us to phrase the automorphic invariance (5) since the states ω_1 and $\omega_1 \circ \alpha$ are not in general unitarily equivalent and therefore not representable as vector states in each other’s GNS representation.

Of course, the lemma also shows that the period 2π of the group ρ_t is, at least mathematically, an arbitrary choice: If instead $\rho_{2\pi/\lambda} = \text{id}$ for some $\lambda > 0$ (but keeping Q with integer spectrum) then $\widetilde{\mathcal{N}}_\rho(\omega_1, \omega_2) := \lambda^{-1} \mathcal{N}_\rho(\omega_1, \omega_2) \in \mathbb{Z}$. Finally, we emphasize again that the requirement that the intertwining unitary u obey $u \in \mathcal{D}(\delta^\rho)$ is not vacuous with the following example.

Example 2.10. Let $\mathcal{A} = \text{CAR}(\ell^2(\mathbb{N}))$ with $\{e_j\}_{j \geq 1}$ the canonical position basis of $\ell^2(\mathbb{N})$ so $a_j \equiv a(e_j)$ are the annihilation operators associated with those basis elements. Then

$$q_N := \sum_{j=1}^N a_j^* a_j$$

defines a sequence of bounded self-adjoint elements in $\mathcal{A}$ which approximates a derivation δ as

$$\delta(a) \equiv \lim_N i[q_N, a] \quad (a \in \mathcal{D}(\delta))$$

where $\mathcal{D}(\delta)$ is the space of elements a for which the limit exists. Define

$$a := \sum_{j=1}^{\infty} j^{-3/2} a_1^* \cdots a_j^*$$

which exists since $\|a\| \leq \sum_j j^{-3/2} < \infty$. We may then calculate

$$[q_N, a_1^* \cdots a_j^*] = \min(\{N, j\}) a_1^* \cdots a_j^*$$

and hence

$$[q_N, a] = \sum_{j=1}^N j^{-1/2} a_1^* \cdots a_j^* + N \sum_{j=N}^{\infty} j^{-3/2} a_1^* \cdots a_j^*.$$

We see that the limit $N \rightarrow \infty$ of the above expression does not exist so that $a \notin \mathcal{D}(\delta)$.

Now take $h := a + a^*$ and $u := (\mathbb{1} + i h)(\mathbb{1} - i h)^{-1}$ to get a unitary element which is not in $\mathcal{D}(\delta)$. As a result, for any $\omega \in \mathcal{P}(\mathcal{A})$ we may define $\widetilde{\omega} := \omega \circ \text{Ad}_u$ such that $\omega, \widetilde{\omega}$ are locally comparable but not ρ -locally comparable.

2.2. Connection with the index of a pair of projections on a Hilbert space

Let $\mathcal{H}$ be a separable Hilbert space and P_1, P_2 be two self-adjoint projections such that

$$P_1 - P_2 \in \mathcal{K}(\mathcal{H}). \quad (17)$$

In this setting it is well-known that a topological index is associated with the pair (P_1, P_2) (see, e.g., [7] and earlier citations within) given by

$$\text{index}(P_1, P_2) \equiv \text{index}_{\text{im}(P_1) \rightarrow \text{im}(P_2)} \left(P_2 P_1|_{\text{im}(P_1)} \right) = \text{index}_{\mathcal{H}}(P_2 P_1 + U P_1^\perp) \quad (18)$$

where the index on the right-hand sides is the Fredholm index and U is any unitary $U : \text{im}(P_1^\perp) \rightarrow \text{im}(P_2^\perp)$. In the special case that $P - Q \in \mathcal{J}_p(\mathcal{H})$, the ideal of p -Schatten class operators, for some p , we may also write

$$\text{index}(P_1, P_2) = \text{tr} \left((P_1 - P_2)^{2p'+1} \right)$$

for any $p' \in \mathbb{N}$ such that $2p' + 1 \geq p$. Finally, if $P_1 - P_2 \in \mathcal{J}_2(\mathcal{H})$ then by [4, Theorem 3], we may write

$$\text{index}(P_1, P_2) = \text{tr}(P_2(P_1 - P_2)P_2) + \text{tr}(P_2^\perp(P_1 - P_2)P_2^\perp). \quad (19)$$

We turn to the relation of $\text{index}(P_1, P_2)$ with the newly introduced many-body index $\mathcal{N}_\rho(\omega_1, \omega_2)$.

Remark 2.11. Since any two projections P_1, P_2 on an infinite-dimensional Hilbert space with infinite $\ker P_j$, $\text{im } P_j$ admit a unitary U which conjugates them, that is, $P_2 = U^* P_1 U$, one may be tempted to conclude that if $P - Q$ is “small” then so is $U - \mathbb{1}$ (so as to have any hope to implement it in $\mathcal{A}$). However, it is well-known (see, e.g., [41, Theorem 1.2] or [21]) that a unitary $U \in \mathcal{U}(\mathcal{H})$ can be found such that both $U - \mathbb{1} \in \mathcal{K}(\mathcal{H})$ and $P_2 = U^* P_1 U$ iff

$$\text{index}(P_1, P_2) = 0.$$

This indicates that to capture a nontrivial index we most likely need to find a unitary in the CAR algebra which is not the second quantization of a Hilbert space one.

As we shall see below, there will be additional obstructions beyond the purely Hilbert space index obstruction.

Let now $\mathcal{A} = \text{CAR}(\mathcal{H})$ (see Section 3.2 for more details) be the algebra of observables corresponding to systems of many fermions. Any self-adjoint projection P induces a pure state $\omega_P \in \mathcal{P}(\mathcal{A})$, the so-called quasi-free state associated to P , given by the Gaussian formula

$$\omega_P(a(f_1)^* \cdots a(f_m)^* a(g_n) \cdots a(g_1)) := \delta_{n,m} \det(\{\langle g_i, P f_j \rangle\}_{i,j=1,\dots,n}), \quad (20)$$

where $f_i, g_i \in \mathcal{H}$, and extended linearly to all polynomials and by continuity to all of $\mathcal{A}$.

The canonical ‘charge’ $U(1)$ *-automorphism ρ_t is defined by

$$\rho_t(a(f)) = e^{-it} a(f), \quad \rho_t(a^*(f)) = e^{it} a^*(f), \quad (21)$$

for any $f \in \mathcal{H}$ and $t \in \mathbb{R}$. Note that $\rho_{2\pi} = \mathbb{1}$. For any orthogonal projection P , the state ω_P is automatically invariant under ρ_t .

To contextualize the ensuing discussion, let us recall a few basic facts:

- (i) The Shale-Stinespring condition for quasi-free states on $\text{CAR}(\mathcal{H})$ [44]: $\omega_{P_1}, \omega_{P_2}$ have unitarily equivalent GNS representations if and only if $P_1 - P_2 \in \mathcal{J}_2(\mathcal{H})$.
- (ii) The Kadison transitivity theorem for general C*-algebras [29, Corollary 8]: If two pure states $\omega, \tilde{\omega} \in \mathcal{P}(\mathcal{A})$ have unitarily equivalent GNS representations then there exists some $u \in \mathcal{U}(\mathcal{A})$ such that $\tilde{\omega} = \omega \circ \text{Ad}_u$.
- (iii) The condition on Bogoliubov automorphisms of $\text{CAR}(\mathcal{H})$ to be inner [2, Theorem 5]: $U - \mathbb{1} \in \mathcal{J}_1(\mathcal{H})$ if and only if $\exists \Gamma(U) \in \mathcal{U}(\mathcal{A})$ such that $a(Uf) = \Gamma(U)^* a(f) \Gamma(U)$ for all $f \in \mathcal{H}$.

We are now ready to connect the index in (18) with the one defined in (3). For this, given the basic facts stated above, and in particular (iii), we need to strengthen the assumption on $P_1 - P_2$ from compact to trace-class.

Theorem 2.12. *Let P_1, P_2 be self-adjoint projections such that*

$$P_1 - P_2 \in \mathcal{J}_1(\mathcal{H}).$$

Then ω_{P_1} and ω_{P_2} are ρ -locally comparable and

$$\text{index}(P_1, P_2) = \mathcal{N}_\rho(\omega_{P_1}, \omega_{P_2}).$$

This theorem identifies $\mathcal{N}_\rho$ as the many-body analog of the notion of an index of pair of projections, albeit with a certain mathematical gap: if $P_1 - P_2 \in \mathcal{K}(\mathcal{H})$ but is not Hilbert-Schmidt, then by the above basic facts ω_{P_1} and ω_{P_2} cannot be locally comparable: indeed, if they were, they would be GNS unitarily equivalent and hence violate the Shale-Stinespring condition. If $P_1 - P_2$ is Hilbert-Schmidt but not trace-class, then by the above there would be a unitary conjugating them in the CAR algebra, but since that unitary is given abstractly by the Kadison transitivity theorem and is not the second quantization of any unitary, we do not know how to establish that that unitary necessarily lies in the domain of δ^ρ .

Ultimately, a generalization of the many-body index may exist, which relies on a weaker condition that (ω_1, ω_2) be locally comparable, and which would always be defined for quasi-free states $(\omega_{P_1}, \omega_{P_2})$ as soon as $P_1 - P_2 \in \mathcal{K}$. This problem is not entirely academic since in the interesting case of the integer quantum Hall effect, $P - L^*PL \in \mathcal{J}_3 \setminus \mathcal{J}_2$ where

$$L \equiv e^{i \arg(X_1 + iX_2)} \quad (22)$$

is the Laughlin unitary [6] implementing one quanta of a magnetic flux insertion. See also Example 2.14 below for a simple example of this type.

Example 2.13 (Nonzero index yet unitary exists thanks to interactions). The obstruction outlined in Remark 2.11 does *not* imply that we cannot implement nonzero indices. Indeed, pick $\mathcal{H} := \ell^2(\mathbb{Z})$ and let P be the multiplication operator by the indicator function $\chi_{\mathbb{N}}$, namely the projection onto the RHS of space. Let R be the bilateral right shift operator $R : \delta_x \mapsto \delta_{x+1}$. Then $P_R = R^*PR$ is the projection given by $\chi_{\mathbb{N} \cup \{0\}}$ and $P_R - P$ is the finite rank projection given by $\chi_{\{0\}}$, and $\text{index}(P, P_R) = -1$. Of course this means $P - P_R \in \mathcal{J}_2(\mathcal{H})$, equivalently, $[R, P] \in \mathcal{J}_2(\mathcal{H})$.

