Characterization of spectral measures and hyperrigidity

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History

Theorem (Weierstrass Approximation Theorem, 1885)

Suppose f is a continuous function defined on the interval $[0,1]$. For every $\epsilon > 0$, there exists a polynomial p such that for all $x \in [0,1]$, we have $|f(x) - p(x)| < \epsilon$, or equivalently, the supremum norm $\|f - p\| < \epsilon$.

A constructive proof of this theorem uses **the Bernstein polynomials**.

Define sequence $B_n \colon C[0,1] \to [0,1]$, $(n \in \mathbb{N})$ by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad f \in C[0,1], x \in [0,1].$$

It turn out that $\lim_{n\to\infty} \|B_n(f) - f\| = 0$ for all $f \in C[0,1]$.

History

An element $f \in C[0,1]$ is called positive if $f(x) \geqslant 0$ for all $x \in [0,1]$. If this happens, we write $f \geqslant 0$.

A linear operator $\Phi\colon C[0,1] \to C[0,1]$ is called **positive** if

$$f \geqslant 0 \implies \Phi(f) \geqslant 0.$$

Theorem (P. P. Korovkin, 1953)

Let $\Phi_n\colon C[0,1] \to C[0,1]$, $n \in \mathbb{N}$ be a sequence of positive operators such that

$$\Phi_n(g) \to g, \quad g \in \{1, x, x^2\},$$

Then

$$\Phi_n(f) \to f, \quad f \in C[0,1].$$

History

- In other words, the asymptotic behavior of the sequence $\{\Phi_k\}_{k=1}^{\infty}$ on the C^*-algebra $C[0,1]$ is uniquely determined by the vector space G spanned by $\{1, x, x^2\}$.
- Another major achievement was the discovery of geometric theory of Korovkin sets by Y. A. Šaškin in 1967. Namely, Šaškin observed that the key property of G is that its Choquet boundary coincides with $[0,1]$.

Recall: Let X be a compact set, $V \subseteq C(X)$ be a vector subspace. A point $x \in X$ is called a **boundary point** of V if for any measure μ on X

$$f(x) = \int_X f d\mu, \quad \forall f \in V \implies \mu = \delta_x$$

The set of all boundary points of V is called the **Choquet Boundary** of V.

Example

Let $B_n\colon C[0,1] \to [0,1]$ be the Bernstein polynomials defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \quad f \in C[0,1], x \in [0,1].$$

Then B_n is linear, and positive. Also

$$\begin{aligned} B_n(e_0) &= e_0 \\ B_n(e_1) &= e_1 \\ B_n(e_2) &= \frac{1}{n}e_1 + \frac{n-1}{n}e_2, \end{aligned}$$

where $e_k(x) = x^k$.

History

Theorem (Fejér's Theorem, 1904)

For $f \in C(\mathbb{T})$, define $\{\sigma_n(f)\}_{n=1}^{\infty}$ by

$$\sigma_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} s_k(f), \quad n \in \mathbb{N},$$

with

$$s_n(f) = \sum_{k=-n}^{n} \left(\int_{\mathbb{T}} f \bar{\xi}^k \mathrm{d}\nu \right) \xi^k, \quad n \in \mathbb{Z}_+,$$

where ν is the normalized Lebesgue measure on $\mathbb{T}$ and ξ is defined by $\xi(z) = z$ for $z \in \mathbb{T}$. Then $\{\sigma_n(f)\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{T}$ to f.

Completely positive maps

Any linear map $\Phi\colon \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras induces another map $id_{\mathbb{C}^{k\times k}} \otimes \Phi\colon \mathbb{C}^{k\times k} \otimes \mathcal{A} \to \mathbb{C}^{k\times k} \otimes \mathcal{B}$ in a natural way. If $\mathbb{C}^{k\times k} \otimes \mathcal{A}$ is identified with the C^*-algebra $\mathcal{A}^{k\times k}$ of $k \times k$-matrices with entries in $\mathcal{A}$ then $id_{\mathbb{C}^{k\times k}} \otimes \Phi$ acts as

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{bmatrix} \to \begin{bmatrix} \Phi(a_{1,1}) & \cdots & \Phi(a_{1,k}) \\ \vdots & \ddots & \vdots \\ \Phi(a_{k,1}) & \cdots & \Phi(a_{k,k}) \end{bmatrix}$$

We say that Φ is **completely positive** if $id_{\mathbb{C}^{k\times k}} \otimes \Phi$ is positive for all $k \in \mathbb{N}$. The map Φ is called **unital** if it preserves the units.

