To Birman-Krein-Vishik Theory. Solution to the Birman problem

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Krein Extension theory of a symmetric operator $A \geqslant 0$

Extreme extensions

Let $A \geqslant 0$ be a closed non-negative densely defined symmetric op-r in $\mathfrak{H}$. By Friedental-Stone-Friedrichs theorem the set $\mathrm{Ext}_A(0,\infty)$ of all nonnegative selfadjoint ex-s $\widetilde{A} = \widetilde{A}^*$ of A is nonempty. Complete theory of ext-s of $A \geqslant 0$ was built by Krein [Kre47]. In particular, he proved there that the set $\mathrm{Ext}_A(0,\infty)$ contains the maximal (the Friedrichs) and the minimal (the Krein-von Neumann) ext-s $\widehat{A}_F$ and $\widehat{A}_K$. They are uniquely characterized by means of either ineq-s: $\mathfrak{t}_{\widehat{A}_K} \leqslant \mathfrak{t}_{\widetilde{A}} \leqslant \mathfrak{t}_{\widehat{A}_F}$ or inequal-s

$$(\widehat{A}_F + a)^{-1} \leqslant (\widetilde{A} + a)^{-1} \leqslant (\widehat{A}_K + a)^{-1}, \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty), \quad a > 0. \tag{1}$$

The Krein theory has substantially been completed by Vishik [Vis52] and Birman [Bir56]. Nowadays following Alonso–Simon it is called the Birman-Krein-Vishik theory (see [AloSim80]). An important contrib-n to the theory was made also by Grubb [Gru68,Gru74]

Reduced Krein extension

In what follows we systematically use the orthogonal decomposition:

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_1^{\perp}, \quad \text{where} \quad \mathfrak{H}_1 := \operatorname{ran}(I+A) \quad \text{and} \quad \mathfrak{H}_1^{\perp} = \ker(I+A^*) =: \mathfrak{N}_{-1}(A \tag{2}$$

Next, assuming A to be positive definite, $A \geqslant m_A > 0$, Krein characterized the extension $\widehat{A}_K$ as follows:

$$\widehat{A}_K = A^* \upharpoonright \operatorname{dom}(\widehat{A}_K), \quad \operatorname{dom}(\widehat{A}_K) = \operatorname{dom} A \dotplus \mathfrak{N}_0 \tag{3}$$

where $\mathfrak{N}_0 = \mathfrak{N}_0(A) := \ker A^*$. Hence Krein's ext-n $\widehat{A}_K$ admits repres-n:

$$\widehat{A}_K = \widehat{A}'_K \oplus (\mathbb{O} \upharpoonright \mathfrak{N}_0) \text{ where } \quad \widehat{A}'_K := \widehat{A}_K \upharpoonright \mathfrak{M}_0, \quad \mathfrak{M}_0 := \mathfrak{N}_0^{\perp} = \operatorname{ran} A. \tag{4}$$

The op-r $\widehat{A}'_K$ is called the reduced Krein extension and is important in the theory. If $M(z)$ is the Weyl function cor-ing to a BTr for A^*, then:

$$\operatorname{dom}(\widehat{A}_K) = \ker(\Gamma_1 - M(-0)\Gamma_0).$$

If $M(0) = 0$, then this f-la turns into Krein's for-la (3).

List of the problems that are tried to be discussed.

1. Around Krein's theorem. Its inversion.
2. Comparison of eigen-s of $P_1(I+A)^{-1}$ and $(I+\widehat{A}'_K)^{-1}$, P_1– orthopr-n on $\mathfrak{H}_1$ in $\mathfrak{H}$. Their trace ideal properties. Complete vers. of Krein Th-m.
3. Comparison of asymptotics of $P_1(I+A)^{-1}$ and $(I+\widehat{A}'_K)^{-1}$.
4. Abstract version of Alonso-Simon problem:
   comparison of eigenv-s asymptotics of $\widehat{A}_F$ and $\widehat{A}'_K$.
5. Improvement of the Grubb result on LSB-problem.

6. Birman's Problem for an abstract symmetric oper-r $A \geqslant 0$:

Is it true that the following implication holds

$$(I+A)^{-1} \in \mathfrak{S}_\infty \implies (I+\widehat{A}_F)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})? \tag{5}$$

7. Explicit solution to the Birman problem for Schrödinger oper-r.

Krein's theorem

We start with the following result discovered by Krein [Kre47].

Krein's theorem

Let A be a positive definite symmetric op-r in $\mathfrak{H}$. If its Friedrichs' ext-n $\widehat{A}_F$ has compact inverse, then so is the reduced Krein ext-n $\widehat{A}'_K$:

$$(I_{\mathfrak{H}} + \widehat{A}_F)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \implies (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}_p(\mathfrak{M}_0), \quad p \in (0, \infty] \tag{6}$$

It happens that existence of inv-se to (6) impl-n is rel-d to Birman prob.