The operator

$$u := a_0 + a_0^*.$$

is unitary since $u^*u = u^2 = (a_0 + a_0^*)^2 = \mathbb{1}$, such that

$$u^*a_0u = a_0^*, \quad \text{while} \quad u^*a_mu = -a_m \quad \text{whenever } m \neq 0.$$

and we claim that

$$\omega_{P_R} = \omega_P \circ \text{Ad}_u.$$

Indeed, if $n, m \neq 0$,

$$\omega_P(u^*a_m^*a_nu) = \omega_P(a_m^*a_n) = \langle \delta_n, P\delta_m \rangle = \delta_{m,n}\chi_{\mathbb{N}}(n) = \omega_{P_R}(a_m^*a_n)$$

and if $n = m = 0$ then

$$\omega_P(u^*a_0^*a_0u) = \omega_P(a_0a_0^*) = \omega_P(\mathbb{1} - a_0^*a_0) = 1 = \omega_{P_R}(a_0^*a_0),$$

while both $\omega_P(u^*a_m^*a_nu)$ and $\omega_{P_R}(a_m^*a_n)$ vanish if $n \neq m$. Since

$$\delta^\rho(a_0) = \frac{d}{dt}\rho_t(a_0)\Big|_{t=0} = -ia_0$$

we see that $u \in \mathcal{D}(\delta^\rho)$ and $-i\delta^\rho(u) = -a_0 + a_0^*$, so that

$$\mathcal{N}_\rho(\omega_P, \omega_{P_R}) = i\omega_P(u^*\delta^\rho(u)) = -\omega_P(-a_0^*a_0 + a_0a_0^*) = -\omega_P(\mathbb{1} - 2a_0^*a_0) = -1.$$

Example 2.14 (Compact but not Hilbert-Schmidt difference of projections so no unitary exists). Let $\mathcal{H} := \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$. Let P be given by

$$P_{nm} := \delta_{nm} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

that is, projection onto the top of each dimer. Let U be the unitary given by

$$U_{nm} := \delta_{nm} \begin{bmatrix} \cos(\theta_n) & \sin(\theta_n) \\ -\sin(\theta_n) & \cos(\theta_n) \end{bmatrix}$$

where $\{\theta_n\}_n$ is some sequence of angles to be determined below. If $Q := U^*PU$, then

$$P_{nm} - Q_{nm} = \delta_{nm} \begin{bmatrix} \sin^2 \theta_n & -\cos \theta_n \sin \theta_n \\ -\cos \theta_n \sin \theta_n & -\sin^2 \theta_n \end{bmatrix}.$$

Since the singular values of $P - Q$ are $\{|\sin(\theta_n)|\}_{n \geq 1}$ it is clear that if we pick $\theta_n := \arcsin(x_n)$ for some sequence $\{x_n\}_n \subseteq (0, 1)$ we can engineer whichever summability we want. For example if we take $x_n = n^{-\beta}$ for any $\beta \in (1/3, 1/2]$ then we guarantee that $\{x_n^2\}_n$ is not summable but $\{x_n^3\}_n$ is. Note that as soon as $x_n \rightarrow 0$ we have $P - Q \in \mathcal{K}(\mathcal{H})$.

Now, for $\{x_n\}_n$ in ℓ^3 but not ℓ^2 there does not exist a unitary $u \in \mathcal{U}(\mathcal{A})$ such that $\omega_Q = \omega_P \circ \text{Ad}_u$. Indeed, the existence of such a unitary would violate the Shale-Stinespring condition [44].

Before presenting the proof of Theorem 2.12, let us make a final remark. Suppose we knew that ω_1 and ω_2 (with $\omega_j \equiv \omega_{P_j}$) are ρ -locally comparable. Then the equivalence of the two indices would be immediate. Indeed, starting from (9) and approximating $\delta^\rho = \lim_N i[q_N, \cdot]$ with $q_N := \sum_{j=1}^N a^*(e_j)a(e_j)$ where $\{e_j\}_j$ is any ONB of $\mathcal{H}$, we obtain using (20)

$$\begin{aligned} \mathcal{N}_\rho(\omega_1, \omega_2) &= \lim_N (\omega_1(q_N) - \omega_2(q_N)) = \lim_N \sum_{j=1}^N (\langle e_j, P_1 e_j \rangle - \langle e_j, P_2 e_j \rangle) \\ &= \lim_N \text{tr}((P_1 - P_2)\chi_{\{1, \dots, N\}}(X)) \end{aligned}$$

where X is the position operator w.r.t. the basis $\{e_j\}_j$, that is, $Xe_j \equiv je_j$ for any $j \in \mathbb{N}$. Since we have $\chi_{\{1, \dots, N\}}(X) \rightarrow \mathbb{1}$ strongly as $N \rightarrow \infty$, and since $P_1 - P_2 \in \mathcal{J}_1$ we are finished, since the product of a strongly convergent sequence with a trace-class operator is a sequence converging in trace-class norm, that is, we find

$$\mathcal{N}_\rho(\omega_1, \omega_2) = \text{tr}(P_1 - P_2) = \text{index}(P_1, P_2).$$

We also note that in the case $P_1 - P_2 \in \mathcal{J}_2(\mathcal{H})$ the proof given (using a particular choice of the ONB $\{e_j\}_j$) goes through by invoking the explicit formula (19). In the proof below, we rely on the trace-class assumption to construct the unitary intertwiner between ω_1 and ω_2 , which lies in $\mathcal{D}(\delta^\rho)$. So the main difficulty is to establish the ρ -local-comparability of ω_1, ω_2 , that is, to find the unitary which conjugates them and show it lies in $\mathcal{D}(\delta^\rho)$.

The proof of Theorem 2.12 relies on two steps.

- (i) We first extract the ‘excess states’ out of the difference $P_1 - P_2$ which are responsible for a nonzero index, see Lemma 2.15 below. Because of the obstruction in Remark 2.11, the unitary which implements this unfolding is necessarily not the second quantization of a Hilbert space one, but it can be constructed explicitly. In particular, it is in $\mathcal{D}(\delta^\rho)$ since the charge deficiency is finite. Indeed, it is very much in the spirit of Example 2.13.
- (ii) As a result, we obtain two projections whose index is zero and whose difference is trace-class. It remains to use the second quantization provided by Lemma 2.16, which does not change the charge deficiency. Since all unitaries are explicit, the equality of indices follows from a computation.

Proof of Theorem 2.12. Using Lemma 2.15 below, since $P_1 - P_2 \in \mathcal{J}_1(\mathcal{H})$, we write

$$P_2 = \tilde{P} + N_+ - N_-,$$

with $\tilde{P} = VP_1V^*$ and $V - 1 \in \mathcal{J}_1(\mathcal{H})$. By Lemma 2.16 below, $\omega_{\tilde{P}}$ and ω_{P_1} are ρ -locally comparable with $\Gamma(V^*) \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\delta^\mathcal{Q})$ and

$$\omega_{\tilde{P}} = \omega_{P_1} \circ \text{Ad}_{\Gamma(V^*)}. \quad (23)$$

since $\omega_{\tilde{P}}(a^*(f)a(g)) = \langle g, VP_1V^*f \rangle = \omega_{P_1}(a^*(V^*f)a(V^*g))$. Let $n_\pm = \text{rank}(N_\pm)$ and $\{f_i\}$ and $\{g_i\}$ be orthonormal bases of the range of N_- and N_+ respectively. We consider

$$v_+ = \prod_{i=1}^{n_+} (a(g_i) + a^*(g_i)), \quad v_- = \prod_{i=1}^{n_-} (a(f_i) + a^*(f_i)).$$

One checks that $v_\pm \in \mathcal{U}(\mathcal{A})$ as products of unitary factors, see again Example 2.13. Moreover, $v_\pm \in \mathcal{D}(\delta^\mathcal{Q})$ as polynomials in creation and annihilation operators. We claim that

$$u := v_+v_- \Gamma(V)^* \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\delta^\mathcal{Q}).$$

is such that

$$\omega_{P_2} = \omega_{\tilde{P}} \circ \text{Ad}_{v_+v_-} = \omega_{P_1} \circ \text{Ad}_u. \quad (24)$$

We prove the first equality in (24), the second one is just (23). Since

$$(a^*(\xi) + a(\xi))a(\phi) = \langle \phi, \xi \rangle \mathbb{1} - a(\phi)(a^*(\xi) + a(\xi))$$

we see that for $v_\xi = a^*(\xi) + a(\xi)$,

$$\omega_{\tilde{P}} \circ \text{Ad}_{v_\xi}(a^*(\psi)a(\phi)) = \langle \phi, (\tilde{P} + P_\xi - (\tilde{P}P_\xi + P_\xi\tilde{P}))\psi \rangle$$

where $P_\xi = |\xi\rangle\langle\xi|$. In particular,

$$\begin{cases} \omega_{\tilde{P}} \circ \text{Ad}_{v_\xi}(a^*(\psi)a(\phi)) = \langle \phi, (\tilde{P} - P_\xi)\psi \rangle & \text{if } \tilde{P}P_\xi = P_\xi, \\ \omega_{\tilde{P}} \circ \text{Ad}_{v_\xi}(a^*(\psi)a(\phi)) = \langle \phi, (\tilde{P} + P_\xi)\psi \rangle & \text{if } \tilde{P}P_\xi = 0. \end{cases}$$

Applying this recursively with ξ running through the basis of N_+ and then the basis of N_- and recalling that $\tilde{P}N_- = N_-$ and $\tilde{P}N_+ = 0$ yields

$$\omega_{\tilde{P}} \circ \text{Ad}_{v_+v_-}(a^*(\psi)a(\phi)) = \langle \phi, (\tilde{P} + N_+ - N_-)\psi \rangle = \omega_{P_2}(a^*(\psi)a(\phi))$$

which proves (24). Thus ω_{P_2} and ω_{P_1} are ρ -locally comparable.

For completeness let us show the equivalence of the indices using the intrinsic decomposition just exhibited. We compute $\mathcal{N}_\rho(\omega_{P_1}, \omega_{P_2}) = i\omega_1(u^*\delta^\rho(u))$. Since $\delta^\rho(\Gamma(V)) = 0$, we focus on $v_-^*v_+^*\delta^\rho(v_+)v_- + v_-^*v_+^*\delta^\rho(v_-)$. Since

$$\delta^\rho(a^*(f) + a(f)) = i(a^*(f) - a(f))$$

for any normalized f , and therefore

$$(a^*(f) + a(f))\delta^\rho(a^*(f) + a(f)) = i(\mathbb{1} - 2a^*(f)a(f))$$

we conclude that $v_+^*\delta^\rho(v_+) = n_+ - 2Q_+$ and so

$$iv_-^*v_+^*\delta(v_+)v_- + iv_-^*v_+^*\delta^\rho(v_-) = -(n_- - 2Q_-) - (n_+ - 2Q_+).$$

Here $Q_- = \sum_{i=1}^{n_-} a^*(f_i)a(f_i)$ and $Q_+ = \sum_{i=1}^{n_+} a^*(g_i)a(g_i)$. Hence

$$\begin{aligned}\mathcal{N}_\rho(\omega_{P_1}, \omega_{P_2}) &= i\omega_{P_1}(\Gamma(V)((n_- - 2Q_-) + (n_+ - 2Q_+))\Gamma(V^*)) \\ &= -n_- - n_+ + 2\omega_{\bar{P}}(Q_- + Q_+) = n_- - n_+\end{aligned}$$

since $\omega_{\bar{P}}(Q_-) = n_-$ while $\omega_{\bar{P}}(Q_+) = 0$ by (20). $\square$

Lemma 2.15. *If P_1, P_2 are two orthogonal projections on a Hilbert space $\mathcal{H}$ such that $P_1 - P_2 \in \mathcal{J}_p(\mathcal{H})$ for $p \in \mathbb{N}^*$, there exists finite-rank orthogonal projections N_+, N_- with $N_+N_- = 0$ and a unitary V with $V - \mathbb{1} \in \mathcal{J}_p(\mathcal{H})$ such that*

$$P_2 = VP_1V^* + N_+ - N_- \quad (25)$$

Moreover $\text{index}(P_1, P_2) = \text{tr}(N_+ - N_-)$, and for $\tilde{P} = VP_1V^*$ one has $\tilde{P}N_+ = 0$ and $\tilde{P}N_- = N_-$.

The proof of this lemma can be found in [10, Proof of Prop. 5.2], see also [18, Sec 4.2.1] or [22, Sec VII.B].

Lemma 2.16. *If V is a unitary operator on $\mathcal{H}$ such that $V - \mathbb{1} \in \mathcal{J}_1(\mathcal{H})$, the $*$ -automorphism defined by $a(f) \mapsto a(Vf)$ is unitarily implementable by $\Gamma(V) \in \mathcal{A}$, namely*

$$a(Vf) = \Gamma(V)^*a(f)\Gamma(V)$$

for all $f \in \mathcal{H}$. Moreover, $\delta^\rho(\Gamma(V)) = 0$.

This is a known result up to possibly the last property, see, e.g., [2, Theorem 5]. For convenience of the reader we include its proof here.

