Landscape Functions And Agmon-Type Estimates For Magnetic Schrödinger Operators On Infinite Graphs

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Outline

Motivation and Background

Magnetic Schrödinger Operators

Main Results

Summary and Outlook

Agmon's Estimates (Continuous Setting)

Agmon 1982

Each eigenpair (E, φ) of $H = -\Delta + V$ on $\mathbb{R}^d$ satisfies

$$\int_{\mathbb{R}^d} \mathrm{e}^{2(1-\epsilon)\rho_E(x,\tilde{K}_E)} |\varphi(x)|^2 \, \mathrm{d}x < \infty$$

where

$$\rho_E(x, y) := \inf_{\gamma \in AC_{x,y}} \int_0^1 \sqrt{(V(\gamma(t)) - E)_+} |\dot{\gamma}(t)| \, \mathrm{d}t,$$

$$\tilde{K}_E := \{x : V(x) \leq E\}.$$

- connected components of $\tilde{K}_E$ $\equiv$ classically allowed region ("potential wells")
- describes tunnelling between potential wells
- on metric graphs: **Harrell–Maltsev 2018–20**

Schmuel Agmon (1922–2025)

The Landscape Function Idea

Landscape function

Given $H = -\Delta + V$ and a source $\rho \geq 0$, solve

$$Hu = \rho.$$

If $\rho = 1$: u is the *torsion function*:

- **Saint-Venant 1850s:** Modelling of torsional rigidity problems
- **Pólya 1948:** Proof of a Faber–Krahn-type result for $\|u\|_1$.

Why is it a "landscape function"?

Proposition (Filoche–Mayboroda 2012, van den Berg 2012)

Let (E, φ) be an eigenpair of $H = -\Delta + V$ with Dirichlet BC on μ, $V \geq 0$, s.t. $\varphi \in L^\infty$. Then

$$\frac{|\varphi(x)|}{\|\varphi\|_\infty} \leq |E| \cdot (H^{-1}\mathbf{1})(x).$$

Proof.

$\varphi = EH^{-1}\varphi$, hence

$$|\varphi| \leq |E||H^{-1}\varphi| \leq |E|H^{-1}|\varphi| \leq |E|H^{-1}(\|\varphi\|_\infty \mathbf{1}) \leq |E|\|\varphi\|_\infty H^{-1}\mathbf{1} \qquad \square$$

- φ can only be supported in $\{x : |E| \cdot (H^{-1}\mathbf{1})(x) \leq 1\}$: $H^{-1}\mathbf{1}$ can predict localization regions *before* computing eigenfunctions.

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- φ can only be supported in $\{x : |E| \cdot (H^{-1}\mathbf{1})(x) \leq 1\}$: $H^{-1}\mathbf{1}$ can predict localization regions *before* computing eigenfunctions.
- **Hoskins–Quan–Steinerberger 2023:** Extensions to magnetic Schrödinger operators
- **M. 2024:** Extensions to operators with dominated inverse
- Can be extended from $\mathbf{1}$, $\varphi \in L^\infty$ to $0 < \rho$ s.t. $\varphi \leq \|\varphi\|_{\infty,\rho}\rho$.

A reminder of domination theory on Hilbert lattices

$T \in \mathcal{L}(H)$ is *dominated* by $T^\sharp$ if $|Tf| \le T^\sharp |f|$ for all $f \in H$.

$(T(t)^\sharp)_{t\ge 0} \subset \mathcal{L}(H)$ is the *modulus semigroup* of $(T(t))_{t\ge 0} \subset \mathcal{L}(H)$ if for each $t > 0$

- $T(t)^\sharp$ dominates $T(t)$
- $T(t)^\sharp$ is dominated by any other positivity preserving operator $\tilde{T}$ that also dominates $T(t)$.