History

A natural non-commutative analogue of Korovkin-type rigidity would be a subset G of a unital C^*-algebra $\mathscr{A}$ with the property that for any sequence of unital completely positive (UCP) maps

$\Phi_k \colon \mathscr{A} \to \mathscr{A}$ $(k \in \mathbb{N})$,

$$\lim_{k\to\infty} \|\Phi_k(g) - g\| = 0 \;\; \forall g \in G \implies \lim_{k\to\infty} \|\Phi_k(a) - a\| = 0 \;\; \forall a \in \mathscr{A}.$$

In fact, Arveson introduced even more non-commutativity in this picture.

Definition (W. B. Arveson, 2011)

A finite or countably infinite set G of generators of a unital C^*-algebra $\mathscr{A}$ is said to be **hyperrigid** if for any faithful representation $\pi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ and for any sequence $\Phi_k\colon \boldsymbol{B}(\mathcal{H}) \to \boldsymbol{B}(\mathcal{H})$ ($k \in \mathbb{N}$) of UCP maps,

$$\lim_{k\to\infty} \|\Phi_k(\pi(g)) - \pi(g)\| = 0 \ \ \forall g \in G \implies \lim_{k\to\infty} \|\Phi_k(\pi(a)) - \pi(a)\| = 0 \ \ \forall a \in \mathscr{A}.$$

History

- if $T \in \boldsymbol{B}(\mathcal{H})$ is a self-adjoint operator and $\mathscr{A}$ is the C^*-algebra generated by T, then $G = \{T, T^2\}$ is hyperrigid in $\mathscr{A}$, but $\{T\}$ is not hyperrigid.
- if $V \in \boldsymbol{B}(L^2[0,1])$ is a Voltera operator

$$V(f)(x) = \int_0^x f(t)dt, \quad f \in L^2[0,1], x \in [0,1]$$

  and $\mathscr{A}$ is the C^*-algebra generated by V, then $G = \{V, V^2\}$ is hyperrigid in $\mathscr{A}$.
- if $V_1, \dots, V_n \in \boldsymbol{B}(\mathcal{H})$ is a finite set of isometries that generates a C^*-algebra $\mathscr{A}$, then $G = \{V_1, \dots, V_n, V_1 V_1^* + \cdots + V_n V_n^*\}$ is a hyperrigid set of generators for $\mathscr{A}$.

History

Arveson asked whether the set $\{I, T, f(T)\}$ is hyperrigid in $C^*(\{T\})$ for a selfadjoint operator T, where f is a strictly convex continuous function on the closed subinterval $[a, b]$ of the real line $\mathbb{R}$ containing the spectrum of T. A positive answer was given by L. G. Brown using the following theorem.

Theorem (L.G. Brown, 2016)

Let f be a strictly convex continuous function on a closed interval $I \subseteq \mathbb{R}$, let $\mathcal{H}$ be a Hilbert space, and let $\{T_i\}_{i\in\Sigma}$ be a net of selfadjoint operators in $\boldsymbol{B}(\mathcal{H})$ such that $\sigma(T_i) \subseteq I$ for all $i \in \Sigma$. If $\{T_i\}_{i\in\Sigma}$ converges weakly to a selfadjoint operator $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subseteq I$, and if also $\{f(T_i)\}_{i\in\Sigma}$ converges weakly to $f(T)$, then $\{\varphi(T_i)\}_{i\in\Sigma}$ converges strongly to $\varphi(T)$ for every bounded continuous function φ on I. In particular if the net $\{T_i\}_{i\in\Sigma}$ is bounded, then $\{T_i\}_{i\in\Sigma}$ converges strongly to T.