Proof of Lemma 2.16. We recall the explicit construction of the implementer $\Gamma(V) \in \mathcal{A}$ given in [5] whenever $V - \mathbb{1}$ is of finite rank. For rank-one operators $A_i = |f_i\rangle\langle g_i|$, ($i = 1, \dots, n$), we set

$$d\Gamma(A_1, \dots, A_n) = a^*(f_n) \cdots a^*(f_1) a(g_1) \cdots a(g_n). \quad (26)$$

The definition is extended by multilinearity to operators A_i of finite rank. The result is independent of the particular decomposition into rank one operators. Then if $V - \mathbb{1}$ has finite rank, we set

$$\Gamma(V) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} d\Gamma(V - \mathbb{1}, \dots, V - \mathbb{1}). \quad (27)$$

The sum is finite, because the terms with $n > \text{rank}(V - \mathbb{1})$ vanish by the CAR.

A calculation, see [5], then shows that $\Gamma(U)$ is indeed an implementer

$$\Gamma(V)a^*(f) = a^*(Vf)\Gamma(V) \quad (28)$$

and that $\Gamma(V_1)\Gamma(V_2) = \Gamma(V_1V_2)$. In particular, $\Gamma(V)$ is unitary if V is. Moreover, the definition (26) immediately implies that $\rho_t(d\Gamma(A_1, \dots, A_n)) = d\Gamma(A_1, \dots, A_n)$, where ρ_t is the $U(1)$ -automorphism defined in ((21)). This yields in turn that $\rho_t(\Gamma(V)) = \Gamma(V)$ by the definition (27). Hence $\delta^\rho(\Gamma(V)) = 0$.

It remains to extend the map Γ to unitaries such that $V - \mathbb{1}$ is trace class. For rank-one operators, (26) and the CAR yield

$$\|d\Gamma(A_1, \dots, A_n)\| \leq \prod_{i=1}^n \|f_i\| \|g_i\|$$

and so if A_i is a rank r_i operator, we have that

$$\|d\Gamma(A_1, \dots, A_n)\| \leq \inf_{\{f\}, \{g\}} \sum_{i_1=1}^{r_1} \|f_{i_1}\| \|g_{i_1}\| \cdots \sum_{i_n=1}^{r_n} \|f_{i_n}\| \|g_{i_n}\| = \prod_{i=1}^n \|A_i\|_1$$

where the infimum is over all decomposition of the operators $A_j = \sum_{i_j=1}^{r_j} |f_{i_j}\rangle \langle g_{i_j}|$. Hence $d\Gamma$ is a bounded linear transformation from finite-rank operators equipped with the trace norm to the algebra, and it extends to a bounded linear transformation $d\Gamma : \mathcal{J}_1(\mathcal{H})^{\otimes n} \rightarrow \mathcal{A}$, with $\|d\Gamma\| = 1$ for all $n \in \mathbb{N}$. This now implies that the series in (27) is convergent since

$$\left\| \frac{1}{n!} d\Gamma(V - \mathbb{1}, \dots, V - \mathbb{1}) \right\| \leq \frac{1}{n!} \|V - \mathbb{1}\|_1^n.$$

In fact, we have that $\|\Gamma(V)\| \leq e^{\|V - \mathbb{1}\|_1}$. □

3. Integer quantum Hall effect

We now turn to a concrete setting where the abstract index defined and analyzed in Section 2 is realized as the charge deficiency associated with the adiabatic pumping of a unit flux through a two-dimensional electron gas. In the present interacting setting this is closely related to [33]. The key difficulty is to find an appropriate class of states on which flux insertion can be realized by a unitary element of the algebra. A construction of the unitary was however implemented in [10] upon which we shall rely in this section.

3.1. Construction overview and comparison with previous work

The construction being rather long, we summarize its main steps. First, we set in Section 3.2 the algebraic framework by defining concretely the CAR algebra $\mathcal{A}$ and its local structure, 0-chains and their associated dynamics (locally generated automorphism or LGA), and the charge operator Q associated to $U(1)$ -gauge transformations ρ . This allows us to define Q -symmetric and short-ranged-entangled (SRE) states, which we aim to classify up to homotopy. The SRE property encodes the locality property of a state ω . In particular, the existence of a parent Hamiltonian H , for which ω is a gapped ground state, is a direct consequence of the SRE property. Moreover, in contrast to single particle systems where electric charge is the identity operator, its many-body counterpart Q is not always preserved, so we focus on the class of Q -symmetric states to study the integer quantum Hall effect.

Then, a central step is the construction of the flux insertion automorphism from Section 3.4, which starts from an SRE state ω and ends with the defect state ω^D , where a unit of magnetic flux has been inserted at the origin; see Figure 1. Flux insertion relies on gauge transformation, quasi-adiabatic flow and truncation. The main goal is to localize the operations to a half-line, and mimics the half-line gauge

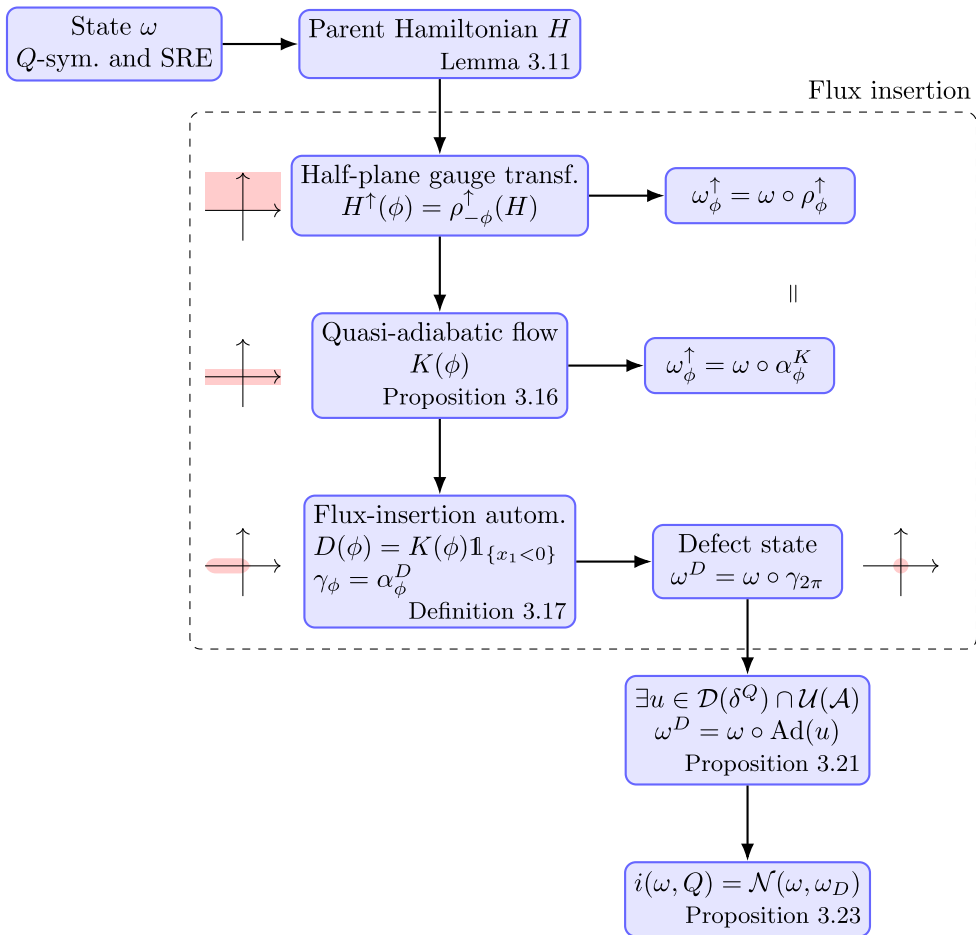


Figure 1. Overview of the construction from the SRE state ω to the defect state ω^D via magnetic flux insertion. The 0-chains are progressively localized near the half-line. At $\phi = 2\pi$, the two states only differ near the origin, which allows for the existence of a unitary u so that they are ρ^Q -locally comparable.

flux insertion from single particle picture [24]. At one quantum of flux $\phi = 2\pi$, the two states ω and ω^D only differ near the origin, allowing the existence of a unitary $u \in \mathcal{A}$ relating the two. Namely, they are ρ -locally comparable and the formalism of Definition 2.4 applies for $\mathcal{N}_\rho(\omega, \omega^D)$. This construction may look convoluted at first, but it is to our current knowledge the only way to construct a defect state which is ρ -locally comparable to ω (see also the discussion at the beginning of Section 3.4). An analogous version is presented in [10], on which we rely.

However, it is actually not possible to provide a quantum Hall model for which this index on SRE states is nonzero. Indeed, for a translation-invariant system of noninteracting electrons, the (stable) SRE condition is equivalent to the triviality of the bundle, hence the vanishing of the Hall conductance and accordingly the vanishing of our index. To model nontrivial Hall states, in Section 3.5 we extend the construction to invertible states. Those are states ω for which there exists some auxiliary state ω' over $\mathcal{A}'$ such that $\omega \hat{\otimes} \omega'$ is itself SRE on $\mathcal{A} \hat{\otimes} \mathcal{A}'$. $\mathcal{A}'$ is another CAR algebra which implements extra degrees of freedom on each lattice site, and $\hat{\otimes}$ is the stacking operation. In the noninteracting case, this amounts to considering the direct sum of two bundles having opposite Chern numbers; the Chern number is additive under direct sums. We then reproduce the previous construction, with the main subtlety of having two charge operators Q and Q' for each layer. The flux insertion is performed on the two layers

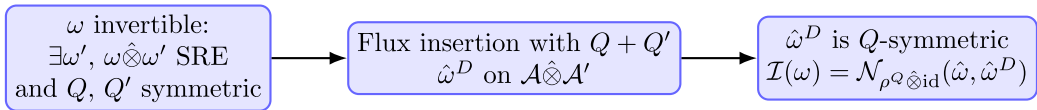


Figure 2. Extending the construction to invertible states.

with $Q + Q'$, but the symmetry appearing in our index is Q , acting on the first layer only. Physically, this amounts to a bona fide flux insertion across both layers, which is essential to have ρ -locally comparable states at the end of the process, but a measurement of charge transport only on the first, physical one. The construction is summarized in Figure 2. It leads to Definition 3.30 and Theorem 3.31, main result of Section 3.5.

We consider here exclusively the ‘bulk picture’ of the quantum Hall effect. As pointed out in the introduction, the technicalities of the construction we provide in this section are heavily inspired by [10]. There, the additional assumption of time reversal invariance implies that there is no net charge transport and the interest is on the remaining $\mathbb{Z}_2$ -valued index which can be nontrivial even for SRE states. In the infinite planar geometry we consider here, the construction of a flux insertion automorphism for invertible states goes back to [33], which in turn was inspired by the flux threading procedure of [9, 8] on large but finite tori. From the point of view of these works, the goal of this section is merely to point out that they provide a concrete physical setting in which the general abstract index of the previous section is realized. Still in the many-body, interacting setting, the Hall conductance and its quantization has also been described as a Thouless pump and [13, 3] provide a different, albeit related picture of its quantization. For weakly interacting fermions, [28] provides a rigorous renormalization group approach to the quantization of the quantum Hall conductance: while the previous works rely on the assumption of a spectral gap, this result proves the stability of the gap in the perturbative regime. Recently, [46] have extended the strongly interacting framework to some cases where the gap closes. This is in fact closely related to [31], which initiated the present line of work. While these works are all concerned with microscopic lattice systems, the quantum Hall effect was analyzed earlier from a field theoretic point of view, see, for example, [27] or the reviews [25, 26]. Finally, Section 4 will establish the equality of the Hall index defined in this work for invertible states with the Hall index for a pair of projections, and through it with the many indices long known to be equivalent to it, whenever they can be compared: spectral flow, Chern number or K-theoretic indices. In particular, we will prove that the trace-class assumption of the general Theorem 2.12 follows from the locality of the Hamiltonian and the gap assumption.