Examples

- $(\mathrm{e}^{t\Delta^{\mathrm{D}}})_{t\ge 0}$ *is dominated by* $(\mathrm{e}^{t\Delta^{\mathrm{N}}})_{t\ge 0}$.
- *A positivity preserving semigroup is its own modulus semigroup.*
- **Kato 1972, Simon 1977:** $(\mathrm{e}^{-t((i\nabla - A)^2 + V)})$ *is dominated by* $(\mathrm{e}^{-t(-\Delta + \mathrm{Re}\, V)})$
- **Schaefer 1974:** *Not every operator has a modulus; if $M \equiv m_{ij}$ is a matrix, then the modulus semigroup of $(\mathrm{e}^{tM})_{t\ge 0}$ is generated by*

$$\tilde{M} \equiv \tilde{m}_{ij} := \begin{cases} \mathrm{Re}\, m_{ii} & i = j, \\ |m_{ij}| & i \ne j \end{cases}$$

Landscape theory for $-\Delta\varphi + V\varphi = E\varphi$

- If V is large $\Rightarrow$ u is small $\Rightarrow$ ρ/u is large: ρ/u as *effective potential*:useful e.g. in Anderson models
- **Filoche–Mayboroda 2012:** behavior of the φ at various energy levels E is controlled to a good extent by the landscape function $u := (-\Delta + V)^{-1}\mathbf{1}$
- **Arnold–David–Filoche–Jerison–Mayboroda 2019:**

$$-\Delta\varphi + V\varphi = E\varphi$$

is equivalent to

$$-\frac{1}{u^2}\nabla\left(u^2\nabla\psi\right) + \frac{1}{u}\psi = E\psi, \qquad \psi := \frac{\varphi}{u}$$

Agmon metric wrt potential wells $K_E := \{x : \frac{1}{u(x)} \leq E\}$ instead of $\tilde{K}_E = \{x : V(x) \leq E\}$:

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Agmon metric wrt potential wells $K_E := \{x : \frac{1}{u(x)} \leq E\}$ instead of $\tilde{K}_E = \{x : V(x) \leq E\}$: useful if V is very rough (Anderson!)

Agmon-type estimates vs landscape theory

Theorem (Arnold–David–Filoche–Jerison–Mayboroda 2019)

Every eigenpair (E,φ) of H on $\Omega \subset \mathbb{R}^d$ with Neumann BCs satisfies, for any $\alpha \in (0,1)$,

$$\int_\Omega (H^{-1}\mathbf{1})^2 \left|\nabla\left(\frac{\chi e^{\alpha h}\varphi}{H^{-1}\mathbf{1}}\right)\right|^2 \mathrm{d}x + (1-\alpha)^2 \int_\Omega \left(\frac{1}{H^{-1}\mathbf{1}} - E\right)_+ (\chi e^{\alpha h}\varphi)^2 \,\mathrm{d}x$$
$$\leq (1+2\alpha)e^{2\alpha}\left(\|V\|_\infty - E\right) \int_{\{x:0\leq h(x)\leq 1\}} \varphi^2 \,\mathrm{d}x$$

Here:

- $w_E(x) := \left(\frac{1}{H^{-1}\mathbf{1}(x)} - E\right)_+$
- $\rho_E(x,y) := \inf_\gamma \int_0^1 \sqrt{w_E(\gamma(t))}|\dot\gamma(t)|\,\mathrm{d}t$
- $h(x) := \rho_E(x, K_E)$
- $\chi(x) := 1 \wedge h(x)$

Extension to matrices

Real B is a <u>M-matrix</u> if $b_{ij} \leq 0$ for all $i \neq j$ and $\operatorname{Re} E \geq 0$ for all eigenvalues E.
Examples: discrete Laplacians on un/directed graphs (by Geršgorin)

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Theorem (Filoche–Mayboroda–Tao 2021)

If B is a real, symmetric, non-singular $n \times n$ M-matrix with connectivity W

$$\sum_{k=1}^{n} \phi_k^2 \mathrm{e}^{\frac{2\rho_E(k,K_E)}{\sqrt{W}}} \left(\frac{1}{(B^{-1}\mathbf{1})_k} - E \right)_+ \leq W \max_{1\leq i,j\leq n} |b_{ij}|$$

Here:

- ρ_E shortest path pseudometric wrt
  $w(x,y) := \ln \left(1 + \sqrt{\frac{\sqrt{v_{i_\ell} v_{i_{\ell+1}}}}{|b_{i_\ell i_{\ell+1}}|}} \right)$
- $v_i := \left(\frac{1}{B^{-1}\mathbf{1}_i} - E \right)_+$
- $K_E := \left\{ i : \frac{1}{(B^{-1}\mathbf{1})_i} \leq E \right\}$
- $W := \max_i \#\{j \neq i : b_{ij} \neq 0\} \wedge \max_j \#\{i \neq j : b_{ij} \neq 0\}$

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Proof:
Dissipativity for double commutator of B, $\mathrm{e}^{\alpha \rho_E(k,K_E)}$

Infinite Weighted Graphs

Definition

Let X be a finite or countably infinite set, $m : X \to [0, \infty)$. A graph over (X, m) is a function $\mu \colon X \times X \to [0, \infty)$ such that

(i) $\mu_{xy} = \mu_{yx}$ *for all* $x, y \in X$,

(ii) $\mu_{xx} = 0$ *for all* $x \in X$,

(iii) $\sum_{y \in X} \mu_{xy} < \infty$ *for every* $x \in X$.

Hilbert space:

$$\ell^2_m(X) := \{f : X \to \mathbb{C} \mid \sum_{x \in X} m(x)|f(x)|^2 < \infty\}$$

The standard Dirichlet form on graphs

Consider

$$\mathcal{Q}(f) := \frac{1}{2} \sum_{x,y \in X} \mu_{xy} |f(x) - f(y)|^2$$

with domain

$$D(\mathcal{Q}) := h^1(X) := \{f \in \ell^2_m(X) \mid \mathcal{Q}(f) < \infty\}$$

Keller–Lenz 2012: $\mathcal{Q}$ is a closed form.

$\mathcal{L}$ is the self-adjoint operator on $\ell^2_m(X)$ associated with $\mathcal{Q}$.

$\mathcal{L}$ has pure point spectrum if

- **Bonnefont–Golenia–Keller 2015**; **Keller–Lenz–Wojciechowski 2021:** X is *weakly sparse* and

$$\sup_{X' \subset X} \inf_{x \in X \setminus X'} \deg_\mu(x) = \infty.$$

- **Georgakopoulos–Haeseler–Keller–Lenz–Wojciechowski 2015:** $m(X) < \infty$ and $\mathrm{diam}_{\mu^{-1}}(X) < \infty$;
- **Hofmann–Kennedy–M.–Plümer 202?** $m(X) < \infty$ and $h^1(X) \hookrightarrow \ell^p_m(X)$ for some $p \in (2, \infty)$.

Elementary: $h^1(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$

Varopoulos 1985: $h^1(\mathbb{Z}^d) \hookrightarrow \ell^{\frac{2d}{d-2}}(\mathbb{Z}^d)$ for $d \geq 3$
(for $\mu \equiv 1$, $m \equiv 1$)

Theorem (Keller–Pogorzelski 2025)

If (E, φ) is a generalised eigenpair of $H := \mathcal{L} + V$. If $H - E$ satisfies a Hardy inequality with weight w, and if (X, ρ) is complete, then for $|\alpha| \approx 0$ there holds

$$\varphi \in \ell^2_{we^{-\alpha\rho}}(X) \quad \Rightarrow \quad \varphi \in \ell^2_{we^{\alpha\rho}}(X).$$

Here

$$\rho(x,y) := \inf_{\gamma=\{x_1,\ldots,x_\ell\}\in\mathcal{P}_{x,y}} \sum_{i=1}^{\ell} \sqrt{\frac{w(x_i)\wedge w(x_{i+1})}{\deg_{\max}}} \wedge 1$$
$$\rho(x) := \rho(x,o), \quad o \in X$$

Schrödinger Operators on Graphs

Formal magnetic Schrödinger operator

For $\alpha_{xy} = -\alpha_{yx} \in [0, 2\pi)$:

$$(\mathcal{L}_\alpha + V)f(x) = \frac{1}{m(x)}\left(\sum_y \mu_{xy}\left(f(x) - \mathrm{e}^{i\alpha_{xy}} f(y)\right) + V(x)f(x)\right), \quad f \in C_c(X)$$

Studied by Lieb–Loss 1993, ..., Colin de Verdière–Torki-Hamza–Truc 2011, Golénia 2014, ...