POV, semispectral and spectral measure

Recall that a map $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ defined on a Borel σ-algebra $\mathfrak{B}(\mathbb{R})$ is said to be:

- a **positive operator valued measure** (*POV measure*) if $\langle F(\cdot)h, h\rangle$ is a positive measure for every $h \in \mathcal{H}$,
- a **semispectral measure** if F is a POV measure such that $F(\mathbb{R}) = I$,
- a **spectral measure** if F is a semispectral measure such that $F(\varDelta)$ is an orthogonal projection for every $\varDelta \in \mathfrak{B}(\mathbb{R})$,

where $\boldsymbol{B}(\mathcal{H})$ is the collection of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I is the identity operator on $\mathcal{H}$.

The celebrated Naimark's dilation theorem

Theorem (Naimark 1943)

Suppose that $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ *is a semispectral measure. Then there exist a Hilbert space* $\mathcal{K}$ *containing* $\mathcal{H}$ *and a spectral measure* $E\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{K})$ *such that*

$$F(\Delta) = PE(\Delta)|_{\mathcal{H}}, \quad \Delta \in \mathfrak{B}(\mathbb{R}).$$

Operator moments of POV measure I

- For an integer $n \geqslant 1$ and a Borel POV measure F on $\mathbb{R}$ with compact support, the integral

$$\int_{\mathbb{R}} x^n F(\mathrm{d}x)$$

is a (bounded) self-adjoint operator, which is called the ***n*****th operator moment** of F.

- a Borel POV measure on $\mathbb{R}$ with compact support is uniquely determined by its operator moments.
- if E is a Borel spectral measure on $\mathbb{R}$ with compact support, then the following identities hold

$$\Big(\int_{\mathbb{R}} x E(\mathrm{d}x)\Big)^n = \int_{\mathbb{R}} x^n E(\mathrm{d}x), \quad n = 1, 2, \ldots. \tag{1}$$

History

- all operator moments of spectral measure are determined by the first one, and according to the spectral theorem there is a one-to-one correspondence between Borel spectral measures on $\mathbb{R}$ and their first operator moments. This is no longer true for general Borel semispectral measures on $\mathbb{R}$.

It turns out, however, that the single equality in (1) with $n = 2$ guarantees spectrality.

Theorem (J. Kiukas, P. Lahti, K. Ylinen, 2006)

A Borel semispectral measure F on $\mathbb{R}$ with compact support is spectral if and only if

$$\Big(\int_{\mathbb{R}} x F(\mathrm{d}x)\Big)^2 = \int_{\mathbb{R}} x^2 F(\mathrm{d}x).$$

History

- this theorem was developed for the needs of quantum physics
- the main purpose of the quantization proposed by J. Kiukas, P. Lahti, K. Ylinen was to construct observables that are not spectral measures
- it was important to be able to use these moments to determine whether a given observable is or is not a spectral measure.

Kiukas-Lahti-Ylinen Problem

This led Kiukas, Lahti and Ylinen to the following question :

Open Problem

When is a positive operator measure projection valued?

Main results

Theorem (P.P., J. Stochel, 2021)

Let $F\colon \mathfrak{B}(\mathbb{R}_+) \to \boldsymbol{B}(\mathcal{H})$ *be a semispectral measure with compact support and* p *be a positive real number other than* 1. *Assume that*

$$\Big(\int_{\mathbb{R}_+} x F(\mathrm{d}x)\Big)^p = \int_{\mathbb{R}_+} x^p F(\mathrm{d}x).$$

Then F *is a spectral measure.*

Main results

Criterion for the spectrality of a Borel semispectral measure on $\mathbb{R}$ compactly supported in $[0,\infty)$, written in terms of its two operator moments.