3.2. Algebraic framework

Fermionic observables are elements of the Canonical Anti-commutation Relation (CAR) algebra $\mathcal{A}$ over the one-particle Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ where $n \in \mathbb{N}$ takes into account internal degrees of freedom such as spin. The algebra $\mathcal{A}$ is the unital C^* -algebra generated by an identity element $\mathbb{1}$ and the annihilation operators $\{a(f) : f \in \mathcal{H}\}$, which satisfy the CAR

$$\{a(f), a^*(g)\} = \langle g, f \rangle \mathbb{1}, \quad \{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0 \quad (29)$$

for all $f, g \in \mathcal{H}$, and where $a^*(f) = (a(f))^*$ are the creation operators. By picking the orthonormal basis $\{\delta_x \otimes e_i : x \in \mathbb{Z}^2, i \in \{1, \dots, n\}\}$ of $\mathcal{H}$, the algebra is also generated by $a_{x,i}, a_{y,j}^*$ for $x, y \in \mathbb{Z}^2$ and $i, j \in \{1, \dots, n\}$, where $a_{x,i}^\# = a^\#(\delta_x \otimes e_i)$. Here and in the following $a^\#$ stands for either a or a^* .

For any (possibly infinite) $\Lambda \subset \mathbb{Z}^2$ we denote by $\mathcal{A}_\Lambda$ the unital C^* -subalgebra of $\mathcal{A}$ generated by $\mathbb{1}$ and the $a_{x,i}$ for $x \in \Lambda$ only. When $\Lambda = \{x\}$, we denote $\mathcal{A}_x = \mathcal{A}_{\{x\}}$. There is a natural hierarchy of inclusions: if $\tilde{\Lambda} \subseteq \Lambda$ then $\mathcal{A}_{\tilde{\Lambda}}$ is identified with the subalgebra of $\mathcal{A}_{\tilde{\Lambda}} \otimes \mathbb{1}_{\Lambda \setminus \tilde{\Lambda}}$ of $\mathcal{A}_\Lambda$. Let $\mathcal{F}$ be the set of finite subsets of $\mathbb{Z}^2$. An operator $A \in \mathcal{A}_\Lambda$ with $\Lambda \in \mathcal{F}$ is called *local*. The smallest (in the sense of set

inclusion) $\Lambda \in \mathcal{F}$ such that $A \in \mathcal{A}_\Lambda$ is called the *support* of A , denoted by $\text{supp}(A)$. The $*$ -subalgebra

$$\mathcal{A}^{\text{loc}} = \bigcup_{\Lambda \in \mathcal{F}} \mathcal{A}_\Lambda$$

is called the algebra of local observables. By the definition of $\mathcal{A}$, $\mathcal{A} \equiv \overline{\mathcal{A}^{\text{loc}}}$, in the $\|\cdot\|$ -topology.

Almost local observables.

While $\mathcal{A}^{\text{loc}}$ is extremely convenient for keeping track of where in space observables act, as we shall recall below, it is in general not invariant under the Schrödinger time evolution. For that reason, we work with a standard invariant algebra, originally introduced in [14], which still keeps track of the “support center” of observables. It can be defined as follows. Let

$$\mathcal{L} = \{f : [0, \infty) \rightarrow (0, \infty) \mid f \text{ is bounded, nonincreasing, and } \forall p > 0, \lim_{r \rightarrow \infty} r^p f(r) = 0\}.$$

For $x \in \mathbb{Z}^2$ and $r \in \mathbb{N}$ we denote by $B_x(r)$ the ball of center x and radius r within $\mathbb{Z}^2$. An observable $A \in \mathcal{A}$ called *f-localized* near $x \in \mathbb{Z}^2$ if there exists $f \in \mathcal{L}$, a sequence $A_n \in \mathcal{A}_{B_x(n)}$ such that

$$\|A - A_n\| \leq f(n)\|A\|$$

for all $n \in \mathbb{N}$. An observable is *almost local* if it is *f-localized* for some function $f \in \mathcal{L}$ and some $x \in \mathbb{Z}^2$. We denote by $\mathcal{A}^{\text{al}}$ the $*$ -algebra of almost local observables. Since $\mathcal{A}^{\text{loc}} \subset \mathcal{A}^{\text{al}}$, the latter is also dense in $\mathcal{A}$.

Parity.

The *fermionic parity* is the unique $*$ -automorphism $\theta : \mathcal{A} \rightarrow \mathcal{A}$ which satisfies $\theta(a_{x,i}) = -a_{x,i}$ for all $x \in \mathbb{Z}^2$ and $i \in \{1, \dots, n\}$. Due to θ 's local structure, for any $\Lambda \subset \mathbb{Z}^2$ one has $\theta(\mathcal{A}_\Lambda) = \mathcal{A}_\Lambda$. An element $A \in \mathcal{A}$ is called *even* if $\theta(A) = A$, *odd* if $\theta(A) = -A$, and *homogeneous* if it is either odd or even. For $\Lambda \subset \mathbb{Z}^2$ we denote by $\mathcal{A}_\Lambda^+ = \{A \in \mathcal{A}_\Lambda \mid \theta(A) = A\}$ the C^* -subalgebra of even elements in $\mathcal{A}_\Lambda$, and identify $\mathcal{A}_{\mathbb{Z}^2}^+$ with $\mathcal{A}^+$.

0-chains and autonomous dynamics.

Let $f \in \mathcal{L}$ be given. An *f-local 0-chain* F [34] is a sequence $(F_x)_{x \in \mathbb{Z}^2} \subseteq \mathcal{A}^{\text{al}} \cap \mathcal{A}^+$ such that

- For all $x \in \mathbb{Z}^2$, F_x is self-adjoint and *f-localized* near x .
- $\sup_{x \in \mathbb{Z}^2} \|F_x\| < \infty$.

Proposition 3.1. *Let F be an f -local 0-chain. Then the densely defined map $\delta^F : \mathcal{A}^{\text{al}} \rightarrow \mathcal{A}^{\text{al}}$ given by*

$$A \mapsto \mathbf{i}[F, A] := \sum_{x \in \mathbb{Z}^2} \mathbf{i}[F_x, A]$$

is a $$ -derivation on $\mathcal{A}$.*

In general, δ^F is an unbounded operator on $\mathcal{A}$.

Proof. Let A be $f_A \in \mathcal{L}$ localized near some $x_A \in \mathbb{Z}^2$ and let $C := \sup_x \|F_x\|$. We claim that $\delta^F(A)$ is g -localized near x_A for some $g \in \mathcal{L}$. Let $n \geq 2\|x - x_A\|$ then

$$[F_x, A] = [F_{x,n/2}, A_n] + [F_x, A - A_n] + [F_x - F_{x,n/2}, A]$$

with the first term being supported on $B_{x_A}(n)$ and both remaining terms are bounded above by $C\|A\|g(n)$ for some $g \in \mathcal{L}$, where g can be chosen to be independent of x . Hence $[F_x, A]$ is g -localized near x_A . It follows that $\sum_{x \in \mathbb{Z}^2} [F_x, A]$ is summable and defines an element in $\mathcal{A}^{\text{al}}$. $\square$

For a f -local 0-chain F , the derivation δ^F generates a strongly continuous one-parameter group of $*$ -automorphisms $\mathbb{R} \ni t \mapsto \alpha_t^F$ on $\mathcal{A}$ which is defined by $\alpha_t^F := \exp(t\delta^F)$, that is, as the solution to

$$\alpha_0^F(A) = A, \quad \frac{d}{dt}\alpha_t^F(A) = \alpha_t^F(\delta^F(A))$$

on $\mathcal{A}^{\text{al}}$ and extends by continuity to all of $\mathcal{A}$. Furthermore, it satisfies the semigroup property $\alpha_t^F \circ \alpha_s^F = \alpha_{t+s}^F$.

Nonautonomous setting.

The use of quasi-adiabatic flow and parallel transport below requires that we extend the formalism beyond the autonomous setting. A *time-dependent 0-chain* is a family $(F(s))_{s \in \mathbb{R}}$ such that:

- For any $s_0 \in \mathbb{R}$ there exists $f \in \mathcal{L}$ such that $F(s)$ is an f -localized 0-chain for all $|s| < s_0$.
- $\sup_{x \in \mathbb{Z}^2, s \in \mathbb{R}} \|F_x(s)\| < \infty$.
- $s \mapsto F_x(s)$ is norm-continuous for all $x \in \mathbb{Z}^2$.

As in the time-independent case, a time-dependent 0-chain generates a nonautonomous time evolution $\alpha_{s \rightarrow t}^F$ for $s, t \in \mathbb{R}$ on $\mathcal{A}$ defined by

$$\alpha_{s \rightarrow s}^F(A) = A, \quad \frac{d}{dt}\alpha_{s \rightarrow t}^F(A) = \alpha_{s \rightarrow t}^F(\delta^{F(t)}(A)) \quad (A \in \mathcal{A}^{\text{al}}).$$

The semigroup property is replaced by the following cocycle property $\alpha_{t' \rightarrow t} \circ \alpha_{s \rightarrow t'} = \alpha_{s \rightarrow t}$.

Remark 3.2. From now on, unless stated, all the 0-chains that we consider are time-dependent. For convenience we shall slightly abuse notation and denote $\alpha_t^F := \alpha_{0 \rightarrow t}^F$.

The next result is a consequence of the Lieb-Robinson bound (see for example [47] for a strong version that applies in our setting, and the references therein) and it is the main reason for the introduction of the algebra $\mathcal{A}^{\text{al}}$.

Proposition 3.3. *Let F be a 0-chain. Then, for all $t, s \in \mathbb{R}$,*

$$\alpha_{s \rightarrow t}^F(\mathcal{A}^{\text{al}}) \subset \mathcal{A}^{\text{al}}.$$

Moreover, $\alpha_{s \rightarrow t}^F$ is homogeneous in the sense that $\alpha_t^F \circ \theta = \theta \circ \alpha_t^F$.

Definition 3.4 (LGA). An automorphism α is called a *locally generated automorphism* (LGA) if there exists a 0-chain F and some $s \in [0, \infty)$ such that $\alpha = \alpha_s^F$.

Charge and symmetry.

The relevant symmetry to consider in the quantum Hall effect is charge conservation, which was already defined in (21) but we give here a more concrete description. The charge operator at any site $x \in \mathbb{Z}^2$ is given by

$$Q_x = \sum_{i=1}^n a_{x,i}^* a_{x,i}, \quad (30)$$

which accounts for internal degrees of freedom. Since Q_x is supported on the single site x , the collection $\{Q_x : x \in \mathbb{Z}^2\}$ defines a (time-independent) 0-chain which is trivially f -localized. For $\Lambda \in \mathcal{F}$ the local charge $Q^\Lambda = \sum_{x \in \Lambda} Q_x$ is supported in Λ and it has integer spectrum. For $\phi \in \mathbb{R}$ we denote by $\rho_\phi = \alpha_\phi^Q$ and $\rho_\phi^\Lambda = \alpha_\phi^{Q^\Lambda}$ the LGA respectively associated to Q and Q^Λ .

Proposition 3.5. For any $\Lambda \in \mathcal{F}$, $\phi \in \mathbb{R}$ and $A \in \mathcal{A}$, one has $\rho_\phi^\Lambda(A) = e^{i\phi Q^\Lambda} A e^{-i\phi Q^\Lambda}$. In particular, $\rho_{\phi+2\pi}^\Lambda = \rho_\phi^\Lambda$. Moreover,

$$\rho_\phi(A) = \lim_{\Lambda \rightarrow \mathbb{Z}^2} e^{i\phi Q^\Lambda} A e^{-i\phi Q^\Lambda}$$

for all $\phi \in \mathbb{R}$ and $\rho_{\phi+2\pi} = \rho_\phi$.

Proof. The periodicity follows from the integrality of the spectrum of Q^Λ and the group property of $e^{i\phi Q^\Lambda}$. Existence of the limit is immediate for any $A \in \mathcal{A}^{\text{loc}}$ since the sequence is eventually constant. The general case follows by density. $\square$

Notice that the same limiting procedure yields an automorphism ρ_ϕ^Λ for any other infinite subset of $\mathbb{Z}^2$. Below, we shall in particular consider Λ to be a half-plane.