Magnetic Schrödinger operators on infinite graphs

Consider the quadratic form

$$Q_{\mathcal{L}_\alpha+V}(f)=\frac{1}{2}\sum_{x,y}\mu_{xy}|f(x)-\mathrm{e}^{i\alpha_{xy}}f(y)|^2+\sum_x V(x)|f(x)|^2,\qquad f\in C_c(X).$$

- Friedrichs extension of $(\mathcal{L}_\alpha+V)_{|C_c(X)}$ is a self-adjoint, positive semidefinite operator on $\ell^2_m(X)$.
- **Ouhabaz 1996** + **Lenz–Schmidt–Wirth 2021** + discrete diamagnetic inequality $|f(x)-\mathrm{e}^{i\alpha_{xy}}f(y)|\geq|f(x)-f(y)|$ $\rightsquigarrow$ $(\mathrm{e}^{-t(\mathcal{L}_\alpha+V)})_{t\geq 0}$ is dominated by $(\mathrm{e}^{-t(\mathcal{L}+V)})_{t\geq 0}\ldots$

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- or by Schaefer 1974 + approximation
- Laplace transform: If $\inf\sigma(\mathcal{L}_\alpha+V)>0$, then $(\mathcal{L}_\alpha+V)^{-1}$ is dominated by $(\mathcal{L}+V)^{-1}$

Setup: $\inf \sigma(\mathcal{L} + V) > 0$, $0 \leq \rho \in \ell^\infty(X)$, $\rho \not\equiv 0$.
Let $(A_n)_{n\in\mathbb{N}}$ be a monotone exhaustion of X by finite sets.

Landscape function & potential well

$$u(x) := \lim_{n\to\infty} (\mathcal{L} + V)^{-1}(\mathbf{1}_{A_n}\rho)(x) \ \in \ (0, +\infty].$$

$$K_{E,\rho} := \left\{ x \in X : \tfrac{\rho(x)}{u(x)} \leq E \right\}$$

- The limit is independent of the exhaustion (by monotonicity).
- If $\rho \in \ell^2_m(X)$: $u = (\mathcal{L} + V)^{-1}\rho$ in the usual sense.

Towards an Agmon pseudometric

Definition (Frank–Lenz–Wingert 2014)

A pseudo-metric $\sigma : X \times X \to [0,\infty)$ is an <u>*intrinsic metric*</u> *for a graph μ over (X, m) if*

$$\sum_y \mu_{xy}\sigma^2(x,y) \le m(x) \qquad \forall x \in X.$$

Examples

- *combinatorial shortest path metric, if $m = 1$ and $\sup_x \deg(x) < \infty$*
- *combinatorial shortest path metric, if $m(x) := \sum_y \mu_{xy}$*
- *shortest path metric wrt edge weight*

$$w_s(x,y) := \big(\mathrm{Deg}(x) \vee \mathrm{Deg}(y)\big)^{-1/2} \wedge s, \qquad s \in (0,\infty],$$

where $\mathrm{Deg}(x) = \dfrac{1}{m(x)} \displaystyle\sum_y \mu_{xy}.$

The Agmon Pseudometric

Definition

Given the landscape function $u = u_\rho$ and $E > 0$, set

$$\nu(x) := \left(\frac{\rho(x)}{u(x)} - E\right)_+ .$$

For an intrinsic metric σ, let $d = d_{\sigma,\rho,E}$ shortest path pseudometric wrt

$$w(x,y) := \ln\Big(1 + \sqrt[4]{\nu(x)\nu(y)}\,\sigma(x,y)\Big),$$

- $d_\sigma = 0$ on $K_{E,\rho}$ (where $\nu \equiv 0$).

Main Theorem: Agmon-Type Decay Estimate

Theorem (M.–Schonlau–Täufer 202?)