Theorem 1. Improvement of the Krein Th-m

Let A be a pos. definite sym-c op-r in $\mathfrak{H}$ and let P_1 be the orthoproj-n in $\mathfrak{H}$ onto $\mathfrak{H}_1$. Then for any sym-ly normed ideal $\mathfrak{S}$ the fol. eq-ce holds:

$$P_1(I_{\mathfrak{H}} + A)^{-1} \in \mathfrak{S}(\mathfrak{H}_1) \iff (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}(\mathfrak{M}_0), \tag{7}$$

In partic-r, equiv-ce (7) holds for Neumann-Schatten ideals $\mathfrak{S} = \mathfrak{S}_p$, $p \in (0, \infty]$, ideals Σ_p ($\iff s_n(T) = O(n^{-1/p})$), Σ_p^0, ($p \in (0, \infty)$) etc.

Corollaries from Theorem 1.

AGMST-2010 Result

(Ashbaugh, Gesztesy, Mitrea, Shterenberg, and Teschl, MN-2010)
For any (not neces-ly semib-ded) ext-n $\widetilde{A} = \widetilde{A}^*$ of $A \geqslant 0$ the following impl-n holds: $(\widetilde{A} - iI)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longrightarrow (I + \widehat{A}'_K)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$

Proof. Indeed, the following impl-n is obvious
$(\widetilde{A} - iI)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Longrightarrow P_1(A - iI)^{-1} \in \mathfrak{S}_p(\mathfrak{H}_1)$. Now Th.1 completes.

Does the inverse to the Krein impl-n hold?

$$(I_{\mathfrak{H}} + \widehat{A}_F)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \Longleftarrow \quad (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}_\infty(\mathfrak{M}_0)???, \tag{8}$$

Negative sol-n to the Birman problem + Th.1 $\Longrightarrow$ the answer is "NO" , i.e. implication (8) does not hold in general.

Does the inverse to the Krein implic-n (5) hold if $(\widehat{A}_F)^{-1} \in \mathfrak{S}_\infty$?

Let $p > p_1 > 0$. Then there exists a closed densely defined non-negative symmetric operator A in $\mathfrak{H}$ such that its Friedrichs extension $\widehat{A}_F$ and the reduced Krein extension $\widehat{A}'_K$ possess the following property:

$$(\widehat{A}'_K)^{-1} \in \mathfrak{S}_{p_1} \quad \text{while} \quad (\widehat{A}_F)^{-1} \in \mathfrak{S}_p \setminus \mathfrak{S}_{p_1},$$

$$(\widehat{A}'_K)^{-1} \in \Sigma_{p_1} \quad \text{while} \quad (\widehat{A}_F)^{-1} \in \Sigma_p \setminus \Sigma_{p_1},$$

and the implication converse to (6) does not hold for a finite $p > 0$.

Th 2. Compar-n of eigen-s of $P_1(I+A)^{-1}$ and $(I+\widehat{A}'_K)^{-1}$

Let $A \geqslant m_A > 0$ satisfy cond-s of Th.1 and let $c = \|D_V\| \leqslant 1$. Then:
(i) There is an isometry U_1 from $\mathfrak{H}_1$ onto $\mathfrak{M}_0$ such that:

$$U_1^*(I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} U_1 \leqslant c^2 P_1(I_{\mathfrak{H}} + A)^{-1} \leqslant P_1(I_{\mathfrak{H}} + A)^{-1}; \tag{9}$$

(ii) In the class $P_1 \mathrm{Ext}_A(0,\infty) \upharpoonright \mathfrak{H}_1$ the reduced Krein extension $\widehat{A}'_K$ satisfies the following maximal property ($\widetilde{A} \in \mathrm{Ext}_A(0,\infty)$):

$$U_1^*(I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} U_1 \leqslant c^2 P_1(I_{\mathfrak{H}} + A)^{-1} \upharpoonright \mathfrak{H}_1 = c^2 P_1(I_{\mathfrak{H}} + \widetilde{A})^{-1} \upharpoonright \mathfrak{H}_1 \leqslant P_1(I_{\mathfrak{H}} + \widetilde{A} \tag{10}$$

(iii) If $P_1(I_{\mathfrak{H}} + A)^{-1}$ is compact, then the eigenv-s of oper-s $(I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1}$ and $P_1(I_{\mathfrak{H}} + A)^{-1}$ arranged in the decreasing order satisfy the ineq-s

$$\lambda_n\left((I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1}\right) \leqslant c^2 \lambda_n\left(P_1(I_{\mathfrak{H}} + A)^{-1}\right) = c^2 \lambda_n\left(P_1(I_{\mathfrak{H}} + \widetilde{A})^{-1} \upharpoonright \mathfrak{H}_1\right), \tag{11}$$