The LGA ρ_ϕ is the $U(1)$ - (equivalently $\mathbb{S}^1$ -) transformation associated with charge conservation. A 0-chain F is called Q -preserving (or $U(1)$ -symmetric) if $\rho_\phi(F_x(s)) = F_x(s)$ for all $\phi \in \mathbb{R}$, $x \in \mathbb{Z}^2$ and $s \in \mathbb{R}$. If F is a Q -preserving 0-chain, then $\alpha_{s \rightarrow t}^F$ such that $\rho_\phi \circ \alpha_{s \rightarrow t}^F = \alpha_{s \rightarrow t}^F \circ \rho_\phi$ for all $s, t, \phi \in \mathbb{R}$. Consequently, an LGA is called Q -preserving if there exists a Q -preserving 0-chain which generates it.

3.3. Symmetric SRE states

A state $\omega_0 \in \mathcal{S}(\mathcal{A})$ is called a *product state* iff

$$\forall X, Y \subset \mathbb{Z}^d, \quad X \cap Y = \emptyset, \quad A \in \mathcal{A}_X, B \in \mathcal{A}_Y, \quad \omega_0(AB) = \omega_0(A)\omega_0(B).$$

Lemma 3.6. Let ω_0 be a pure state. Then ω_0 is a product state if and only if its restriction to $\mathcal{A}_x$ is pure for any $x \in \mathbb{Z}^2$.

Proof. Let ω_0 be a product state and let ω_0^x be its restriction to $\mathcal{A}_x$. If there is $x \in \mathbb{Z}^2$ such that ω_0^x is not pure, then there are states ϕ^x, ξ^x of $\mathcal{A}_x$ and $\lambda \in (0, 1)$ such that $\omega_0^x = \lambda \phi^x + (1 - \lambda) \xi^x$. It follows that $\omega_0 = \lambda \phi^x \otimes \omega_0^{\mathbb{Z}^2 \setminus \{x\}} + (1 - \lambda) \xi^x \otimes \omega_0^{\mathbb{Z}^2 \setminus \{x\}}$, namely ω_0 is not pure. Conversely, any convex decomposition of ω_0 yields a decomposition of its restriction to any $x \in \mathbb{Z}^2$, so that the purity of ω_0^x yields that of ω_0 . $\square$

Remark 3.7. Pure product states are the simplest states on $\mathcal{A}$ in that they have no entanglement between any two, disjoint sets of sites, and they are convenient representatives of the trivial phase. By analogy, in the noninteracting, single-particle picture, they correspond to ground states of tight-binding Hamiltonians where all lattice sites are disconnected – the atomic limit. Mathematically these correspond to Fermi projections P which are diagonal in the position basis $P_{xy} = P_{xy} \delta_{xy}$. Indeed, in that case the quasi-free state ω_P is product.

Definition 3.8 (SRE). A state $\omega \in \mathcal{S}(\mathcal{A})$ is *short-range entangled* (SRE) iff there exists a pure product state ω_0 and a LGA α such that $\omega = \omega_0 \circ \alpha$.

Note that a SRE state is necessarily pure, since automorphisms preserve purity.

Definition 3.9. A state $\omega \in \mathcal{S}(\mathcal{A})$ is called Q -symmetric iff $\omega \circ \rho_\phi = \omega$ for all $\phi \in \mathbb{R}$. We denote $\mathcal{S}_{\text{sym}}^Q(\mathcal{A})$ the set of Q -symmetric states.

We remark that, with Q defined as in (30), a Q -symmetric state is necessarily homogeneous, namely $\omega \circ \theta = \omega$ since $\theta = e^{i\pi Q}$.

Definition 3.10. Let ω be a pure state and H be a 0-chain. We say that ω is a ground state of H iff $\omega(A^* \delta^H(A)) \geq 0$ for all $A \in \mathcal{A}^{\text{loc}}$. Moreover we say that ω is a locally unique gapped ground state of H with gap $\Delta > 0$ if

$$\omega(A^* \delta^H(A)) \geq \Delta \omega(A^* A)$$

for all $A \in \mathcal{A}^{\text{loc}}$ such that $\omega(A) = 0$.

If ω is a pure state, we say that a 0-chain H is a *parent Hamiltonian* for ω if ω is a locally unique gapped ground state of H . An essential, although simple, result of this section is the following lemma, whose proof may be found in [10, Lemma 3.3].

Lemma 3.11. *Any Q -symmetric SRE state has a Q -preserving parent Hamiltonian.*

Remark 3.12. In the present work, a parent Hamiltonian does not need to have a physical interpretation, but it exists as a mathematical tool. This emphasizes the fact that we are classifying states, not Hamiltonians.

Last but not least, we introduce the relevant notion of ‘deformation’ of states, which is used to define the topological phases. We remind the reader of the overall symmetry assumption introduced above.

Definition 3.13. Two pure and Q -symmetric states ω_1, ω_2 are called Q -equivalent if there exists a Q -preserving LGA α on $\mathcal{A}$ such that $\omega_2 = \omega_1 \circ \alpha$. We shall denote this $\omega_1 \sim \omega_2$.

Remark 3.14. A symmetric SRE state is not necessarily Q -equivalent to a product state because we do not assume the LGA in the definition of SRE to be Q -preserving. The SRE condition encodes the locality of the state, regardless of the charge symmetry, whereas equivalence describes Q -preserving deformations of states. Symmetric SRE states which are not Q -equivalent to a product state are called symmetry-protected topologically (SPT) ordered [48].

3.4. Magnetic flux insertion at the origin

To probe the Hall conductivity associated with a state ω , we follow the Laughlin picture and focus on charge deficiency arising from a magnetic flux insertion in our system. We will show, see also [9, 33], that the Hall conductivity of ω is equal to $\mathcal{N}_\rho(\omega, \omega^D)$ where ω^D is obtained from ω by inserting one unit of magnetic flux.

Before we proceed with the technical construction, we make a few comments to motivate what is ahead. Following the work of [6], in the many-body setting we would like to create ω^D out of ω by analogy to how L^*PL is obtained out of the Fermi projection P , where the operator L is the Laughlin unitary given in (22). For quasi-free states, this lifting is done by considering the outer automorphism $\ell : \mathcal{A} \rightarrow \mathcal{A}$ via

$$a(f) \mapsto a(Lf) \quad (f \in \mathcal{H})$$

and extend linearly, and recalling that $\omega_P \circ \ell = \omega_{L^*PL}$. Hence, also for non-quasi-free-states, the natural choice is then $\omega^D := \omega \circ \ell$ with which one may hope to show that, under suitable assumptions on the initial state ω , the pair ω, ω^D is ρ -locally comparable and thus define the charge deficiency index of ω as $\mathcal{N}_\rho(\omega, \omega \circ \ell)$.

As proposed, this is impossible. Indeed, $P - L^*PL \in \mathcal{J}_3(\mathcal{H})$ but the analysis in Section 2.2 indicates that if $P - Q$ is not Hilbert-Schmidt then (ω_P, ω_Q) cannot be ρ -locally comparable. In fact we made the even stronger assumption $P - Q \in \mathcal{J}_1(\mathcal{H})$.

This is remedied by considering another gauge (the ‘half-line gauge’) for the same flux insertion procedure. For noninteracting Fermions, this was studied in detail in [24]. In the interacting setting, this was carried out in [10], following the closely related [33], where it is shown that the corresponding automorphism applied to an initial SRE state yields a unitarily equivalent state. This extends to so-called invertible states, see Definition 3.26 below. Interestingly, in general the unitary depends on the initial state (unlike the universal L), and we shall see that continuity properties of $\mathcal{N}(\omega, \omega^D)$ must be dealt with carefully. In particular, there is no inner automorphism that directly implements a half-line gauge.

Fundamentally, the implementability of the flux insertion procedure for invertible states is due to the triviality of their superselection sectors [15]. Beyond the invertible setting, one expects flux insertion to produce states that are truly inequivalent to the initial state in the sense that they carry anyonic quasi-particles [11].

For the sake of clarity, we consider the upper half-plane $\mathbb{P}^\uparrow := \mathbb{Z} \times \mathbb{N}$, but in principle one could consider any half-plane, see, for example, [10].

$U(1)$ -transformations on half-planes.

We consider the $U(1)$ -transformation $\rho^\uparrow := \rho^{\mathbb{P}^\uparrow}$ associated with the upper half-plane. Let H be a Q -preserving 0-chain. For $\phi \in \mathbb{R}$ we consider the 0-chain $H^\uparrow(\phi)$ defined by

$$\forall x \in \mathbb{Z}^2, \quad H^\uparrow(\phi)_x = \rho_{-\phi}^\uparrow(H_x),$$

Notice that $H^\uparrow(\phi)$ is Q -preserving for all ϕ .

Lemma 3.15. *Let ω be an SRE state and let H be a Q -preserving parent Hamiltonian. Then $H^\uparrow(\phi)$ is a Q -preserving parent Hamiltonian of $\omega_\phi^\uparrow = \omega \circ \rho_\phi^\uparrow$ for all $\phi \in \mathbb{R}$.*

Note that if the state is ‘rotated forward’ while the Hamiltonian is ‘rotated backwards’, see also the proof in [10, Section 3.1.3].

Quasi-adiabatic continuation

The fact that ω is invariant under $U(1)$ -transformation indicates that $\omega_\phi^\uparrow$ differs from ω only along the boundary of the upper half-plane. This intuition can be made precise by using the fact that ω has a gapped Hamiltonian: This is the role of the spectral flow [32, 16].

Proposition 3.16. *There exists a 0-chain $K = (K(\phi))_{\phi \in \mathbb{R}}$ such that*

$$\omega_\phi^\uparrow = \omega \circ \alpha_\phi^K. \quad (31)$$

The generator satisfies $K_x(\phi) = \theta(K_x(\phi)) = \rho_{\phi'}(K_x(\phi))$ for all $x \in \mathbb{Z}^2$ and $\phi, \phi' \in \mathbb{R}$. Moreover, there exists a function $g \in \mathcal{L}$ such that

$$\forall x = (x_1, x_2) \in \mathbb{Z}^2, \quad \forall \phi \in \mathbb{R}, \quad \|K_x(\phi)\| \leq g(|x_2|). \quad (32)$$

The proof is in [10, Section 3.1.3], but we recall here that

$$K_x(\phi) := \int_{\mathbb{R}} dt W(t) \alpha_t^{H^\uparrow(\phi)} \left(\frac{d}{d\phi} [H^\uparrow(\phi)_x] \right) \quad (33)$$

where $W : \mathbb{R} \rightarrow \mathbb{R}$ is an odd bounded function decaying faster than any power at $t \rightarrow \pm\infty$ and with Fourier transform $\hat{W}(E)$ equals $-i/E$ for $|E| > 1$. Notice that for a given $\phi \in \mathbb{R}$, $\alpha_t^{H^\uparrow(\phi)}$ is the LGA generated by the t -independent generator $H^\uparrow(\phi)$.

This proposition is essential: equation (31) states that $U(1)$ -transformations on half-spaces can be implemented on the state ω by another LGA, which is nonautonomous but acts nontrivially only along the boundary of the upper half-space, see (32).

Definition 3.17 (Magnetic flux insertion). Let K be as in (33). For $\phi \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{Z}^2$ we define

$$D_x(\phi) = K_x(\phi) \mathbb{1}_{\{x_1 < 0\}} := \begin{cases} K_x(\phi), & x_1 < 0, \\ 0, & \text{otherwise.} \end{cases}$$

The associated LGA, which we denote $\gamma_\phi = \alpha_\phi^D$, is called the flux insertion automorphism.