Let $\inf \sigma(\mathcal{L}+V) > 0$, $0 \le \rho \in \ell^\infty(X)$, σ *be an intrinsic metric for* (X, m). *For every eigenpair* (E, φ) *of* $\mathcal{L}_\alpha + V$ *with* $\varphi \in \ell^2_{\deg}(X)$, *any* $0 < \gamma < 1$

$$(1-\gamma^2)\sum_{x\in X}\left(\frac{\rho(x)}{(\mathcal{L}+V)^{-1}\rho(x)} - E\right)_+ |\varphi(x)|^2 \mathrm{e}^{2\gamma d(x,\, K_{E,\rho})} m(x) \le \|\varphi\|^2_{\ell^2_{\deg}(\partial K_{E,\rho})}.$$

- Once $(\mathcal{L}+V)^{-1}\rho$ is computed for $\alpha = 0$, the estimate holds for *all* magnetic potentials α simultaneously.

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- Once $(\mathcal{L}+V)^{-1}\rho$ is computed for $\alpha = 0$, the estimate holds for *all* magnetic potentials α simultaneously.
- It is convenient to take $\gamma \approx 0$ (resp., $\gamma \approx 1$) if $d(x, K_{E,\rho})$ predominantly small (resp., large).
- $\varphi \in \ell^2_{\deg}(X)$ if
  - $\frac{\deg}{m} \in \ell^\infty$;
  - X is weakly sparse (planar, tree, ...);
  - $h^1(X) \hookrightarrow \ell^\infty$ and $\{x : \deg(x) \geq Cm(x)\}$ is finite, for some $C > 0$.

Proof: First Key Lemma

Lemma (Hardy inequality)

If $\inf \sigma(\mathcal{L} + V) > 0$*, then for all* $f \in \mathcal{D}(Q)$*:*

$$\sum_{x\in X} \frac{\rho(x)}{(\mathcal{L}+V)^{-1}\rho(x)} |f(x)|^2 m(x) \ \leq \ Q_{\Delta_\alpha + V}(f).$$

Earlier Hardy inequalities on graphs:

- **Keller–Pinchover–Pogorzelski 2018**
- **Krejčiřík–Štampach 2022**
- **Fischer 2024**

Proof: Second Key Lemma

Lemma (upper bound)

For any eigenpair (E, φ) *of* $\mathcal{L}_\alpha + V$ *with* $\varphi \in \ell^2_{\deg}(X)$ *and* $g \in \ell^\infty(X)$ *real-valued:*

$$Q_{\Delta_\alpha + V}(\varphi g) \ \le \ \frac{1}{2}\sum_{x,y} \mu_{xy} |\varphi(x)\varphi(y)| |g(x) - g(y)|^2 + E \sum_x |\varphi(x)|^2 g(x)^2 m(x).$$

Proof: Combining and the Cut-off Argument

- Lemma 1 + Lemma 2 $\rightsquigarrow$

$$\sum_x \Big(\tfrac{\rho}{(\mathcal{L}+V)^{-1}\rho} - E\Big)|\varphi|^2 g^2 m \;\le\; \frac{1}{2}\sum_{x,y}\mu_{xy}|\varphi(x)\varphi(y)|\,|g(x)-g(y)|^2.$$

- Introduce cut-off function:

$$g_N(x) := \mathbf{1}_{\{x\notin K_{E,\rho}\}}\,\mathrm{e}^{\gamma(d(x,K_{E,\rho})\wedge N)}, \qquad 0<\gamma<1,\; N\in\mathbb{N}.$$

- Because $d_\sigma(x,y) \le \ln(1+\sqrt[4]{m(x)m(y)}\,\sigma(x,y))$ and $(1+t)^\gamma \le 1+\gamma t$:

$$|g_N(x)-g_N(y)| \;\le\; \gamma\, g_N(y)\,\sqrt[4]{m(x)m(y)}\,\sigma(x,y) \qquad \forall x,y\notin K_{E,\rho}$$

$$\rightsquigarrow \quad |g_N(x)-g_N(y)|^2 \;\le\; \gamma^2\, g_N(x)g_N(y)\,\sqrt{m(x)m(y)}\,\sigma^2(x,y)$$

- Claim follows by Cauchy–Schwarz + Fatou.

