Optimization of the lowest Neumann eigenvalues for parallelograms

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Based on joint works with Corentin Léna and Vladimir Lotoreichik

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$\Omega \subset \mathbb{R}^d$ bounded, open, $\lambda_1(\Omega)$ first Dirichlet Laplacian EV

Theorem [Faber 1923, Krahn 1925]

We have

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

where B is a ball with $|B| = |\Omega|$.

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where B is a ball with $|B| = |\Omega|$.

Now assume B ball with $|\partial B| = |\partial \Omega|$. Then $|\Omega| \leq |B|$ (isoperimetric inequality). Thus for the scaled version $\widetilde{\Omega}$ with $|\widetilde{\Omega}| = |B|$, Faber–Krahn gives

$$\lambda_1(\Omega) \geq \lambda_1(\widetilde{\Omega}) \geq \lambda_1(B).$$

[Courant 1918]

$\Omega \subset \mathbb{R}^d$ bounded Lipschitz, $\mu_2(\Omega)$ first positive Neumann Laplacian EV

Theorem [Szegő 1954, Weinberger 1956]

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$\Omega \subset \mathbb{R}^d$ bounded Lipschitz, $\mu_2(\Omega)$ first positive Neumann Laplacian EV

Theorem [Szegő 1954, Weinberger 1956]

We have

$$\mu_2(\Omega) \leq \mu_2(B)$$

where B is a ball with $|B| = |\Omega|$.

No implications for the case $|\partial B| = |\partial \Omega|$.

Now $d = 2$. What bounded, **convex** $\Omega \subset \mathbb{R}^2$ maximizes $\mu_2(\Omega)|\partial\Omega|^2$?

- For any disk, $\mu_2(\Omega)|\partial\Omega|^2 \approx 134$.
- For any square or equilateral triangle, $\mu_2(\Omega)|\partial\Omega|^2 = 16\pi^2 \approx 158$.

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Theorem [Laugesen, Siudeja 2009]

Among triangles $\mathcal{T} \subset \mathbb{R}^2$,

$$\mu_2(\mathcal{T})|\partial\mathcal{T}|^2 \leq 16\pi^2$$

with equality if and only if $\mathcal{T}$ is equilateral.

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Conjecture [Laugesen 2009]

Among bounded, convex $\Omega \subset \mathbb{R}^2$,

$$\mu_2(\Omega)|\partial\Omega|^2 \leq 16\pi^2$$

with equality if and only if Ω is a square or an equilateral triangle.

Theorem [Henrot, Lemenant, Lucardesi 2024]

Among bounded, convex $\Omega \subset \mathbb{R}^2$ **with two axes of symmetry**,

$$\mu_2(\Omega)|\partial\Omega|^2 \leq 16\pi^2,$$

with equality if and only if Ω is a square or an equilateral triangle. The same bound holds for all parallelograms.

Theorem [Léna, R. 2024]

Let $\mathcal{P} \subset \mathbb{R}^2$ be a parallelogram with side lengths $\ell_1 \leq \ell_2$ and one angle φ. Define

$$\lambda_- = \frac{\pi^2}{2|\mathcal{P}|^2}\left(\ell_1^2 + \ell_2^2 - \sqrt{\left(\ell_1^2 - \ell_2^2\right)^2 + \frac{256}{\pi^4}\ell_1^2\ell_2^2\cos^2\varphi}\right)$$

and

$$\eta_- = \frac{6}{|\mathcal{P}|^2}\left(\ell_1^2 + \ell_2^2 - \sqrt{\left(\ell_1^2 - \ell_2^2\right)^2 + 4\ell_1^2\ell_2^2\cos^2\varphi}\right).$$

Then

$$\mu_2(\mathcal{P}) \leq \min\{\lambda_-, \eta_-\}.$$

These bounds imply

$$\mu_2(\Omega)|\partial\Omega|^2 \leq 16\pi^2,$$

with equality if and only if Ω is a square or an equilateral triangle.

What bounded, convex $\Omega \subset \mathbb{R}^2$ maximize $\mu_3(\Omega)|\partial\Omega|^2$?

$\mu_1 = 157.91$ $\mu_2 = 353.48$ $\mu_3 = 492.45$

FIGURE 3. Maximizers for the k-th Neumann eigenvalue among shapes with unit perimeter.

Picture source: Bogosel, Henrot, Michetti 2024

What bounded, convex $\Omega \subset \mathbb{R}^2$ maximize $\mu_3(\Omega)|\partial\Omega|^2$?

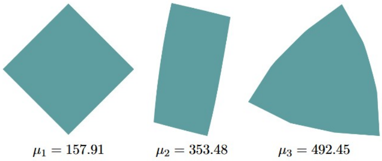

FIGURE 3. Maximizers for the k-th Neumann eigenvalue among shapes with unit perimeter.

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Conjecture [Bogosel, Henrot, Michetti 2024]

Among bounded, convex $\Omega \subset \mathbb{R}^2$,

$$\mu_3(\Omega)|\partial\Omega|^2 \leq 36\pi^2$$

with equality if and only if Ω is a rectangle with side length ratio 2:1.

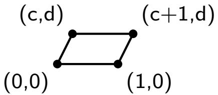
Theorem [Lotoreichik, R. 2025]

Among parallelograms $\mathcal{P} \subset \mathbb{R}^2$

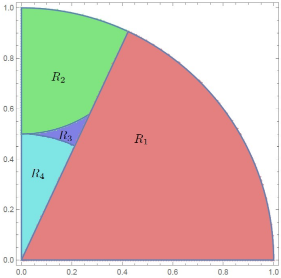
$$\mu_3(\mathcal{P})|\partial\mathcal{P}|^2 \leq 36\pi^2,$$

with equality if and only if Ω is a rectangle with side length ratio 2:1.

Proof strategy



Assume $c^2 + d^2 \leq 1$. Distinguish regions for (c, d):



R_1: Use $\mu_3(\Omega) \leq \left(\frac{2j_{0,1}+\pi}{d_\Omega}\right)^2$ [Kröger 1999]

R_2, R_3, R_4: Variational estimates using the first 5 eigenfunctions of $(0,1) \times (0,d)$

Note: $\mu_3 = \mu_4$ for the optimizer.

Final remarks

- For each k, $\sup\{\mu_k(\Omega)|\partial\Omega|^2 : \Omega \subset \mathbb{R}^2 : \Omega \text{ convex}\}$ is attained.
  [Bogosel, Henrot, Michetti 2024]

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- **Open Problem 1:** Prove that squares and equilateral triangles maximize $\mu_2(\Omega)|\partial\Omega|^2$ among bounded, convex $\Omega \subset \mathbb{R}^2$.

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- **Open Problem 2:** Prove that 2:1-rectangles maximize $\mu_3(\Omega)|\partial\Omega|^2$ among bounded, convex $\Omega \subset \mathbb{R}^2$.

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- **Open Problem 3:** Find optimal shapes for higher eigenvalues.