Which Donut Does the Determinant Like Best?

Global extrema of the zeta-regularized determinant on orthogonal flat tori

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The question, in one picture

Take all n-dimensional orthogonal flat tori of unit volume: $\mathbb{R}^n / \operatorname{diag}(q_1, \ldots, q_n)\mathbb{Z}^n$.

Each has a Laplacian Δ,
spectrum $0 = \lambda_0 < \lambda_1 \leq \ldots \nearrow \infty$,
spectral zeta function $\zeta(s) = \sum_{k\geq 1} \lambda_k^{-s}$,
"**determinant**" $\det_\zeta(\Delta) = e^{-\zeta'(0)}$.

a torus is just a lattice, rolled up

Our beauty contest

Among all these tori, **which one maximizes** $\det_\zeta(\Delta)$?

Answer (this talk). The perfectly *equilateral* torus $\mathbb{R}^n/\mathbb{Z}^n$ — and it wins *uniquely*, in every dimension $n \geq 2$.

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So the "determinant" would be

$$\prod_{k\geq 1} \lambda_k \;=\; \infty \qquad \text{(oops)}.$$

We need a way to make this infinite product *finite and meaningful*.

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots = \infty$?!

Zeta function regularization - Ray & Singer

Spectral zeta function mimics the Riemann zeta function, replacing integers by eigenvalues:

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Definition (Ray 1970; Ray–Singer 1971)

$$\boxed{\det\nolimits_\zeta(\Delta) \;:=\; e^{-\zeta'(0)}}$$

The infinite product is gone; only $\zeta'(0)$ remains. Magic.

det = e^(-zeta'(0))

Why $\zeta'(0)$ makes sense

The bridge is the heat trace $\operatorname{Tr} e^{-t\Delta}$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\operatorname{Tr} e^{-t\Delta} - \Pi_{\ker \Delta} \right) dt.$$

- Short-time heat asymptotics $\Rightarrow$ ζ extends **meromorphically** to $\mathbb{C}$;
- it is **holomorphic at** $s = 0$;
- so $\zeta'(0)$ is well defined — and so is $\det_\zeta(\Delta)$.

This number is called the height of the manifold,

$$\text{height} = \zeta'(0), \qquad \det_\zeta(\Delta) = e^{-\text{height}}.$$

Maximize the determinant $\iff$ minimize the height.

heat kernel $\approx$ a warm cup of tea

A physicist's detour: why anyone cares

Hawking, 1977

Zeta regularization tames **quadratic path integrals** on a curved spacetime: the determinant of the operator in the action is exactly

$$e^{-\zeta'(0)}.$$

- Agrees with *dimensional regularization* (add flat dimensions);
- ubiquitous in QFT, CFT, string theory;
- links to *analytic torsion* (Ray–Singer) and the *Selberg zeta function* (Sarnak).

The same gadget that regularizes a path integral is the one we are about to *optimize over geometry*.

determinants in curved spacetime

A speedy history of extremal determinants

- **1971** Ray–Singer: analytic torsion, metric-independence.
- **1978** Cheeger & Müller: analytic torsion = R-torsion.
- **1988** Osgood–Phillips–Sarnak: *extremals* of determinants; the determinant is monotone along Ricci flow.
- **1988–89** OPS: quantify isospectral sets of surfaces & plane domains.
- **2001** Okikiolu: in *odd* dimensions the story flips — no local maxima if $n \equiv 1 \pmod 4$, no local minima if $n \equiv 3 \pmod 4$.
- **2013** Albin–Aldana–Rochon: Ricci flow $\nearrow$ determinant on certain non-compact surfaces.

The recurring theme

Extremal geometry. The determinant "knows" the shape; the nicest shapes sit at its extrema.