Theorem (P.P., J. Stochel, 2021)

Let $F\colon \mathfrak{B}([0,\infty)) \to \boldsymbol{B}(\mathcal{H})$ *be a compactly supported semispectral measure. Let* $T \in \boldsymbol{B}(\mathcal{H})$ *be a positive operator and* α, β *be two distinct positive real numbers. Then the following conditions are equivalent:*

(i) F *is the spectral measure of* T*,*

(ii) $T^r = \int_{[0,\infty)} x^r F(\mathrm{d}x)$ *for* $r = \alpha, \beta$*.*

Main results

Theorem (P.P., J. Stochel, 2023)

Let $T \in \boldsymbol{B}(\mathcal{H})$ be a selfadjoint operator, $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure with compact support and p, q be positive integers such that $p < q$, p is odd and q is even. Then the following conditions are equivalent:

(i) *F is the spectral measure of T,*

(ii) *$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$ for $k = p, q$.*

Main results

Theorem (P.P., J. Stochel, 2023)

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a unital positive linear map, $a \in \mathcal{A}$ be selfadjoint and p, q be positive integers such that $p < q$, p is odd and q is even. Then the following conditions are equivalent:

(i) *Φ restricted to the unital subalgebra generated by $\{a\}$ is multiplicative,*

(ii) *there exists a selfadjoint element $b \in \mathcal{B}$ such that $b^k = \Phi(a^k)$ for $k = p, q$.*

Moreover, if (ii) *holds, then $b = \Phi(a)$.*

Main results

In the complementary result, we showed that the set (with $\mathbb{N} = \{1, 2, 3, \ldots\}$)

$$\Omega := \{(p, q) \in \mathbb{N}^2 \colon p < q,\ p \text{ odd and } q \text{ even}\}$$

is the largest possible subset of $\{(p, q) \in \mathbb{N}^2 \colon p \leqslant q\}$ for which the previous theorem holds for $\Xi = \{p, q\}$.

Main results

Theorem (P.P., J. Stochel, 2023)

Suppose that $(p,q) \in \mathbb{N}^2 \backslash \Omega$ *and* $p \leqslant q$. *Let* $\tau \in \mathbb{R}\backslash\{0\}$. *Set* $\mathcal{H} = \mathbb{C}$ *and* $T = \tau I$. *Then there exist* $\alpha, \beta \in (0,1)$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that* $\alpha + \beta = 1$, $\lambda_1 \neq \lambda_2$ *and*

(i) *the semispectral measure* $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ *defined by*

$$F(\varDelta) = \alpha\delta_{\lambda_1}(\varDelta)I + \beta\delta_{\lambda_2}(\varDelta)I, \quad \varDelta \in \mathfrak{B}(\mathbb{R}),$$

is not spectral and

$$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k = p, q,$$

Main results

Theorem (P.P., J. Stochel, 2023)

(ii) *the selfadjoint operator* $S \in \boldsymbol{B}(\mathcal{H} \oplus \mathcal{H})$ *defined by*

$$S = \begin{bmatrix} \alpha\lambda_1 + \beta\lambda_2 & \sqrt{\alpha\beta}(\lambda_1 - \lambda_2) \\ \sqrt{\alpha\beta}(\lambda_1 - \lambda_2) & \beta\lambda_1 + \alpha\lambda_2 \end{bmatrix}$$

does not commute with $P := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *and*

$$T^k = PS^k|_{\mathcal{H}}, \quad k = p, q.$$

Theorem (P.P., J. Stochel, 2023)

(iii) *the unital positive linear map* $\Phi\colon C(K) \to \boldsymbol{B}(\mathcal{H})$ *defined by*

$$\Phi(f) = \int_K f \,\mathrm{d}F, \quad f \in C(K),$$

where $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ *defined by*

$$F(\varDelta) = \alpha\delta_{\lambda_1}(\varDelta)I + \beta\delta_{\lambda_2}(\varDelta)I, \quad \varDelta \in \mathfrak{B}(\mathbb{R}),$$

is not multiplicative, $C(K)$ *is the unital algebra generated by* a *and*

$$b^k = \Phi(a^k), \quad k = p, q,$$

$K = \{\lambda_1, \lambda_2\}$, $a(x) = x$ *for* $x \in K$ *and* $b = T$.