Remark 3.18. By construction $D(\phi)$ is g -localized near the half-line $\{x_1 < 0, x_2 = 0\}$. It corresponds to a defect line (or branch cut) of vector potential associated with a ϕ -flux insertion at the origin. The dynamics γ_ϕ is a caricature of the adiabatic switching of a magnetic flux from 0 to ϕ .

Definition 3.19 (Defect state). Let ω be an SRE state. The state $\omega_\phi^D := \omega \circ \gamma_\phi$ is called the *defect state* at flux ϕ . At $\phi = 2\pi$, we denote $\omega^D = \omega_{2\pi}^D$.

We recall that the construction from the symmetric SRE state ω to the defect state ω^D is summarized in Figure 1.

Remark 3.20. Since K is even and Q -preserving, so is D , and so $\gamma_\phi \circ \rho_{\phi'} = \rho_{\phi'} \circ \gamma_\phi$ as well as $\gamma_\phi \circ \theta = \theta \circ \gamma_\phi$. This implies immediately that ω_ϕ^D is Q -symmetric.

The two states ω, ω_ϕ^D differ from each other along the half-line $\{x_1 < 0, x_2 = 0\}$. At $\phi = 2\pi$, the intrinsic locality of the state, namely the SRE assumption, implies that, in fact, ω, ω^D differ from each other only around the flux insertion point. Indeed, far along the half-line, $\omega^D \simeq \omega \circ \rho_{2\pi} = \omega$ because ρ is a $U(1)$ -transformation. This implies that (ω, ω^D) are locally comparable. In fact, they are even ρ -locally comparable because the unitary can be shown to be almost localized near the origin, see Definition 2.1 and Definition 2.3. This is the content of the following proposition.

Proposition 3.21. Let $\omega \in \mathcal{S}_{\text{sym}}^Q(\mathcal{A})$ be SRE and let ω^D be the associated defect state with flux 2π . Then there exists a homogeneous unitary $u \in \mathcal{A}^{\text{al}}$ such that $\omega^D = \omega \circ \text{Ad}_u$. Moreover, $u \in \mathcal{D}(\delta^Q)$.

The proof of the existence of u with all claimed properties except for the last one can be found in [10, Proposition 3.8], with the replacements of $\omega_{-\pi}$, respectively ω_π , there by ω , respectively ω^D , here. See also [3, Appendix A]. Finally, $u \in \mathcal{A}^{\text{al}}$ implies that it lies in the domain of δ^Q , see Proposition 3.1.

Remark 3.22. (i) u is the many-body analog of the Laughlin unitary discussed above. It exists only for $\phi = 2\pi$ (and could also be constructed for other integer fluxes). (ii) If ω is a symmetric product state, then one can take $u = 1$. (iii) The fact that u implements the Q -preserving transformation $\gamma_{2\pi}$ on the symmetric state ω does not imply that $\delta^Q(u) = 0$. In Example 2.13, the bilateral shift reads in second quantization $\alpha^R(a_x) = a_{x+1}$ and for all $x \in \mathbb{Z}$, while $\rho_\phi(a_x) = e^{-i\phi} a_x$. Clearly, we have that $\alpha^R \circ \rho_\phi = \rho_\phi \circ \alpha^R$. However, with $u = a_0 + a_0^*$ given there, we see that

$$-i\delta^Q(u) = -i \frac{d}{d\phi} \rho_\phi(u) \Big|_{\phi=0} = -a_0 + a_0^*.$$

Proposition 3.23. Let $\omega \in \mathcal{S}_{\text{sym}}^Q(\mathcal{A})$ be SRE and let $u \in \mathcal{A}^{\text{al}}$ be the unitary given by Proposition 3.21. Let

$$i(\omega, Q) := i\omega(u^* \delta^Q(u)) \quad (34)$$

If β is a Q -preserving LGA then $i(\omega, Q) = i(\omega \circ \beta, Q)$. Moreover, $i(\omega, Q)$ is independent of the choice of symmetric parent Hamiltonian H for ω .

Proof. The definition (34) is nothing else than the abstract index (3) applied in the present specific context, namely

$$i(\omega, Q) = \mathcal{N}_\rho(\omega, \omega^D).$$

An LGA deformation of the initial state $\omega(s) = \omega \circ \beta_s$ yields a deformation of the flux insertion automorphism γ_s and hence a family of states $\omega^D(s) = \omega(s) \circ \gamma_s$. If ω is a symmetric SRE state, so is $\omega(s)$ and so there is a family u_s such that $\omega^D(s) = \omega(s) \circ \text{Ad}_{u_s}$. In particular, $(\omega^D(s), \omega(s))$ are ρ -locally comparable for all s . Note that, in general, ω and $\omega(s)$ (as well as their defect counterparts) are not locally comparable, even for infinitesimal s . However, the state

$$\tilde{\omega}(s) = \omega \circ \beta_s \circ \text{Ad}_{u_s} \circ \beta_s^{-1}$$

can be expressed as $\tilde{\omega}(s) = \omega \circ \text{Ad}_{\beta_s(u_s)}$, so that $(\tilde{\omega}(s), \omega)$ are ρ -locally comparable for all s and $i(\omega, Q) = \mathcal{N}_\rho(\omega, \tilde{\omega}(0))$. The family of normalized vectors $\Omega_{\tilde{\omega}(s)} = \pi_\omega(\beta_s(u_s))\Omega_\omega$ are vector representatives of the state $\tilde{\omega}(s)$ in the GNS Hilbert space $\mathcal{H}_\omega$ of ω . If we again let Q_ω be the charge

operator in the GNS representation, then $\Omega_{\tilde{\omega}(s)}$ are all eigenvectors of Q_ω . We claim that $s \mapsto \Omega_{\tilde{\omega}(s)}$ is a norm-continuous family of vectors in $\mathcal{H}_\omega$. Since orthogonal unit vectors Ψ, Φ have $\|\Psi - \Phi\|^2 = 2$, we conclude that $\Omega_{\tilde{\omega}(s)}$ are in a constant eigenspace. This in turn means that $\mathcal{N}_\rho(\omega, \tilde{\omega}(s))$ is constant. We conclude that

$$i(\omega, Q) = \mathcal{N}_\rho(\omega, \tilde{\omega}(0)) = \mathcal{N}_\rho(\omega, \tilde{\omega}(s)) = \mathcal{N}_\rho(\omega(s), \omega^D(s))$$

where the last equality is by Theorem 2.5(iv).

It remains to prove the continuity of $\Omega_{\tilde{\omega}(s)}$. We first claim that $s \mapsto \gamma_s$ is strongly continuous. If $\omega = \omega_0 \circ \alpha$, the family $\omega(s)$ is also SRE, with entangler $\beta_s \circ \alpha$. Recalling the summary ??, we see that the parent Hamiltonian H_s is simply given by $H_{s,x} = \beta_s^{-1}(H_x)$ and the interaction terms are continuous. Also, the family H_s has constant spectral gap. It is now a consequence of the Lieb-Robinson bound, see, for example, [20, Theorem 6.2.11], that the dynamics $\alpha_t^{H_s^\dagger(\phi)}$ is strongly continuous with respect to $H^\dagger$. In particular, the interaction terms defined by

$$K_{s,x}(\phi) := \int_{\mathbb{R}} dt W(t) \alpha_t^{H_s^\dagger(\phi)} \left(\frac{d}{d\phi} [H^\dagger(\phi)_{s,x}] \right)$$

are continuous as a function of s . So is restriction D_s of the 0-chain K_s to the half-line, and the claimed strong continuity of $s \mapsto \gamma_{s,\phi}$ follows again by the Lieb-Robinson bound. With this, the formula

$$\tilde{\omega}_2(s) = \omega_1 \circ \beta_s \circ \gamma_{s,2\pi} \circ \beta_s^{-1}$$

implies immediately the weak-* continuity of $s \mapsto \tilde{\omega}_2(s)$. Hence $s \mapsto \Omega_{\tilde{\omega}_2(s)}$ is weakly continuous (in $\mathcal{H}_\omega$) and since this is a family of constant norm, it is in fact norm continuous, concluding that part of the proof.

We turn to the invariance under the choice of parent Hamiltonian. For all $\phi \in [0, 2\pi]$, let $\omega_\phi^{D,j} = \omega \circ \gamma_\phi^j$, $j = 1, 2$ be the defect states generated by the Hamiltonians H_j . We claim that, for all $A \in \mathcal{A}$,

$$|\omega_0(A) - \tilde{\omega}_\phi^D(A)| \leq f(r) \|A\| \quad (35)$$

where $\tilde{\omega}_\phi^D = \omega_0 \circ \alpha \circ \gamma_\phi^1 \circ (\gamma_\phi^2)^{-1} \circ \alpha^{-1}$. Indeed, by the locality of α , we can replace $\gamma_\phi^1 \circ (\gamma_\phi^2)^{-1}$ by $\alpha_\phi^{K_1} \circ (\alpha_\phi^{K_2})^{-1}$ for all $\phi \in [0, 2\pi]$ and all A supported in $(\Lambda \cap B_0(r)) \cup (\Lambda^c \cap B_0(r))$, where Λ is the left half-plane. This is because, far away from the origin, the effect of the truncation $K \rightarrow D$ is irrelevant in the left half-plane and similarly on the right half-plane simply because both γ_ϕ^D and α_ϕ^K act as identity, see [10, Lemma 3.7]. This, and the identity

$$\omega \circ \alpha_\phi^{K_1} \circ (\alpha_\phi^{K_2})^{-1} = \omega_\phi^\uparrow \circ (\alpha_\phi^{K_2})^{-1} = \omega,$$

implies (35) for all $\phi \in [0, 2\pi]$ and all A supported in $(\Lambda \cap B_0(r)) \cup (\Lambda^c \cap B_0(r))$. That this can be lifted to all $A \in \mathcal{A}$ follows from [10, Appendix A], and in turn there is a unitary w_ϕ such that

$$\tilde{\omega}_\phi^D = \omega_0 \circ \text{Ad}_{w_\phi}.$$

The family of Q -symmetric states $\tilde{\omega}_\phi^D$ is weakly-* continuous and so

$$\mathcal{N}_\rho(\omega_0, \tilde{\omega}_\phi^D) = \mathcal{N}_\rho(\omega_0, \omega_0) = 0$$

by the continuity argument in the first part of the proof, with $\omega \rightarrow \omega_0$ and $\tilde{\omega}(s) \rightarrow \tilde{\omega}_\phi^D$, and Theorem 2.5(v). Evaluating this at $\phi = 2\pi$, we conclude by Theorem 2.5(iv) that

$$0 = \mathcal{N}_\rho(\omega_0, \tilde{\omega}_{2\pi}^D) = \mathcal{N}_\rho(\omega^{D,2}, \omega^{D,1}),$$

and by additivity

$$\mathcal{N}_\rho(\omega, \omega^{D,1}) = \mathcal{N}_\rho(\omega, \omega^{D,2}) + \mathcal{N}_\rho(\omega^{D,2}, \omega^{D,1}) = \mathcal{N}_\rho(\omega, \omega^{D,2})$$

which is what we had set to prove. $\square$

Remark 3.24. If ω is a symmetric product state, then the Q -preserving parent Hamiltonian of Lemma 3.11 can be chosen to be purely on-site. Since the $U(1)$ -transformation is also on-site, the Hamiltonian is invariant under $\rho^\uparrow$. Hence the quasi-adiabatic generator K of (33) is uniformly equal to 0. It follows that $\omega^D = \omega$, so that u can be taken to be the identity and we conclude that $i(\omega, Q) = 0$ for an initial product state, as it should.

As we shall see in Section 4 below, the problem with Proposition 3.23 is that $i(\omega, Q) = 0$ for any quasi-free SRE state ω describing the (noninteracting) quantum Hall effect. In the next section, we extend the definition of the index to the larger class of invertible states which covers all topological ground states of noninteracting models. In principle, other physical models might provide examples of many-body SRE states with a nontrivial index, without stacking with an auxiliary system, but we are not aware of them.