The playground: orthogonal flat tori

Definitions

An orthogonal lattice is $D\mathbb{Z}^n$ with $D = \mathrm{diag}(q_1, \ldots, q_n)$ invertible. An orthogonal torus is

$$T = \mathbb{R}^n / \mathrm{diag}(q_1, \ldots, q_n)\mathbb{Z}^n.$$

Laplace eigenvalues are explicit:

$$\lambda_k = 4\pi^2 \sum_{j=1}^{n} \frac{k_j^2}{q_j^2}, \qquad k = (k_1, \ldots, k_n) \in \mathbb{Z}^n.$$

With $Q = \mathrm{diag}(q_1^2, \ldots, q_n^2)$, these are (up to $4\pi^2$) the values of the quadratic form Q^{-1} on the integer lattice.

unit volume: $\prod_j q_j = 1$

Meet the (Paul) Epstein zeta function

Define, for $\Re(s) > n/2$,

$$\zeta_E(s, Q) \;=\; \sum_{k\in\mathbb{Z}^n}{}^{'}\,(Q[k])^{-s}, \qquad Q[k] = k^{\mathsf{T}} Q\, k,$$

the prime meaning "omit $k = 0$". Then the torus zeta function is just

$$\zeta_\circ(s) \;=\; (2\pi)^{-2s}\,\zeta_E(s, Q^{-1}).$$

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Two gifts (for $|Q| = 1$)

$$\zeta_E(0, Q^{-1}) = -1 \;\; \text{(independent of } Q\text{)}, \qquad \zeta_\circ'(0) = 2\log(2\pi) + \zeta_E'(0, Q^{-1}).$$

So our whole problem becomes:

$$\textbf{minimize} \quad \zeta_E'(0, Q^{-1}) \quad \text{over unit-volume } Q.$$

det = e^(-zeta'(0))

Previous results

- **Dim 2 (OPS 1988).** Over *all* flat tori, the hexagonal lattice maximizes the determinant.
- **Dim 3 (Sarnak–Strömbergsson 2006).** Global maximizer identified via heights of flat tori.
- **Dim 2, rectangular (Faulhuber 2021).** Maximizer is the *square* — via the Dedekind η function and Jacobi theta identities.
- **Dim 2, box (Rowlett–Aldana 2017).** Dirichlet determinant on $[0, a] \times [0, 1/a]$ is maximized at the square $a = 1$.

$n \geq 4$: no η to lean on

The catch

All of these lean on the Dedekind η function — which has *no* natural higher-dimensional analogue. We need a new idea for general n.

Sarnak's conjecture

Conjecture (Sarnak; Chiu, Conj. 4.5)

Among all n-dimensional flat tori of volume 1, the height is minimized by the torus whose lattice has the *longest minimal vector*.

Recall: height $= \zeta'(0)$, and

$$\min \text{ height} \iff \max \det_\zeta(\Delta).$$

Among *orthogonal* unit-volume tori, the lattice with the longest minimal vector is exactly the cube $\mathbb{Z}^n$.

So we will prove Sarnak's conjecture in the orthogonal case.

longest shortest vector wins

Theorem 1.3

For every $n \geq 2$, among all orthogonal n-dimensional tori of volume 1, the zeta-regularized determinant of the Laplacian is **uniquely maximized** by the equilateral torus $\mathbb{R}^n/\mathbb{Z}^n$.

The plan of attack

We minimize, over unit-volume diagonal Q, the functional

$$F(Q) \;=\; \sum_{a\in\mathbb{Z}^n}{}' \int_1^\infty \left(e^{-\pi Q^{-1}[a]\,t}\, t^{-1} + e^{-\pi Q[a]\,t}\, t^{n/2-1} \right) dt,$$

which differs from the height only by a universal constant.

1. **Existence.** A minimizer exists (compactness). [Thm 2.2]
2. **Lagrange.** Any minimizer solves a symmetric multiplier system. [§3]
3. **Monotonicity.** The system has a *unique* solution: $Q = I$. [theta functions]

Existence + uniqueness of critical point $\Rightarrow$ the equilateral torus is *the* maximizer.