Main results

Let $\{f_n\}_{n=0}^{\infty}$ be the Fibonacci sequence, that is, $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \in \mathbb{N}$. It is well known that

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}, \quad n \in \mathbb{N}. \tag{2}$$

Set $\mathcal{H} = \mathbb{C}$, $T = I$ and

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then S is a selfadjoint operator and the spectral measure E of S is given by

$$E(\varDelta) = \frac{1}{1+\phi^2}\delta_{1-\phi}(\varDelta)\begin{bmatrix} \phi^2 & -\phi \\ -\phi & 1 \end{bmatrix} + \frac{1}{1+\phi^2}\delta_{\phi}(\varDelta)\begin{bmatrix} 1 & \phi \\ \phi & \phi^2 \end{bmatrix},$$

for $\varDelta \in \mathfrak{B}(\mathbb{R})$,

Main results

The semispectral measure $F := PE|_{\mathcal{H}}$ takes the form

$$F(\mathit{\Delta}) = \frac{\phi^2}{1+\phi^2}\delta_{1-\phi}(\mathit{\Delta})I + \frac{1}{1+\phi^2}\delta_{\phi}(\mathit{\Delta})I, \quad \mathit{\Delta} \in \mathfrak{B}(\mathbb{R}).$$

Then we have

$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$ for $k = 2, 3$ and $T^k \neq \int_{\mathbb{R}} x^k F(\mathrm{d}x)$ for $k \in \mathbb{N} \backslash \{2, 3\}$.

Clearly, F is not a spectral measure.

Main results

To provide C^*-algebra generalization of spectral measure characterization, it seems natural to replace the set $\{x^k \colon k \in \Omega\}$ by a subset G of a unital commutative C^*-algebra $\mathscr{A}$ satisfying the following condition:

for every representation $\pi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ and every regular semispectral measure $F\colon \mathfrak{B}(\mathfrak{M}_{\mathscr{A}}) \to \boldsymbol{B}(\mathcal{H})$,

$$\pi(a) = \int_{\mathfrak{M}_{\mathscr{A}}} \widehat{a}\,\mathrm{d}F \;\; \forall a \in G \implies F \text{ is the spectral measure of } \pi,$$

,

where we denote $\mathfrak{M}_{\mathscr{A}}$, $\mathfrak{B}(\mathfrak{M}_{\mathscr{A}})$ and $\widehat{a}\colon \mathfrak{M}_{\mathscr{A}} \to \mathbb{C}$ the maximal ideal space of $\mathscr{A}$, the σ-algebra of Borel subsets of $\mathfrak{M}_{\mathscr{A}}$ and the Gelfand transform of $a \in \mathscr{A}$, respectively.

Main results

Theorem (P.P., J. Stochel, 2024)

A finite or countably infinite set G of generators of a unital commutative C^*-algebra $\mathscr{A}$ is hyperrigid if and only if for every representation $\pi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ and every regular semispectral measure $F\colon \mathfrak{B}(\mathfrak{M}_{\mathscr{A}}) \to \boldsymbol{B}(\mathcal{H})$,

$$\pi(a) = \int_{\mathfrak{M}_{\mathscr{A}}} \widehat{a}\,\mathrm{d}F \ \ \forall a \in G \implies F \text{ is the spectral measure of } \pi,$$

Main results

Theorem (P.P., J. Stochel, 2024)

Let $X \subset \mathbb{C}$ be compact and G be a countable set of generators of $C(X)$. Then the following are equivalent:

(i) *G is hyperrigid,*

(ii) *for every Hilbert space $\mathcal{H}$, every normal operator $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subseteq X$ and every semispectral measure $F \colon \mathfrak{B}(X) \to \boldsymbol{B}(\mathcal{H})$,*

$$f(T) = \int_X f \, \mathrm{d}F \ \forall f \in G \implies F \text{ is the spectral measure of } T,$$

Main results

Theorem (P.P., J. Stochel, 2024)