3.5. Many-body index

So far, we have shown how to associate an index to an SRE state in the presence of a $U(1)$ -symmetry. The SRE assumption is used in two distinct steps: first to ensure that the state has a gapped parent Hamiltonian, and second to show that the associated defect state is α -locally comparable with the initial state. In both cases, the SRE assumption is a sufficient condition but in no way necessary. In this section, we extend the index to invertible states.

Stacking.

In the following we shall make use of *stacking* which is a procedure to increase the number of internal degrees of freedom. For this, let $\mathcal{A}^1$ and $\mathcal{A}^2$ respectively be the CAR algebras over $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ and $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^m$ for $n, m \in \mathbb{N}$. We denote by $\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2$ the CAR algebra over $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{n+m}$.

Remark 3.25. A word about $\hat{\otimes}$: The standard construction of the tensor product $\mathcal{A}^1 \otimes \mathcal{A}^2$ is also a CAR algebra, but to see that one has to reshuffle its generators. Indeed, consider the simple example of space consisting of a single point and $n = m = 1$. Then $a, \tilde{a}$, the two generators of $\mathcal{A}^1, \mathcal{A}^2 \cong \text{Mat}_2(\mathbb{C})$ respectively, embed naturally into $\mathcal{A}^1 \otimes \mathcal{A}^2$ as $a \otimes \mathbb{1}, \mathbb{1} \otimes \tilde{a}$ respectively, which commute rather than anti-commute. A choice that realizes the CAR is $a_1 = a \otimes \mathbb{1}$ and $a_2 = (-1)^{a^*a} \otimes \tilde{a}$. We use the symbol $\hat{\otimes}$ to denote this reshuffling of the generators (though the underlying C^* -algebraic structure is, of course, identical).

The stacking operation on algebras naturally extends to states: for $\omega_1 \in \mathcal{S}(\mathcal{A}^1)$ and $\omega_2 \in \mathcal{S}(\mathcal{A}^2)$ we define $\omega_1 \hat{\otimes} \omega_2 \in \mathcal{S}(\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2)$ by

$$(\omega_1 \hat{\otimes} \omega_2)(A_1 \hat{\otimes} A_2) := \omega_1(A_1)\omega_2(A_2) \quad (A_1 \in \mathcal{A}^1, A_2 \in \mathcal{A}^2)$$

and extended beyond simple tensors by linearity.

Definition 3.26 (Invertible and stably SRE). A state $\omega \in \mathcal{S}(\mathcal{A})$ is *invertible* iff there is an auxiliary system $\mathcal{A}'$ (namely a CAR algebra over $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^m$ for some $m \in \mathbb{N}$) and a state $\omega' \in \mathcal{S}(\mathcal{A}')$ such that $\omega \hat{\otimes} \omega'$ is SRE on $\mathcal{A} \hat{\otimes} \mathcal{A}'$. If ω' can be taken to be a product state, then ω is called *stably SRE*.

Notice that a stably SRE state is necessarily invertible. The converse is true in one dimension [35]. That these two classes are not equal in two dimensions is a corollary of the results presented below, see Section 4.

Remark 3.27. (i) The relevance of invertible states lies in the observation that they correspond to integer quantum Hall systems. Any quasi-free state is invertible and so all noninteracting models have invertible ground states. However, it is believed that a nonvanishing Hall conductance is an obstruction to being stably SRE. Moreover, as Theorem 3.31 below shows, they do not cover fractional quantum Hall states. In fact, it can be shown that invertibility guarantees the absence of anyons, see [15].

(ii) The stacking with product states, which appears in the notion of stably SRE, is analogous to the adding of some trivial bands in the single-particle setting. It also parallels a common procedure in K-theory.

Let ω be a state on $\mathcal{A}$ which is invertible, with inverse ω' on auxiliary algebra $\mathcal{A}'$. We denote by $\hat{\omega} = \omega \hat{\otimes} \omega'$ and $\hat{\mathcal{A}} = \mathcal{A} \hat{\otimes} \mathcal{A}'$. The algebra $\mathcal{A}'$ is equipped with a charge Q as well, so we are left with 3 distinct charges on $\hat{\mathcal{A}}$:

$$Q := Q \hat{\otimes} 1, \quad Q' = 1 \hat{\otimes} Q, \quad \hat{Q} = Q + Q'. \quad (36)$$

Notice that these three charges commute.

Definition 3.28. We say that ω is symmetric invertible if it is invertible with inverse ω' and if $\hat{\omega} = \omega \hat{\otimes} \omega'$ is both Q and Q' -symmetric.

In particular if ω is symmetric invertible then $\hat{\omega}$ is also $\hat{Q}$ -symmetric. The parent Hamiltonian $\hat{H}$ provided by Lemma 3.11 is $\hat{Q}$ -symmetric. Defining then

$$\tilde{H}_x = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \rho_{\phi_2}^{Q'} \circ \rho_{\phi_1}^Q (\hat{H}_x) d\phi_1 d\phi_2$$

for all $x \in \mathbb{Z}^2$ yields a parent Hamiltonian that is both Q - and Q' -preserving – a fortiori still $\hat{Q}$ -preserving. Note that the commutativity of $Q, Q', \hat{Q}$ is essential here.

With this, we can carry out the construction described in Section 3.4 and define the flux insertion automorphism $\hat{\gamma}_\phi$ and a defect state $\hat{\omega}_\phi^D := \hat{\omega} \circ \hat{\gamma}_\phi$, with respect to the total charge $\hat{Q}$. By inspecting the formula (33), we conclude that $\hat{\gamma}_\phi$ is invariant under both Q, Q' , and so the defect state $\hat{\omega}_\phi^D$ is $Q, Q', \hat{Q}$ -symmetric. Hence we have the following

Proposition 3.29. A symmetric invertible state $\hat{\omega}$ has a symmetric parent Hamiltonian which is both Q - and Q' -symmetric. Moreover the defect state $\hat{\omega}^D$, generated by $\hat{Q}$, remains Q - and Q' -symmetric.

Continuing as in Section 3.4, Proposition 3.21 yields $\hat{u} \in \hat{\mathcal{A}}^{\text{al}}$ such that $\hat{\omega}^D = \hat{\omega} \circ \text{Ad}_{\hat{u}}$. Since $\hat{u} \in \hat{\mathcal{A}}^{\text{al}}$, we further have that $\hat{u} \in \mathcal{D}(\delta^Q)$, namely $(\hat{\omega}^D, \hat{\omega})$ are ρ^Q -locally comparable.

Definition 3.30. Let ω be an invertible state as above. The index of ω is defined by

$$\mathcal{I}(\omega) := \mathcal{N}_{\rho^Q \hat{\otimes} \text{id}}(\hat{\omega}, \hat{\omega}^D) = i \hat{\omega} \left(\hat{u}^* \delta^Q \hat{\otimes} 1(\hat{u}) \right) \quad (37)$$

We display the second expression to emphasize that the charge used in this definition is on the first subsystem $\mathcal{A}$, on which ω is defined, although the flux insertion was carried out using the full charge $\hat{Q}$.

With these preparations, we can now state our main result.

Theorem 3.31. Let ω be a symmetric invertible state. The index $\mathcal{I}(\omega)$ of Definition 3.30 has the following properties:

- (i) *Integrality:* $\mathcal{I}(\omega) \in \mathbb{Z}$.
- (ii) *Well-definedness:* $\mathcal{I}(\omega)$ is independent from the choice of symmetric parent Hamiltonian for $\hat{\omega}$.

- (iii) *Continuity*: If $\tilde{\omega}$ is Q -equivalent to ω , then $\tilde{\omega}$ is symmetric invertible and $\mathcal{I}(\omega) = \mathcal{I}(\tilde{\omega})$.
- (iv) *Additivity*: If ω_1 and ω_2 are symmetric invertible states on $\mathcal{A}^1$ and $\mathcal{A}^2$ then $\omega_1 \hat{\otimes} \omega_2$ is a symmetric invertible state on $\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2$ and $\mathcal{I}(\omega_1 \hat{\otimes} \omega_2) = \mathcal{I}(\omega_1) + \mathcal{I}(\omega_2)$.
- (v) *Surjectivity*: For all $n \in \mathbb{Z}$, there exists an invertible state ω such that $\mathcal{I}(\omega) = n$.

Remark 3.32. (i) If the stacked state $\omega \hat{\otimes} \omega'$ is not only SRE, but it is $\hat{Q}$ -equivalent to a pure product state, then

$$\mathcal{I}(\omega') = -\mathcal{I}(\omega)$$

by (iii) and (iv) of Theorem 3.31.

(ii) At this point, the injectivity of the index remains an open question, namely whether two invertible states with the same index are necessarily Q -equivalent.

Proof. Since

$$\mathcal{I}(\omega) = \mathcal{N}_{\rho_{Q \hat{\otimes} \mathbb{1}}}(\hat{\omega}, \hat{\omega}^D) = i(\hat{\omega}, Q \hat{\otimes} \mathbb{1}),$$

the theorem is a direct consequence of Theorem 2.5 for the abstract index and Proposition 3.23 on $\hat{\mathcal{A}}$. We check that the assumptions hold in the context of invertible states.

(i) The states $\hat{\omega}$ and $\hat{\omega}^D = \hat{\omega} \circ \text{Ad}_{\hat{Q}}$ are $\hat{\rho}$ -locally comparable since $\hat{U} \in \hat{\mathcal{A}}^{\text{al}}$ and $Q = Q \hat{\otimes} \mathbb{1}$ is a 0-chain of $\hat{\mathcal{A}}$. The state $\hat{\omega} = \omega \hat{\otimes} \omega'$ is Q -symmetric since ω is symmetric invertible. Moreover, $\hat{\omega}^D$ is Q -symmetric as well by Proposition 3.29. Thus $\mathcal{I}(\omega) = \mathcal{N}_{\rho_Q}(\hat{\omega}, \hat{\omega}^D) \in \mathbb{Z}$ by Theorem 2.5(ii).

(ii) This follows as in the proof of Proposition 3.23

(iii) Let $\tilde{\omega}$ be Q -equivalent to ω . Let β be a Q -preserving LGA on $\mathcal{A}$ such that $\tilde{\omega} = \omega \circ \beta$. Since ω is invertible, let ω' be its inverse. One has $\omega \hat{\otimes} \omega' = \hat{\omega}_0 \circ \hat{\alpha}$ with an $LGA \hat{\alpha}$ and a pure product state $\hat{\omega}_0$ on $\hat{\mathcal{A}}$. Thus we have

$$\tilde{\omega} \hat{\otimes} \omega' = \hat{\omega}_0 \circ \hat{\alpha} \circ (\beta \hat{\otimes} \mathbb{1})$$

Thus $\tilde{\omega}$ is invertible. Moreover it is Q -symmetric and since $\hat{\omega}$ is symmetric invertible then ω' is Q' -symmetric. Thus $\tilde{\omega}$ is symmetric invertible, so that $\mathcal{I}(\tilde{\omega})$ is well defined. The invariance of the index $\mathcal{I}(\tilde{\omega}) = \mathcal{I}(\omega)$ follows from Theorem 2.5(iv) on $\hat{\mathcal{A}}$ and goes along the same lines of the proof of Proposition 3.23 but on $\hat{\mathcal{A}}$.

