Eigenvalues of self-adjoint extensions for defect larger than one

Jakob Reiffenstein (Stockholm University)

joint with A. Luger (Stockholm University)

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$$(\widetilde{A} - \lambda)^{-1} = (A - \lambda)^{-1} - \gamma(\lambda)(m(\lambda) + \tau)^{-1}\gamma(\bar{\lambda})^*. \tag{1}$$

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The *defect family* $\gamma(\lambda) : \mathbb{C}^n \to H$ maps $\mathbb{C}^n$ onto $\ker(S^* - \lambda)$.

The Weyl function $m(\lambda)$ is a $\mathbb{C}^{n\times n}$-valued Nevanlinna function.

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(1) relates (one-to-one):

- extension $\widetilde{A} = \widetilde{A}^*$ in H
- *parameter* τ, a self-adjoint linear relation in $\mathbb{C}^n$.

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$$P_H(\widetilde{A} - \lambda)^{-1}|_H = (A - \lambda)^{-1} - \gamma(\lambda)(m(\lambda) + \tau(\lambda))^{-1}\gamma(\overline{\lambda})^*. \quad (1)$$

(1) relates (essentially one-to-one):

- extension $\widetilde{A} = \widetilde{A}^*$ in $\widetilde{H} \supseteq H$
- *parameter* $\tau(\lambda) \in \widetilde{\mathcal{N}_0^{n\times n}}$, a Nevanlinna family.

Examples

- $n = 1$: *Schrödinger operator on* $(0, \infty)$ *with* 0 *regular and* ∞ *in limit point case.*
  S is the minimal operator, A the Dirichlet operator. All other s.-a. ext. of S are given by a one-parameter family of boundary conditions at 0. In Krein's formula: $\tau \in \mathbb{R} \cup \{\infty\}$.

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  shrinking graph-like thin manifolds may lead to an exit-space extension of the minimal graph Laplacian [Exner-Post 2005]

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- $n > 1$: *finite compact quantum graphs*
  shrinking graph-like thin manifolds may lead to an exit-space extension of the minimal graph Laplacian [Exner-Post 2005]

- Krein's formula is also valid for operators in Pontryagin spaces (with a finite-dimensional negative component).

Let α in $\mathbb{R}$. Is α an eigenvalue of $\widetilde{A}$?

$$P_H(\widetilde{A} - \lambda)^{-1}|_H = (A - \lambda)^{-1} - \gamma(\lambda)(m(\lambda) + \tau(\lambda))^{-1}\gamma(\overline{\lambda})^*.$$

Goal: Find a criterion in terms of m and τ
(Assuming that the extension $\widetilde{A}$ is H-minimal).

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- $n=1$ [Behrndt-Luger 2007]: α is an EV of $\widetilde{A}$ iff it is either:
  - a generalized zero of $m+\tau$, or
  - a generalized pole of both m and τ.

Guess for $n \geq 1$:

Theorem?
$\alpha \in \mathbb{R}$ is an EV of $\widetilde{A}$ iff it is either:

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Is this correct?

- YES if $\widetilde{A}$ acts in a Hilbert space;
- MAYBE if $\widetilde{A}$ acts in an indefinite Pontryagin space.

Generalized poles and zeros

$\mathcal{N}_0^{n\times n}$ denotes the set of all $\mathbb{C}^{n\times n}$-valued Nevanlinna functions.

Definition

Let $Q \in \mathcal{N}_0^{n\times n}$, $\alpha \in \mathbb{R}$ and let $\mathcal{U}_\alpha$ be a neighborhood of α in $\mathbb{C}$.
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$$\lim_{\lambda \hat{\to} \alpha} f(\lambda) = 0, \qquad \lim_{\lambda \hat{\to} \alpha} Q(\lambda) f(\lambda) \neq 0$$

and

$$\left(\frac{Q(\lambda) - Q(\lambda)^*}{\lambda - \bar{\lambda}} f(\lambda), f(\lambda) \right)_{\mathbb{C}^n} \quad \text{is bounded as } \lambda \hat{\to} \alpha.$$

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If such f exists then α is called *generalized pole* of Q.

