Boundary Value Problems for Dirac Operators on Graphs

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Fredholm index

Definition

The (Fredholm) index of a linear operator D is given by

$$\operatorname{ind} D = \dim \ker D - \dim \ker D^*.$$

An operator is called Fredholm if this index is finite.

Index theory on closed manifolds

Consider ellitpic differential operators on closed (compact+no boundary) manifolds:

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- Atiyah-Singer index theorem $\implies$ index depends on the topology of the underlying manifold
- The Gauss-Bonnet theorem is a special case:

$$\operatorname{ind}(d + d^*) = \chi(M).$$

Index theory on compact manifolds with boundary

Consider a first-order elliptic differential operator D on a compact manifold with boundary of dimension ≥ 2:

- D is not Fredholm if not subjected to boundary conditions

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Consider a first-order elliptic differential operator D on a compact manifold with boundary of dimension ≥ 2:

- D is not Fredholm if not subjected to boundary conditions
- Atiyah, Patodi and Singer found a boundary condition with respect to which D is Fredholm, and gave a formula for the index (if D is Dirac-type).

Bär-Ballmann-Bandara framework

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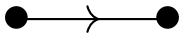
- Obtain a classification of all boundary conditions for which D is Fredholm.
- Simplification of the index theory for such problems.
- Led to a decomposition theorem, a general version of the relative index theorem and a generalization of the cobordism-invariance for indices.

What about them?

What about one-dimensional manifolds with boundary?

Bär-Ballmann-Bandara on 1-dim manifolds?



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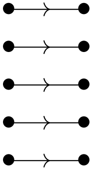

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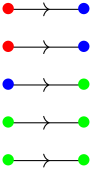

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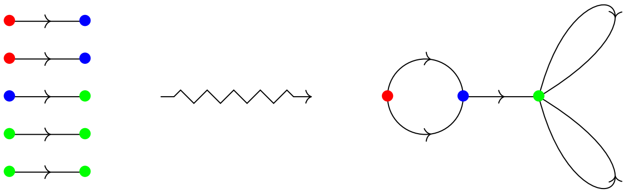

Spectral theory on graphs

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- The spectral theory of Schrödinger operators on graphs is a classical subject with applications in physics, engineering and combinatorics.
- In order to account for the effect of spin on the quantum mechanics of graphs, first order differential operators, such as the Dirac operator, have also been studied on graphs.
- Most existing approaches focus on the spectral analysis of specific operators.

Aim/Motivation

Look at boundary value problems for metric graphs through the lens of the Bär-Ballmann framework.

In order to define first order differential operators we need to parametrize each edge, which is equivalent to

- an orientation on each edge,
- the assignment of a length to each edge.

We allow for parallel edges and loops. We thus work on finite **metric multidigraphs**.

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- Function spaces on G are given by the direct sum of the functions spaces on the intervalls $[0, l_e]$, e.g. $L^2(G) := \bigoplus_{e \in E} L^2([0, l_e])$
- There is a trace map $tr : H^1(G, \mathbb{C}^r) \to \mathbb{C}^{2rn}$ mapping $f \in H^1(G, \mathbb{C}^r)$ to

$$tr(f) = (f_{e_1}(0), ..., f_{e_n}(0), f_{e_1}(l_{e_1}), ..., f_{e_n}(l_{e_n}))$$

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- A boundary condition is a linear subspace $B \subseteq \mathbb{C}^{2rn}$, set $H^1_B(G, \mathbb{C}^r) = tr^{-1}(B)$.

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$$U_v := \left\{ (v_1, \dots, v_n, w_1, \dots, w_n) \mid \substack{v_i = 0 \text{ if } \partial_- e_i \neq v \\ w_i = 0 \text{ if } \partial_+ e_i \neq v} \right\} \subseteq \mathbb{C}^{2rn}$$

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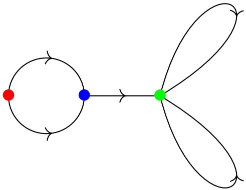
Definition

A boundary condition B is local if for each $v \in V$ there exists a subspace $B_v \subseteq U_v$ such that $B = \bigoplus_{v \in V} B_v$