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$$(1-\gamma^2)\sum_{x\in X} |\varphi(x)|^2 \mathrm{e}^{2\gamma d(x,\, K_{E,\rho})} \left(\frac{\rho(x)}{(\mathcal{L}+V)^{-1}\rho(x)} - E\right)_+ m(x) \leq \|\varphi\|^2_{\ell^2_{\text{deg}}(\partial K_{E,\rho})}.$$

where d = *shortest path pseudometric wrt* $w(x,y) := \ln\left(1 + \sqrt[4]{\nu(x)\nu(y)}\,\sigma(x,y)\right)$

Theorem (Filoche–Mayboroda–Tao 2021)

If B *is a real, symmetric non-singular* $n \times n$ *M-matrix with connectivity* W

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⚠ If $(\mathcal{L} + V)^{-1}\rho(x) = \infty$: $x \in K_{E,\rho}$ for *all* $E \geq 0$ $\rightsquigarrow$ no exponential decay ☹

Finiteness criterion for landscape function

Let $\inf \sigma(\mathcal{L} + V) > 0$ and

$$G(x, y) := [(\mathcal{L} + V)^{-1} \mathbf{1}_{\{y\}}](x)$$

Theorem

$(\mathcal{L} + V)^{-1}\rho(x) < \infty$ for all $x \in X$ if $G(x, \cdot) \in \ell^1(X)$ for all $x \in X$.

The converse holds if $\inf_z \rho(z) > 0$.

For $\Omega \subset \mathbb{R}^d$: **Poggi 2024**

Finiteness criterion

Proposition

Let $M := \sup_{x\in X} \deg(x) < \infty$*, and let* $a := \inf \sigma(\Delta + V) > 0$*.*
If for all $x \in X$ *there exists a finite set* $F_x \subset \mathbb{N}$ *and* $C = C_x > 0$ *such that for some* $0 < \kappa < 1 + a/M$

$$|S_N(x)| \leq C\kappa^N \qquad \forall N \in F_x^{\complement},$$

then $G(x, \cdot) \in \ell^1(X)$ *for all* $x \in X$*.*

Proof.

Combes–Thomas estimate from **Aizenman–Warzel 2015** □

Examples (with $\mu \equiv 1$, $m \equiv 1$)

1. On any graph of sub-exponential volume growth, $G(x, \cdot) \in \ell^1(X)$, hence
$$u(x) < \infty \qquad \text{for all } x \in X.$$
E.g.: on $\mathbb{Z}^d$, $|S_n(x)| \sim CN^{d-1} \rightsquigarrow u < \infty$
2. Tree with root o where each vertex at level N has exactly 2^{N+1} children:
$$|S_N(o)| = \prod_{k=1}^{N} 2^k = 2^{N(N+1)/2}.$$
For $V \equiv 1$, $\sum_{y \in X} G(o, y) = \infty$, so $u(o) = \infty$. By connectedness,
$$u(x) = \infty \qquad \text{for all } x \in X.$$

Alternative metric for dense Graphs

⚠ $d(x,\ K_{E,\rho}) = 0$ if $x \notin K_{E,\rho}$, $x \sim K_{E,\rho}$ $\rightsquigarrow$ no exponential decay ☹

Alternative pseudometric d_δ

- $\delta_0 := \sup\{\delta : \sum_y \mu_{xy}^{1-\delta} < \infty\ \forall x \in X\}$; pick $\delta \in (0, \delta_0)$.
- Let σ be an intrinsic metric wrt $\mu^{1-\delta}$.
- d_δ shortest path metric wrt edge weight

$$w_\delta(x,y) := \ln\left(1 + \mu_{xy}^{-\delta}\, h(m(x))\, h(m(y))\, \sigma^2(x,y)\right), \quad h(t) := \begin{cases} \sqrt{t} & t > 0 \\ 1 & t = 0 \end{cases}$$

Theorem (M.–Schonlau–Täufer 202?)