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Operator representations

Every $Q \in \mathcal{N}_0^{n\times n}$ has a minimal operator realization. Under additional conditions, it has the simple form

$$Q(\lambda) = \Theta + \gamma^*(T - \lambda)^{-1}\gamma \tag{2}$$

where
$\Theta = \Theta^* \in \mathbb{C}^{n\times n}$,
T is a self-adjoint operator in a Hilbert space H,
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“minimal” means

$$H = \overline{\operatorname{span}\{(T-\lambda)^{-1}\gamma x \mid \lambda \in \mathbb{C}\setminus\mathbb{R},\ x \in \mathbb{C}^n\}}.$$

If (H, T, γ) is a minimal realization of $Q \in \mathcal{N}_0^{n\times n}$ then the eigenvalues of T are precisely the generalized poles of Q.

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In the situation of Krein's formula:

- $m(\lambda)$ has the minimal realization (H, A, γ).
- If τ is a constant Hermitian matrix such that $\det(m(\lambda) + \tau) \not\equiv 0$ then $-(m(\lambda) + \tau)^{-1}$ has a minimal realization with $\widetilde{A}$.

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- If $-(m(\lambda) + \tau(\lambda))^{-1}$ exists, it always has a realization with $\widetilde{A}$. But: in general not minimal.

Directional functions

Definition

Let $Q \in \mathcal{N}_0^{n\times n}$, $\alpha \in \mathbb{R}$.
An analytic function $f : \mathcal{U}_\alpha \cap \mathbb{C}^+ \to \mathbb{C}^n$ is called *directional function* of Q at α if

$$\exists\, f_0 := \lim_{\lambda \hat{\to} \alpha} f(\lambda), \qquad \exists g_0 := \lim_{\lambda \hat{\to} \alpha} Q(\lambda) f(\lambda)$$

and

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Directional pairs

Definition

Let $f_0, g_0 \in \mathbb{C}^n$. We say that $(f_0; g_0)$ is a *directional pair* of Q at α if there is a directional function f of Q at α s. t.

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With this terminology:
α is a gen. pole of Q iff $\exists g_0$ s. t. $(0; g_0)$ is a directional pair.
α is a gen. zero of Q iff $\exists f_0$ s. t. $(f_0; 0)$ is a directional pair.

Theorem (Luger-R. 2025)

Let $Q \in \mathcal{N}_0^{n\times n}$ *have the minimal operator realization*

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For $\alpha \in \mathbb{R}$ *and* $f_0, g_0 \in \mathbb{C}^n$ *the following are equivalent:*

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Here $Q(\alpha)$ is defined as a linear relation.

The eigenvalue criterion

Recall Krein's formula

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with an H-minimal extension $\widetilde{A}$.

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Theorem (Luger-R. 2025, Hassi 2008)

A point $\alpha \in \mathbb{R}$ is an eigenvalue of $\widetilde{A}$ if and only if there exist vectors $f_0, g_0 \in \mathbb{C}^n$, not both trivial, such that

$$(f_0; g_0) \in m(\alpha) \quad \text{and} \quad (f_0; -g_0) \in \tau(\alpha).$$

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Is the first alternative equivalent to $(f_0; 0) \in (m + \tau)(\alpha)$?

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Definition

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Theorem (Luger-R. 2025)

Let $\alpha \in \mathbb{R}$. If $m, \tau \in \mathcal{N}_0^{n \times n}$ then

$$m(\alpha) + \tau(\alpha) = (m + \tau)(\alpha). \tag{4}$$

If m, τ are generalized Nevanlinna functions then (4) *may fail.*

$$P_H(\widetilde{A}-\lambda)^{-1}|_H = (A-\lambda)^{-1} - \gamma(\lambda)(m(\lambda)+\tau(\lambda))^{-1}\gamma(\overline{\lambda})^*.$$

Corollary

Assume that all operators act in Hilbert spaces (S, A in H and $\widetilde{A}$ in $\widetilde{H} \supseteq H$).
Then $\alpha \in \mathbb{R}$ is an eigenvalue of $\widetilde{A}$ if and only if either:

- *α is a generalized zero of $m + \tau$, or*
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Conjecture

The above criterion still holds in the Pontryagin space setting.