Local boundary conditions



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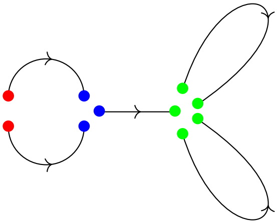

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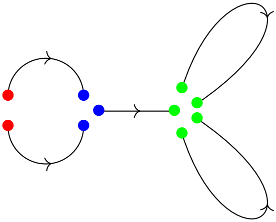

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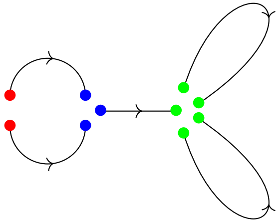

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Example: The matching condition imposes that the values of a function at a vertex all agree.

Differential operators & boundary conditions

- Let $D : H^1(G, \mathbb{C}^r) \to L^2(G, \mathbb{C}^r)$ be a elliptic differential operator of first order, i.e.

$$D = \sigma \cdot d/dt + V, \text{ where } \sigma, V \in C^\infty(G, \mathbb{C}^{r\times r})$$

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- For a boundary condition B denote the restriction of D to $H^1_B(G, \mathbb{C}^r)$ by D_B
- Let $D_{\text{min}} = D_0$ and $D_{\text{max}} = D_{\mathbb{C}^{2rn}}$.

Adjoint boundary conditions

The principal symbol σ of D defines a map $\sigma_0 : \mathbb{C}^{2rn} \to \mathbb{C}^{2rn}$

$$\sigma_0(v_1, ..., v_n, w_1, ..., w_n) = \\ (\sigma_{e_1}(0)v_1, ..., \sigma_{e_n}(0)v_n, \sigma_{e_1}(l_{e_1})w_1, ..., \sigma_{e_n}(l_{e_n})w_n).$$

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Definition

For a boundary condition B define the **adjoint boundary condition**

$$B^{\mathrm{ad}} := (\sigma_0 B)^{\perp}.$$

Adjoint boundary conditions II

Proposition

The adjoint of D_B is $(D^\dagger)_{B^{\mathrm{ad}}}$, where $D^\dagger$ is the formal adjoint of D, we have $(D_{\max})^* = (D^\dagger)_{\min}$ and $(D_{\min})^* = (D^\dagger)_{\max}$.

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- $\mathrm{ind} D_{\max} = rn$ and $\mathrm{ind} D_{\min} = -rn$

Index theory

Lemma

Let B be a subspace of $\mathbb{C}^{2rn}$, $B^\perp$ be the orthogonal complement to B and let $P : \mathbb{C}^{2rn} \to \mathbb{C}^{2rn}$ be the orthogonal projection onto $B^\perp$. Then,

$$L : H^1(G, \mathbb{C}^r) \to L^2(G, \mathbb{C}^r) \oplus B^\perp, \quad g \mapsto (D_{\max} g, P \operatorname{tr} g)$$

is a Fredholm operator with the same index as D_B.

The index formula

Theorem

Let $B_1 \subseteq B_2 \subseteq \mathbb{C}^{2rn}$ be boundary conditions. Then,

$$\operatorname{ind} D_{B_2} = \operatorname{ind} D_{B_1} + \dim B_2/B_1.$$

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 & L^2(G,\mathbb{C}^r) \oplus B_2^\perp \arrow[d, hook, "\mathrm{id} \oplus \iota"] \\
H^1(G,\mathbb{C}^r) \arrow[ur, "{(D,P_2)}"] \arrow[r, "{(D,P_1)}"'] & L^2(G,\mathbb{C}^r) \oplus B_1^\perp
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$$
\begin{aligned}
\operatorname{ind} D_{B_1} &= \operatorname{ind}(D, Q_1) = \operatorname{ind}(D, Q_2) + \operatorname{ind} \mathrm{id} \oplus \iota \\
&= \operatorname{ind} D_{B_2} - \dim B_2/B_1.
\end{aligned}
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Theorem

Let B be any boundary condition, the index of D_B is given by

$$\operatorname{ind}(D_B) = \dim B - r|E|. \tag{1}$$

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Corollary

The index of D with respect to M is equal to the Euler characteristic of the graph, i.e.

$$\mathrm{ind}\, D_M = |V| - |E| = \chi(G).$$

The scalar Dirac operator and graph-like boundary conditions

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We say that A is local if $\Gamma(A)$ is a local boundary condition.