Step 1 — a maximizer exists

If the torus *degenerates* — some side $q \to 0$ — a short computation gives

$$F(Q) \;\geq\; \int_1^\infty e^{-\pi q^2 t}\,dt \;=\; \frac{e^{-\pi q^2}}{\pi q^2} \;\xrightarrow[q\to 0]{}\; \infty.$$

So minimizers can't escape to the boundary. Combined with the volume constraint $\prod_j q_j = 1$, the search reduces to a compact set

$$\{\,\epsilon < |q_j| < \epsilon^{-1}\ \forall j\,\}.$$

Continuity $\Rightarrow$ a minimum is attained. ✓

lasso the runaway donut

Step 2 — Lagrange multipliers

Minimize $F(Q)$ subject to $g(Q) = \prod_j q_j^2 - 1 = 0$. The system reads

$$\alpha_j(Q)^{-1} \frac{\partial F}{\partial q_j} = \lambda \quad \forall j, \qquad \alpha_j(Q) = 2q_j \prod_{k \neq j} q_k^2.$$

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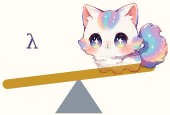
- "$\Leftarrow$": if $Q = \mu^2 I$, a *permutation of coordinates* swaps the i- and j-equations $\Rightarrow$ they agree. So the cube is a critical point.
- "$\Rightarrow$": the hard direction — *uniqueness*. Next slide.



Step 3 — the theta-function reformulation

Introduce the humble 1-D theta function and its derivative

$$\theta_q(t) = \sum_{a\in\mathbb{Z}} e^{-\pi q^2 a^2 t} = \theta(q^2 t), \qquad \theta_q'(t) = -\pi q^2 \sum_{a\in\mathbb{Z}} a^2 e^{-\pi q^2 a^2 t}.$$

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Sums over $\mathbb{Z}^n$ factor into products of these 1-D series. Comparing the i- and j-equations collapses everything to one clean condition:

$$\int_1^\infty \left[\left(\frac{\theta'_{1/q_i}}{\theta_{1/q_i}} - \frac{\theta'_{1/q_j}}{\theta_{1/q_j}} \right) \prod_k \theta_{1/q_k} + \left(\frac{\theta'_{q_j}}{\theta_{q_j}} - \frac{\theta'_{q_i}}{\theta_{q_i}} \right) \prod_k \theta_{q_k}\, t^{n/2} \right] dt = 0.$$

Everything hinges on the single map $q \longmapsto \dfrac{\theta_q'(t)}{\theta_q(t)}$.

Step 3 — monotonicity closes the case

Key monotonicity

With $u = q^2 t$,

$$\frac{\theta_q'(t)}{\theta_q(t)} = \frac{1}{t}\, u\, \frac{\theta'(u)}{\theta(u)},$$

and $u \mapsto u\,\theta'(u)/\theta(u)$ is *strictly increasing*. [Faulhuber 3.13]

Hence $q \mapsto \theta_q'(t)/\theta_q(t)$ is strictly increasing.

If some $q_j > q_i$, *both* bracketed differences are > 0, so the integrand is strictly positive — it cannot integrate to 0. Contradiction!

Thus $q_i = q_j$ for all i, j; with $\prod q_j = 1$,

$$\boxed{q_1 = \cdots = q_n = 1.}$$

equilateral = unique champion

Part II: Boxes (cats love boxes)

Now add *walls*. What changes when the torus becomes a Euclidean box with boundary conditions?

From torus to box: a reflection trick

On the box $B = [0, q_1] \times \cdots \times [0, q_n]$ the boundary breaks the torus' symmetry. But box eigenfunctions are torus eigenfunctions of definite parity under reflection:

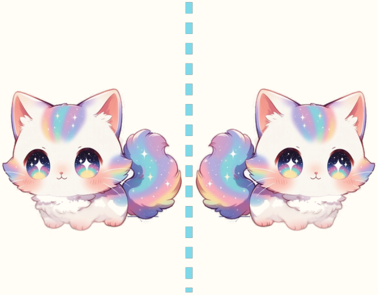
$$\textbf{Dirichlet} \leftrightarrow \text{odd}, \qquad \textbf{Neumann} \leftrightarrow \text{even}.$$

Inclusion–exclusion over subsets $J \subseteq \{1, \ldots, n\}$ writes

each box zeta as a signed sum of *lower-dimensional torus* Epstein zetas $\zeta_E(s, Q_J^{-1})$.