(iii) *for all Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$, and all normal operators $T \in \boldsymbol{B}(\mathcal{H})$, $N \in \boldsymbol{B}(\mathcal{K})$ with spectra in X,*

$$f(T) = Pf(N)|_{\mathcal{H}} \ \forall f \in G \implies PN = NP,$$

where P stands for the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$,

(iv) *for every Hilbert space $\mathcal{H}$, every sequence $\{T_n\}_{n=1}^{\infty} \subseteq \boldsymbol{B}(\mathcal{H})$ of subnormal operators such that $\sigma(T_n) \subseteq X$ for all $n \in \mathbb{N}$, and every normal operator $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subseteq X$,*

$$(\text{WOT}) \lim_{n\to\infty} f(T_n) = f(T) \ \forall f \in G$$
$$\implies (\text{SOT}) \lim_{n\to\infty} f(T_n) = f(T) \ \forall f \in C(X),$$

Main results

Theorem (P.P., J. Stochel, 2024)

(iv) *for every Hilbert space* $\mathcal{H}$, *every sequence* $\{T_n\}_{n=1}^{\infty} \subseteq \boldsymbol{B}(\mathcal{H})$ *of subnormal operators such that* $\sigma(T_n) \subseteq X$ *for all* $n \in \mathbb{N}$, *and every normal operator* $T \in \boldsymbol{B}(\mathcal{H})$ *with* $\sigma(T) \subseteq X$,

$$(\textsc{wot})\lim_{n\to\infty} f(T_n) = f(T) \;\; \forall f \in G \implies (\textsc{sot})\lim_{n\to\infty} f(T_n) = f(T) \;\; \forall f \in C(X),$$

(v) *for every Hilbert space* $\mathcal{H}$, *every sequence* $\{T_n\}_{n=1}^{\infty} \subseteq \boldsymbol{B}(\mathcal{H})$ *of normal operators such that* $\sigma(T_n) \subseteq X$ *for all* $n \in \mathbb{N}$, *and every normal operator* $T \in \boldsymbol{B}(\mathcal{H})$ *with* $\sigma(T) \subseteq X$, *implication in* (iv) *holds.*

Main results

Theorem (P.P., J. Stochel, 2024)

Let $\varXi$ be a set satisfying the following condition:

$\{(p,q),(r,r)\} \subseteq \varXi \subseteq \mathbb{N}^2$ *for some* $p,q,r \in \mathbb{N}$ *such that* $p<q$, $\frac{p+q}{2} < r$, *and* $\gcd\{m-n\colon (m,n)\in\varXi\} = 1$. *Let* $\mathscr{A}$ *be the unital commutative* C^**-algebra generated by a single element* $t \in \mathscr{A}$. *Then the set*

$$G = \{t^{*m}t^n\colon (m,n)\in\varXi\}$$

is hyperrigid.

Main results

Theorem (P.P., J. Stochel, 2024)

Let $P \in \boldsymbol{B}(\mathcal{K})$ *be the orthogonal projection of a Hilbert space* $\mathcal{K}$ *onto its closed subspace* $\mathcal{H}$, $T \in \boldsymbol{B}(\mathcal{H})$ *and* $N \in \boldsymbol{B}(\mathcal{K})$ *be normal operators and* $p, q, r \in \mathbb{N}$ *be such that* $p < q$ *and* $\frac{p+q}{2} < r$*.*
Suppose $\{(p, q), (r, r)\} \subseteq \Omega \subseteq \mathbb{N}^2$ *and*

$$T^{*m} T^n = P N^{*m} N^n|_{\mathcal{H}}, \quad (m, n) \in \Omega.$$

Then

(i) $T^{*n} T^n = P N^{*n} N^n|_{\mathcal{H}}$ *for all* $n \in \mathbb{Z}_+$,

(ii) P *commutes with* N^d, *where* $d := \gcd\{m - n \colon (m, n) \in \Omega\}$; *equivalently*: P *commutes with* $|N|$ *and* U^d, *where* $N = U|N|$ *is the unitary polar decomposition of* N.

Let $J \subseteq \mathbb{R}$ be an interval. A continuous function $f\colon J \to \mathbb{R}$ is said to be **operator monotone** if $f(A) \leqslant f(B)$ for any two selfadjoint operators $A, B \in \boldsymbol{B}(\mathcal{H})$ such that $A \leqslant B$ and the spectra of A and B are contained in J.