Continuation:

Estimates for eigenvalues of $\widehat{A}_F$, $P_1(I+A)^{-1}$, and $\widehat{A}'_K$

(iv) If, in addition, $(I_{\mathfrak{H}}+\widehat{A}_F)^{-1}$ is compact, then the constant c in inequalities (7) and (11) is one, $c=1$. Besides, in this case the eigenvalues arranged in the increasing order satisfy the inequalities

$$\lambda_n(\widehat{A}_F) \leqslant \lambda_n\left(\left(P_1(I_{\mathfrak{H}}+A)^{-1}\right)^{-1}\right) - 1 \leqslant \lambda_n(\widehat{A}'_K), \qquad n\in\mathbb{N}. \tag{12}$$

Estimates for eigenvalues of $\widehat{A}_F$ and $\widehat{A}'_K$

The fol-ing estimates were first estab-ed by Alonso-Simon, [OT,1980]

$$\lambda_n(\widehat{A}_F) \leqslant \lambda_n(\widehat{A}'_K), \qquad n\in\mathbb{N},$$

and then by K. Schmüdgen (Book, 2012).

The ineq-s are strict if A is simple and the spectrum of $\widehat{A}_F$ is simple!

Comparison of asymptotics of eigen-s for A^{-1} and $(\widehat{A}'_K)^{-1}$

Theorem 2. Asymptotics do not coincide in general

Let $A \geqslant m_A > 0$, $P_1 A^{-1} \in \mathfrak{S}_\infty$ and let $\widehat{A}'_K$ be as above. Then:

$$\mathbf{1}.\ \lambda_n\big(P_1 A^{-1}\big) = a n^{-\frac{1}{p}}(1 + o(1)) \implies (\widehat{A}'_K)^{-1} \in \Sigma_p,\ \Big(n^{\frac{1}{p}}\lambda_n\big(\widehat{A}'_K\big)^{-1} = O(1 \tag{13}$$

2. For any $b \in (0, 1]$ exists $A \geqslant m_A$ such that the fol-ing equiv-ce holds

$$\lambda_n\big(P_1 A^{-1}\big)\big) = a n^{-1/p}(1 + o(1)) \implies \lambda_n\big(\widehat{A}'_K\big)^{-1} = abn^{-1/p}(1 + o(1))$$

3. There exists $A \geqslant m_A > 0$ such that the asympt-c in (15) does not imply existence of the asymptotic for $\{\lambda_n\big(\widehat{A}'_K\big)^{-1}\}$, i.e.

$$\underline{\lim}_{n\to\infty} n^{1/p}\lambda_n\big(\widehat{A}'_K\big)^{-1} \neq \overline{\lim}_{n\to\infty} n^{1/p}\lambda_n\big(\widehat{A}'_K\big)^{-1}. \tag{14}$$

The problem is: When the limits in (14) coincide?

Theorem 3. Coincidence of asymptotics of eigen-s of A^{-1} and $(\widehat{A}'_K)^{-1}$

The power asymptotic behaviour of the eigenvalues of operators A^{-1} and $(\widehat{A}'_K)^{-1}$ coincide, i.e. the following equiv-nce holds as $n \to \infty$:

$$\lambda_n\big(P_1 A^{-1})\big) = an^{-1/p}(1 + o(1)) \iff \lambda_n\big((\widehat{A}'_K)^{-1}\big) = an^{-1/p}(1 + o(1)) \tag{15}$$

if and only if $\big(\widehat{A}_F\big)^{-1}$ is compact.

Relation to the Birman problem

Both results (Theorem 2 and 3) give another negative solution to the Birman problem on the abstract level.

Abstract Alonso-Simon Prob: Compar-n eigen. of $\widehat{A}_F$, $\widehat{A}'_K$

Proposition. Different asymptotics of $\widehat{A}_F$ and $\widehat{A}'_K$

Let $p > p_1 > 0$ and $a, a_1 > 0$ be two pairs of positive numbers. Then there exists a closed densely defined symmetric operator $A \geqslant m_A > 0$ in $\mathfrak{H}$ such that eigenv-s of $\widehat{A}'_K$ grow faster(!!!) than the eigenv-s of $\widehat{A}_F$:

$$\lambda_n(\widehat{A}_F) = an^{1/p}(1 + o(1)) \text{ and } \lambda_n(\widehat{A}'_K) = a_1 n^{1/p_1}(1 + o(1)). \qquad (16)$$

Theorem 3A. Let A and $\widehat{A}'_K$ be as above.