Curious constants

$$\zeta_{\mathrm{Dir}}(0, Q) = \left(-\tfrac{1}{2}\right)^n, \qquad \zeta_{\mathrm{Neu}}(0, Q) = 1 - \left(\tfrac{1}{2}\right)^n.$$



Dirichlet = odd, Neumann = even

Neumann: the equilateral box wins again

Theorem 4.3

For $n \geq 2$, among orthogonal boxes of volume 1, the Neumann determinant is **uniquely maximized** by the equilateral cube (all sides $= 1$).

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The height splits over sub-boxes

$$\zeta'_{\mathrm{Neu}}(0, Q) = -2\log(\pi)\left[\left(\frac{1}{2}\right)^n - 1\right] + 2^{-n} \sum_{K \subseteq \{1,\dots,n\}} \zeta'_E(0, Q_K^{-1}).$$

No longer unit volume, but after rescaling, beautiful cancellation occurs, leaving a *sum* of unit-volume torus heights, each minimized at the cube by Theorem 1.3. So the total is too:

$$\zeta'_{\mathrm{Neu}}(0, Q) \;\geq\; \zeta'_{\mathrm{Neu}}(0, I).$$

Theorem 1.3, reused term by term

Dirichlet: the subtle sibling

$$\zeta'_{\mathrm{Dir}}(0,Q) = -2\log(\pi)\left(-\frac{1}{2}\right)^{n} + 2^{-n}\sum_{m=0}^{n}(-1)^{n-m}\sum_{\substack{J\subseteq\{1,\dots,n\}\\ |J|=m}}\zeta'_E(0,Q_J^{-1}).$$

The Dirichlet sum has alternating signs, and the sub-box tori are *not* volume-constrained. The clean cancellation of the Neumann case is lost.

What we *can* prove [Thm 4.6]:

- Precise height asymptotics as sides $\to 0$ and $\to\infty$ (Lemmas 4.4–4.5) make the height **coercive**;
- hence a global maximizer exists;
- the equilateral box is a **critical point** — the natural candidate.

the determinant keeps a secret

Scorecard

Setting	Equilateral is...	Status
Orthogonal torus, $n \geq 2$	unique *maximizer*	**proved**
Neumann box, $n \geq 2$	unique *maximizer*	**proved**
Dirichlet box, $n \geq 2$	a *critical point*; max exists	**open**

Bottom line

In every settled case, **symmetry maximizes the determinant** — Sarnak's intuition, made rigorous in the orthogonal world, in *all* dimensions.

Open doors

- **Dirichlet boxes.** Is the equilateral box a local max? The global max? (We suspect yes.)
- **General flat tori.** Drop “orthogonal”: prove Sarnak’s full conjecture beyond $n = 2, 3, 4, 8, 24$.
- **Physics.** Consequences for energies, CFT/QFT, partition functions on optimal tori?

plenty left to spot

Thank you!

The equilateral torus: symmetric and undefeated.

Francesconi & Rowlett · questions very welcome

Selected references

- D. Ray, I. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math. 7 (1971).
- S. W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Comm. Math. Phys. 55 (1977).
- B. Osgood, R. Phillips, P. Sarnak, *Extremals of determinants of Laplacians*, J. Funct. Anal. 80 (1988).
- P. Sarnak, A. Strömbergsson, *Minima of Epstein's zeta function and heights of flat tori*, Invent. Math. 165 (2006).
- K. Okikiolu, *Critical metrics for the determinant of the Laplacian in odd dimensions*, Ann. of Math. 153 (2001).
- P. Chiu, *Height of flat tori*, Proc. Amer. Math. Soc. 125 (1997).
- C. Aldana, J. Rowlett, *A Polyakov formula for sectors*, J. Geom. Anal. 28 (2017).
- M. Faulhuber, *Extremal determinants of Laplace–Beltrami operators for rectangular tori* (2020).
- P. Epstein, *Zur Theorie allgemeiner Zetafunctionen*, Math. Ann. 56 (1903).
- F. Francesconi & J. Rowlett, *Global Extrema of the Zeta Regularized Determinant on Orthogonal Flat Tori,* arxiv preprint June 2026.