(iv) If ω_1 and ω_2 are symmetric invertible with inverses ω'_1 and ω'_2 , then $\omega_1 \hat{\otimes} \omega_2$ is symmetric invertible on $\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2$ with inverse $\omega'_1 \hat{\otimes} \omega'_2$. The defect state construction factors on each algebra. In particular, up to the reshuffling

$$\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2 \hat{\otimes} (\mathcal{A}^1)' \hat{\otimes} (\mathcal{A}^2)' \cong \mathcal{A}^1 \hat{\otimes} (\mathcal{A}^1)' \hat{\otimes} \mathcal{A}^2 \hat{\otimes} (\mathcal{A}^2)' = \hat{\mathcal{A}}^1 \hat{\otimes} \hat{\mathcal{A}}^2,$$

one has $(\hat{\omega}_1 \hat{\otimes} \hat{\omega}_2)^D = (\hat{\omega}_1)^D \hat{\otimes} (\hat{\omega}_2)^D$ so that, by Theorem 2.5(viii),

$$\begin{aligned} \mathcal{I}(\omega_1 \hat{\otimes} \omega_2) &= \mathcal{N}_{\hat{\rho}_{Q_1} \hat{\otimes} \hat{\rho}_{Q_2}}(\omega_1 \hat{\otimes} \omega_2, (\hat{\omega}_1)^D \hat{\otimes} (\hat{\omega}_2)^D) \\ &= \mathcal{N}_{\hat{\rho}_{Q_1}}(\omega_1, (\hat{\omega}_1)^D) + \mathcal{N}_{\hat{\rho}_{Q_2}}(\omega_2, (\hat{\omega}_2)^D) \\ &= \mathcal{I}(\omega_1) + \mathcal{I}(\omega_2). \end{aligned}$$

(v) This part is an immediate consequence of the quasi-free case, see Theorem 4.3 and Proposition 4.4 below. $\square$

Remark 3.33. One may wonder if the abstract index could also find some relevance for the fractional quantum Hall effect. We believe that it may be the case in following situation, which is similar to the quantization argument in [11]: Even if the initial state is not invertible, it may be that the flux insertion operation becomes inner after q units of quantum flux have been inserted. If this is the case,

the arguments above yield a fractional Hall conductance of the form $\frac{n}{q}$. It should be noted that in such a framework there is no prediction about the possible fractions one may obtain, see [27].

4. Quasi-free fermions and invertibility

We now return to the discussion at the beginning of Section 3.4 and we turn to the relation between the index of a pair of projections from Theorem 2.12 and the many-body index for the quantum Hall effect from Theorem 3.31, when applied to quasi-free states. We shall show that the two indices coincide when they are simultaneously well-defined.

Let $\mathcal{H} = \ell^2(\mathbb{Z}^2)$ (for simplicity, we ignore spin here) and let P be the Fermi projection associated to some gapped local Hamiltonian H . Typically, we assume

$$|H_{x,y}| := |\langle \delta_x, H \delta_y \rangle| \leq C e^{-\mu|x-y|}, \quad (38)$$

and the kernel of P has the same property whenever the Fermi energy lies in a gap. Consider the quasi-free state ω_P on $\mathcal{A} = \text{CAR}(\mathcal{H})$, see (20) above. We first recall

Proposition 4.1 [10, Proposition 2.5]. *Assume H is translation-invariant on $\mathcal{H}$ and consider the associated Chern number $c(P)$ for the Fermi projection. If $c(P) = 0$ then ω_P is stably SRE.*

Note that the assumption there is about time-reversal invariance, but the argument (in [10, Appendix B]) goes through the consequence thereof, which is that the Chern number vanishes.

Conversely, if a quasi-free state is stably SRE and the deformation is a family of quasi-free states, then the smooth path of states yields a smooth deformation of the bundle to a trivial bundle and so the Chern number must vanish.

Translation invariance is not expected to be relevant for this question. Indeed, [23] proves that if P and Q have the same Chern number and both satisfy a certain nontriviality assumption ([23, Definition 1.2]), then there exists a path of local unitaries $[0, 1] \ni t \mapsto U_t$ with $U_0 = \mathbb{1}$ such that $Q = U_1^* P U_1$, that is, P and Q are homotopic within the space of local projections.

Now suppose P has zero Chern number. By the same result, we can connect P to a projection Q that is diagonal in space (and hence also has zero Chern number). By lifting $t \mapsto U_t$ to an LGA, this yields a continuous path from ω_P to ω_Q (a product state), and therefore shows that ω_P is an SRE state.

The problem with this argument is that the notion of locality used in [23] is a far cry from (38): there, P is local iff $[P, L] \in \mathcal{K}$, where L is the Laughlin unitary $L \equiv \exp(i \arg(X_1 + i X_2))$. Bridging this gap would require formulating an analogous notion of locality in the many-body setting (and proving a Lieb–Robinson bound for it). Since this line of study is rather tangential to the present paper we do not pursue it further here and instead merely phrase a

Conjecture 4.2. *Let H be a gapped Hamiltonian obeying (38) for which the Fermi projection P has a well-defined topological index $c(P)$. Then $c(P) = 0$ if and only if ω_P is stably SRE.*

Note that the ‘topological index’ $c(P)$ is, strictly speaking, a ‘Chern number’ only in the periodic setting, but we continue to use the same symbol for it.

We return to the notion of invertibility in the quasi-free setting. Let $\mathcal{H} \hat{\oplus} \mathcal{H}$ be the external direct sum and consider the projection $P \hat{\oplus} P^\perp$, where $P^\perp = \mathbb{1} - P$. One then has $c(P \hat{\oplus} P^\perp) = c(P) + c(P^\perp) = 0$. It follows that, in the translation-invariant setting, the state $\omega_{P \hat{\oplus} P^\perp} = \omega_P \hat{\otimes} \omega_{P^\perp}$ is stably-SRE. Consequently, the quasi-free state ω_P is invertible. In particular, we can associate to it the many-body index $\mathcal{I}(\omega_P)$.

We use the independence of the index on the parent Hamiltonian provided by Theorem 3.31(ii) to choose $H \hat{\oplus} (-H)$ as parent Hamiltonian to $P \hat{\oplus} P^\perp$. Note that this direct sum structure is not what would arise from the general construction presented in Lemma 3.11 because the automorphism mapping $\omega_{P \hat{\oplus} P^\perp}$ to a product state would in general mix the two layers. Now if the parent Hamiltonian is a direct sum, then the flux insertion unitary similarly factorizes as $U_{\phi_1, \phi_2}^{\text{qa}} \hat{\otimes} U_{\phi_1, \phi_2}^{\text{qa}}$. Here $U_{\phi_1, \phi_2}^{\text{qa}}$ is the

flux-insertion unitary given by

$$U_{\phi_1, \phi_1}^{\text{qa}} = \mathbb{1}, \quad i \partial_{\phi_2} U_{\phi_2, \phi_1}^{\text{qa}} = K_{\phi_2} U_{\phi_2, \phi_1}^{\text{qa}},$$

where $K_\phi = \chi_{\text{left}} K_\phi^\uparrow \chi_{\text{left}}$ and $K_\phi^\uparrow = -i[\partial P_\phi^\uparrow, P_\phi^\uparrow]$ is the Kato generator of the family of projections $P_\phi^\uparrow = e^{i\phi\chi_\uparrow} P e^{-i\phi\chi_\uparrow}$. $\chi_\uparrow$ and χ_{left} are the respective projections to the upper and left half-planes. See [10, Sec 5.1] for more details.

As a result, the defect state is again a quasi-free state, corresponding to a projection $P_{2\pi}^{\text{qa}} \hat{\oplus} (P_{2\pi}^{\text{qa}})^\perp$, where $P_{2\pi}^{\text{qa}} = U_{0, 2\pi}^{\text{qa}} P (U_{0, 2\pi}^{\text{qa}})^*$. With this, the construction in the proof of Theorem 2.12 completely factorizes. However, in the final step of the proof when one computes the index, the charge automorphism ρ must be taken with $Q \hat{\otimes} \mathbb{1}$ rather than $Q \hat{\otimes} \mathbb{1} + \mathbb{1} \hat{\otimes} Q$, see the definition in (37) and the discussion before it. Hence δ^ρ acts trivially on the second factor and we conclude

Theorem 4.3. *If the difference $P - P_{2\pi}^{\text{qa}}$ is trace class then*

$$\mathcal{I}(\omega_P) = \text{index}(P, P_{2\pi}^{\text{qa}}).$$

One may wonder when $P - P_{2\pi}^{\text{qa}}$ is trace class, so that the index on the right-hand side is well defined. A typical case is given by a local Hamiltonian as described above with Fermi level in a spectral gap¹ and Proposition 4.4 below. Consider then the instantaneous Fermi projection $P_\phi^\mu = \chi_{(-\infty, \mu]}(H_\phi)$ for the Hamiltonians H_ϕ defined by

$$H_\phi(x, i; y, j) = \begin{cases} e^{i\phi \text{sgn}(x_2 - y_2)} H(x, i; y, j) & \text{if } x_1, y_1 \leq 0 \\ H(x, i; y, j) & \text{otherwise} \end{cases} \quad (39)$$

Notice that, by construction, $H_{2\pi} = H_0$ so that $H_{2\pi}$ is gapped around μ as well, and moreover $P_{2\pi}^\mu = P_0^\mu = P$.

Proposition 4.4 [10, Proof of Prop. 5.1]. *Let μ be in the spectral gap of H_0 and such that all eigenvalue crossings of $\phi \mapsto H_\phi$ across μ are simple. Then for all ϕ the difference $P_\phi^{\text{qa}} - P_\phi^\mu$ is trace class and*

$$\text{SF}_\mu([0, 2\pi] \ni \phi \mapsto H_\phi) = \text{Tr}(P_{2\pi}^{\text{qa}} - P_{2\pi}^\mu).$$

We recall that the spectral flow counts the signed number of eigenvalue crossings with the fiducial line at μ . See for example [24] for a detailed study of spectral flow in the single-particle and discrete setting for Quantum Hall effect.

Summarizing, under the hypothesis of Proposition 4.4, one has various single-particle interpretations of the index,

$$\mathcal{I}(\omega_P) = \text{index}(P_{2\pi}^{\text{qa}}, P_{2\pi}^\mu) = \text{Tr}(P_{2\pi}^{\text{qa}} - P_{2\pi}^\mu) = \text{SF}_\mu([0, 2\pi] \ni \phi \mapsto h_\phi)$$

which ultimately appears as a many-body version of spectral flow induced by magnetic flux insertion.

4.1. Beyond quasi-free states

We now have a complete picture of the relationship between the many-body index of a pair of pure states, the index of a pair of projections, the spectral flow associated with flux insertion and the spectral flow in a noninteracting picture. We point out that the stability of the many-body index and many-body perturbation theory allow one to go beyond the free setting. The proposition below is phrased

¹Note that if we assume that $|P_{x,y}|$ is exponentially decaying in $|x - y|$, then $-P \hat{\oplus} P$ is a bonafide parent Hamiltonian of $P \hat{\oplus} P^\perp$.

in a translation-invariant setting to be able to use Proposition 4.1. This assumption can be dropped if Conjecture 4.2 is true.

Proposition 4.5. *Let $H_\lambda = H_0 + \lambda V$ where H and V are two Q -preserving 0-chains in $\mathcal{A}$. Assume that for all $\lambda \in [0, 1]$, H_λ has a locally unique gapped ground state ω_λ , and that ω_0 is a quasi-free translation-invariant state. Then ω_1 is invertible and $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_0)$.*

Note that a locally unique gapped ground state is necessarily pure, see [45, Theorem A.3].

Proof. Since H_λ has a uniform gapped ground state then the spectral flow [16], generated by a 0-chain K_λ , is such that $\omega_\lambda = \omega_0 \circ \alpha_{0 \rightarrow \lambda}^K$. Then since ω_0 is quasi-free then it admits an inverse ω'_0 on $\mathcal{A}'$. Now,

$$(\omega_1 \hat{\otimes} \omega'_0) \circ ((\alpha_{0 \rightarrow \lambda}^K)^{-1} \hat{\otimes} \mathbb{1}) = \omega_0 \hat{\otimes} \omega'_0,$$

and since $\omega_0 \hat{\otimes} \omega'_0$ is stably SRE, this shows that ω_1 is invertible. Finally, since H_λ is Q -preserving, then α^K is Q -preserving. Thus ω_1 is Q -equivalent to ω_0 and so $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_0)$ by Theorem 3.31(iii). $\square$

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