Eigenvalues of elliptic operators

Comparing apples and oranges

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Proof idea: find functions $u \in \mathrm{dom}(\sqrt{L_{\mathrm{N}}})$ solving $Lu = \lambda u$.

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- S: if Ω and ρ convex, then $\quad \tilde\mu_d \le \tilde\lambda_1$.

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Corollary

If Ω is simply connected and $\Delta \frac{1}{\rho} \leq 0$, then $\tilde{\mu}_3 \leq \lambda_1 \leq \tilde{\lambda}_1$.

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Proof: If $-\frac{1}{\rho}\Delta u = \tilde\lambda u$ and $u|_{\partial\Omega} = 0$, then $G = \nabla^\perp u$ satisfies $G \cdot \nu = \partial_\tau u = 0$, $\operatorname{curl} G|_{\partial\Omega} = \operatorname{curl} \nabla^\perp u|_{\partial\Omega} = -\tilde\lambda u|_{\partial\Omega} = 0$ and

$$MF = -(\nabla \tfrac{1}{\rho} \operatorname{div} \nabla^\perp u + \nabla^\perp \tfrac{1}{\rho} \operatorname{curl} \nabla^\perp u) = -(0 + \nabla^\perp(-\tilde\lambda u)) = \tilde\lambda G\,.$$

So $\{\tilde\mu_n : n \geq 2\} \cup \{\tilde\lambda_n : n \geq 1\} \subseteq \sigma_p(M)$.

Reverse inclusion follows from $\mathbb{L}^2 = \nabla H^1(\Omega) \oplus^\perp \nabla^\perp H^1_0(\Omega)$ $\square$

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Nehari, Bandle: if $\Omega \subseteq \mathbb{R}^2$ simply connected and $\Delta\log\rho \geq 0$, then $\tilde{\mu}_2 \leq \tilde{\lambda}_1$.

Thank you for your attention