Theorem (F. Hansen 1980, M. Uchyiama 1993)

Let $A \in \boldsymbol{B}(\mathcal{H})$ be a positive operator, $T \in \boldsymbol{B}(\mathcal{H})$ be a contraction and $f\colon [0,\infty) \to \mathbb{R}$ be a continuous operator monotone function such that $f(0) \geqslant 0$. Then

$$T^* f(A) T \leqslant f(T^* A T). \tag{3}$$

Moreover, if f is not an affine function and T is an orthogonal projection such that $T \neq I_{\mathcal{H}}$, then equality holds in (3) if and only if $TA = AT$ and $f(0) = 0$.

Lieb-Ruskai inequality

Theorem (Lieb, Ruskai 1974)

Let $R \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ *and let* $\Phi \colon \boldsymbol{B}(\mathcal{K}) \to \boldsymbol{B}(\mathcal{H})$ *be the positive linear map defined by*

$$\Phi(X) = R^* X R, \quad X \in \boldsymbol{B}(\mathcal{K}).$$

Then for all $A, B \in \boldsymbol{B}(\mathcal{K})$, *the net* $\{\Phi(A^*B)(\Phi(B^*B) + \varepsilon I)^{-1}\Phi(B^*A)\}_{\varepsilon>0}$ *is convergent in the strong operator topology as* $\varepsilon \downarrow 0$ *and*

$$\Phi(A^*A) \geqslant (\textsc{sot})\lim_{\varepsilon\downarrow 0}\Phi(A^*B)(\Phi(B^*B) + \varepsilon I)^{-1}\Phi(B^*A).$$

Main results

Theorem (P.P., J. Stochel, 2024)

Let $T \in \boldsymbol{B}(\mathcal{H})$ be a normal operator on a Hilbert space $\mathcal{H}$ and $\varXi$ be as in previous theorem. Assume that $F\colon \mathfrak{B}(\mathbb{C}) \to \boldsymbol{B}(\mathcal{H})$ is a semispectral measure with compact support. Then the following conditions are equivalent:

(i) *F is the spectral measure of T,*

(ii) *$T^{*m}T^n = \int_{\mathbb{C}} \bar{z}^m z^n F(\mathrm{d}x)$ for all $(m, n) \in \varXi$.*

Main results

Theorem (P.P., J. Stochel, 2024)

Let X be a nonempty compact subset of $\mathbb{C}$, let $\{T_k\}_{k=1}^{\infty} \subseteq \boldsymbol{B}(\mathcal{H})$ be a sequence of subnormal operators such that $\sigma(T_k) \subseteq X$ for every $k \in \mathbb{N}$, and let $T \in \boldsymbol{B}(\mathcal{H})$ be a normal operator with $\sigma(T) \subseteq X$. Suppose that Ξ is as in previous theorem and

$$(\mathrm{WOT})\lim_{k\to\infty} T_k^{*m} T_k^{n} = T^{*m} T^{n}, \quad (m,n) \in \Xi.$$

Then

$$(\mathrm{SOT})\lim_{k\to\infty} f(T_k) = f(T), \quad f \in C(X).$$

In particular,

$$(\mathrm{SOT})\lim_{k\to\infty} T_k^{*m} T_k^{n} = T^{*m} T^{n}, \quad (m,n) \in \mathbb{Z}_+^2.$$

Characterisation of operator convex functions

Theorem (F. Hansen, G. K. Pedersen 1982)

If f is a continuous, real function on the half-open interval $[0, \alpha)$ (with $\alpha \leqslant \infty$), the following conditions are equivalent:

(i) f *is operator convex and* $f(0) \leqslant 0$,

(ii) $f(PXP) \leqslant Pf(X)P$ *for every projection* P *and every self-adjoint* X *with spectrum in* $[0, \alpha)$,

(iii) $f(A^*XA) \leqslant A^*f(X)A$ *for all* A *with* $\|A\| \leqslant 1$ *and every self-adjoint* X *with spectrum in* $[0, \alpha)$,

(iv) $f(A^*XA + B^*YB) \leqslant A^*f(X)A + B^*f(Y)B$ *for all* A, B *with* $A^*A + B^*B \leqslant I$ *and all* X, Y *with spectra in* $[0, \alpha)$.