Let $\inf \sigma(\mathcal{L} + V) > 0$, $0 \le \rho \in \ell^\infty(X)$.
For every eigenpair (E, φ) *of* $\mathcal{L}_\alpha + V$ *with* $\varphi \in \ell^2_{\deg}(X) \cap \ell^2(X)$ *and any* $0 < \gamma < \sqrt{5} - 2$

$$(1-4\gamma-\gamma^2)\sum_x \left(\frac{\rho}{(\mathcal{L}+V)^{-1}\rho} - E\right)_+ |\varphi|^2 \mathrm{e}^{2\gamma d_\delta(x,K_{E,\rho})} \le \|\varphi\|^2_{\ell^2_{\deg}(\partial K_{E,\rho})} + \|\varphi\|^2_{\ell^2(\partial K_{E,\rho})}.$$

Example: Fractional Laplacian on $\mathbb{Z}^d$ ($m \equiv 1$, $\mu \equiv 1$)

- $\Delta^\theta_{\mathbb{Z}^d}$ defined via spectral calculus for $\theta \in (0,1)$;
- $\mathbb{Z}^d$ sparse, but $\Delta^\theta_{\mathbb{Z}^d}$ full matrix with $|\Delta^\theta_{\mathbb{Z}^d}(x,y)| \sim |x-y|^{-(d+2\theta)}$ (**Gebert–Rojas-Molina 2020**; **Keller–Post–Sabri–Täufer 2025**)
- $\sigma(\Delta^\theta_{\mathbb{Z}^d}) = [0,(4d)^\theta]$, but $\inf \sigma(\sigma(\Delta^\theta_{\mathbb{Z}^d}))$ if $V > 0$
- If V strong enough: $\Delta^\theta_{\mathbb{Z}^d} + V$ has pure point spectrum (**Aizenman–Molchanov 1993**) and

$$\sum_{x\in\mathbb{Z}^d} \left(\frac{1}{(\Delta^\theta_{\mathbb{Z}^d}+V)^{-1}\mathbf{1}} - E \right)_+ |\varphi|^2 \mathrm{e}^{2\gamma d_\delta(x,K_E)} \le C_d \|\varphi\|^2_{\ell^2(\partial K_{E,\rho})}.$$

- Let $\Lambda_L \subset \mathbb{Z}^d$ box, $V : \Lambda_L \to [0,1]$: then $\inf \sigma(\Delta^\theta_{\Lambda_L} + V) > 0$ and

$$\sum_{x\in\Lambda_L} \left(\frac{1}{(\Delta^\theta_{\Lambda_L}+V)^{-1}\mathbf{1}} - E \right)_+ |\varphi|^2 \mathrm{e}^{2\gamma d(x,K_E)} \le C_{\theta,d} \|\varphi\|^2_{\ell^2(\Lambda_L)}.$$

for $C_{\theta,d}$ independent on L and V!

Summary

- Landscape function for $\alpha = 0$ (defined as a monotone limit) governs all $\Delta_\alpha + V$ simultaneously.
- If $u \neq \infty$: ρ/u is an effective potential for $\mathcal{L}_\alpha + V$; any eigenfunction φ of $\mathcal{L}_\alpha + V$ with eigenvalue E will decay exponentially away from $K_{E,\rho} := \{x : \frac{u}{\rho} \leq E\}$.
- Bounds independent of the graph connectivity.

Outlook

- **David–Filoche–Mayboroda 2021:** Landscape law for IDSs.
  Can this be extended to infinite graphs?
- **Carmona–Simon 1981:** $|\varphi(x)| \lesssim \mathrm{e}^{-\gamma d(x,\tilde{K}_E)} \|\varphi\|_{2,*}$
  Steinerberger 2023: $|\varphi(x)| \leq \mathrm{e}^{-\rho_E(x,\tilde{K}_E)} \|\varphi\|_\infty$,
  where $\rho_E =$ shortest path pseudometric wrt $w(x,y) := \log\left(1 + \frac{(V(x_i)-E)_+}{\deg(x_i)}\right)$
  Can this be obtained for infinite graphs?
- Sharper estimates in terms of the oscillation of the magnetic field

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Thank you for your attention!