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- The existence of a invertible local map A implies that at every vertex there exist as many ingoing edges as outgoing ones.
- Local self-adjoint boundary conditions therefore exist only on Eulerian digraphs.

The spectrum of graph-like boundary conditions

- For $A : \mathbb{C}^n \to \mathbb{C}^n$, define the characteristic multinomial

$$P_A(x_1, ..., x_n) = \det \left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} - A \right).$$

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- The spectrum of $D_{\Gamma(A)}$ consists of all $\lambda \in \mathbb{C}$ such that

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- If $l_1 = ... = l_n = l > 0$, then the spectrum of $D_{\Gamma(A)}$ is equal to

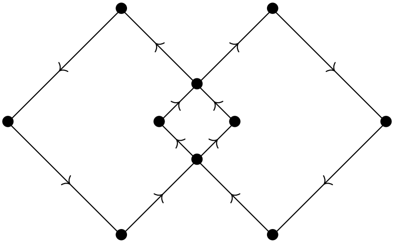
$$\bigcup_{i=1}^{m} \{(\arg(\lambda_i) - i \log(|\lambda_i|) + 2\pi k)/l \mid k \in \mathbb{Z}\},$$

where $\lambda_1, ..., \lambda_m$ are the non-zero eigenvalues of A.

Which ones?

Which boundary conditions can we impose?

Eulerian digraphs and trail decompositions



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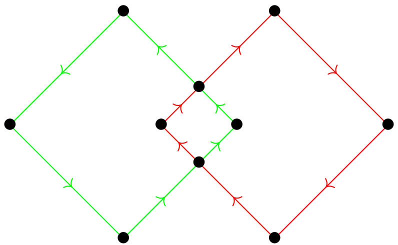

Examples for boundary conditions: Decompositions of Eulerian digraphs

- Every decomposition of a graph into closed directed trails yields a local permutation matrix

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- In fact, there is a 1:1 correspondence between decompositions into closed directed trails and local permutation matrices
- The spectrum of $D_{\Gamma(P)}$ for a local permutation matrix P is given by

  $$\bigcup_{i=1}^{m} \{2\pi k/L_i \mid k \in \mathbb{Z}\},$$

  where $L_1, ..., L_m$ are the lengths of the closed trails in the corresponding decomposition.

The directed incidence matrix

Definition

The directed incidence matrix $I(G)$ of a multidigraph G is given by

$$I(G)_{ij} = \begin{cases} 1 & \text{if } \partial_+ e_j = \partial_- e_i, \\ 0 & \text{otherwise.} \end{cases}$$

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- $I(G)$ defines a local boundary condition, which in general is not self-adjoint.
- The spectrum of $D_{\Gamma(I(G))}$ is determined by the characteristic multinomial of $I(G)$, if all edges have the length 1, then it is determined by it the characteristic polynomial.

The characteristic multinomial of $I(G)$

- Let $\mathcal{C}(G) := \{C \subseteq E \mid C \text{ is a disjoint collection of cycles}\}$, for $C \in \mathcal{C}(G)$, let $\alpha(C)$ denote the amount of connected components of C.

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- For $C \in \mathcal{C}(G)$, put $x^C = \prod_{i:e_i \notin C} x_i$, we then have

$$P_{I(G)}(x_1, \dots, x_n) = \sum_{C \in \mathcal{C}(G)} (-1)^{\alpha(C)} x^C$$

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- For the characteristic polynomial $p_{I(G)}(t) = \sum_{k=0}^{n} a_k t^k$, we have

$$a_k = \sum_{C \in \mathcal{C}(G), |C| = n-k} (-1)^{\alpha(C)}$$

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- $a_{n-3} = -\#\{\text{3-cycles}\} + \#\{\text{disjoint pairs of loops and 2-cycles}\} - \#\{\text{disjoint triples of loops}\}$

Thanks for your attention!