Main results

Theorem (P.P., J. Stochel 2024)

Suppose that X is a nonempty compact subset of $\mathbb{C}$, μ is a Borel probability measure on X and G is a finite or countably infinite set of generators of $C(X)$ such that $\int_X f \,\mathrm{d}\mu = 0$ for every $f \in G$. Fix an integer $n \geqslant 2$. Then the following statements are equivalent:

(i) *G is hyperrigid,*

(ii) *for all Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$, and all normal operators $T \in \boldsymbol{B}(\mathcal{H})$, $N \in \boldsymbol{B}(\mathcal{K})$ with spectra in X,*

$$f(T) = Pf(N)|_{\mathcal{H}} \ \forall f \in G \implies PN = NP,$$

where P stands for the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$,

Main results

Theorem (P.P., J. Stochel 2024)

(ii) *for all Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$, *all normal operators* $T \in \boldsymbol{B}(\mathcal{H})$ *and* $N \in \boldsymbol{B}(\mathcal{K})$ *with spectra in* X, *and every contraction* $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$,

$$f(T) = A^* f(N) A \ \forall f \in G \implies NA = AT,$$

(iii) *for all Hilbert spaces* $\mathcal{H}$, $\mathcal{K}_1, \ldots, \mathcal{K}_n$, *all normal operators* $T \in \boldsymbol{B}(\mathcal{H})$, $N_1 \in \boldsymbol{B}(\mathcal{K}_1), \ldots, N_n \in \boldsymbol{B}(\mathcal{K}_n)$ *with spectra in* X, *and all operators* $A_1 \in \boldsymbol{B}(\mathcal{H}, \mathcal{K}_1), \ldots, A_n \in \boldsymbol{B}(\mathcal{H}, \mathcal{K}_n)$ *such that* $\sum_{i=1}^n A_i^* A_i \leqslant I$,

$$f(T) = \sum_{i=1}^n A_i^* f(N_i) A_i \ \forall f \in G \implies N_i A_i = A_i T \ \forall i = 1 \ldots n.$$

Main results

Theorem (P.P., J. Stochel 2024)

Moreover, the following hold: if the set G is hyperrigid and there are T, S, A (resp., T, $S_1, \ldots, S_n$, $A_1, \ldots, A_n$) satisfying the if-clause with $A^*A \lneqq I$ (resp., $\sum_{i=1}^n A_i^* A_i \lneqq I$), then μ is the Dirac measure at some (uniquely determined) point of X.

Main results

Theorem (P.P., J. Stochel, 2024)

Let G be a finite or countably infinite set of generators of a unital C^*-algebra $\mathscr{A}$. Suppose there exists a state ψ such that $G \subseteq \ker\psi$ (i.e., G is not rigid at 0). Then the following conditions are equivalent:

(i) *G is hyperrigid,*

(ii) *for all Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, all representations $\pi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ and $\rho\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{K})$ and every contraction $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$,*

$$\pi(g) = A^*\rho(g)A \ \ \forall g \in G \implies A\pi(a) = \rho(a)A \ \ \forall a \in \mathscr{A}.$$

Main results

Theorem (P.P., J. Stochel, 2024)

If ψ is a character of $\mathscr{A}$, then (i) *is equivalent to any of the following conditions*:

(iii) *for all Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, all representations $\pi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ and $\rho\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{K})$ and every contraction $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$,*

$$\pi(g) = A^*\rho(g)A \ \ \forall g \in G \implies \pi(a) = A^*\rho(a)A \ \ \forall a \in \ker\psi,$$

(iv) *for every Hilbert space $\mathcal{H}$, every representation $\pi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ and every CCCP map $\Psi\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$,*

$$\pi(g) = \Psi(g) \ \ \forall g \in G \implies \pi(a) = \Psi(a) \ \ \forall a \in \ker\psi.$$

Moreover, if G is hyperrigid and there exists a triplet (π, ρ, A) for which A is non-isometric, then ψ is a character of $\mathscr{A}$.

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Tack för uppmärksamheten!