Let $(I + \widehat{A}_F)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$, $p \in (0, \infty)$, and $a \geqslant 0$. If in adition

$$P_{-1}(I + \widehat{A}_F)^{-1} \upharpoonright \mathfrak{N}_{-1}(A) \in \Sigma_p^0(\mathfrak{N}_{-1}(A)), \qquad (17)$$

then the following equivalence holds as $n \to \infty$:

$$\lambda_n(\widehat{A}_F)) = a^{-1} n^{1/p}(1 + o(1)) \iff \lambda_n(\widehat{A}'_K) = a^{-1} n^{1/p}(1 + o(1)).$$

Vishik Theorem. Resolvable Ext-s.

The fol-ing result by Vishik [?] complements Krein's Theorem.

Vishik's Theorem (Its "symmetric part")

Let A be a sym-c (not neces-ly semib-ed) op-r in $\mathfrak{H}$ with $\ker A = \{0\}$. If the inverse op-r A^{-1} is bounded (compact), then there exists a selfad-t ext-n $\widetilde{A} = (\widetilde{A})^* \in \mathrm{Ext}_A$ with a bounded (compact) inverse.

Further development)

Vishik discovered this res-t for dual pairs (adjoint pairs). An ext-n $\widetilde{A}$ with the bounded (compact) inverse $(\widetilde{A})^{-1}$ is called resolvable (compl-ly res-le). In [?] Vishik employed such an exten-n to prove two for-s that replaced clas. 1-st and 2-nd J.von Neumann for-las and turned out to be more suitable for invest-n BVPs for elliptic equations. Later on Hörmander [?] substantially used resolvable extensions for description resolvable (completely resolvable) extensions for dual pairs of general partial differential operators with constant coefficients.

Birman Problem and its solution. Abstract level

Somehow close to Vishik's Th-m is the following Birman problem:

Birman's Problem.

Let A be a closed nonnegative densely defined operator in $\mathfrak{H}$. Is it true that the following implication holds

$$(I + A)^{-1} \in \mathfrak{S}_\infty \implies (I + \widehat{A}_F)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})? \qquad (18)$$

In [?] we announced the following result giving a complete negative solution to the Birman problem on abstract level.

Theorem 4. Abstract solution to the Birman Problem.

Let $R = R^* \geqslant 0$ be arbitrary self-adjoint operator in $\mathfrak{H}$. Then there exists a nonnegative symmetric operator $A \geqslant 0$ in $\mathfrak{H}$ with $n_\pm(A) = \infty$ and the compact preresolvent $(I + A)^{-1} \in \mathfrak{S}_\infty$ and such that the operators $\widehat{A}_F$ and R are unitarily equivalent, $\widehat{A}_F \sim R$.

Complement to Vishik Theorem

Negative solution to the Birman Problem allows to complement Vishik's Theorem ?? as follows.

Proposition Complementing Vishik Th-m.

Let A be a non-negative symmetric operator in $\mathfrak{H}$ such that $\ker A = \{0\}$. Let also $A^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}_1, \mathfrak{H})$ although $\widehat{A}_F^{-1} \notin \mathfrak{S}_\infty(\mathfrak{H})$. Then every selfadjoint extension $\widetilde{A} \in \mathrm{Ext}_A$ of A with a discrete spectrum is not lower semibounded.

LSB property and generalization of Grubb's result

Following Vicik and Birman, with any extension $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ such that $-1 \notin \sigma_p(\widetilde{A})$ one associates a selfadjoint operator B :

$$B^{-1} := (I + \widetilde{A})^{-1} - (I + \widehat{A}_F)^{-1} \ \left(= (B^{-1})^*\right) \in \mathcal{C}(\mathfrak{N}_{-1}). \qquad (19)$$

The operator B is called a boundary op-r for $\widetilde{A}$ and put $\widetilde{A} = A_B (= A_B^*)$. Note also that B^{-1} is bounded, $B^{-1} \in \mathcal{B}(\mathfrak{N}_{-1})$ exactly if $-1 \in \rho(\widetilde{A})$. In his seminal paper [Bir56] Birman discovered the fol-ing rel-on between semibounded exten-s and their boundary relations/oper-s. We present a gener-n of Birman's result using a concept of a bound. triple.

Birman's Proposition

Let $A \geqslant m_A > 0$ be an op-r in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a b-ry triple for A^* with $A_0 = \widehat{A}_{\mathrm{F}}$ and $A_1 = \widehat{A}_{\mathrm{K}}$. Let also $B = B^* \in \mathcal{C}(\mathcal{H})$ and let $A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. Then $A_B = A_B^*$ and the fol-ing impl-n holds

$$A_B = A_B^* \text{ is lower semi-bounded} \quad \Longrightarrow \quad B = B^* \text{ is lower semi-bounded.} \qquad (20)$$

LSB- Property

In other words, the boundary relation (operator) Θ of the realization $\widetilde{A} = A_\Theta = A_\Theta^*$ inherits the property of A_Θ to be lower semibounded. The converse stat-t is not valid in general (see ex-es in [DM91]). This circums-ce motivates introducing the following def-n.

Definition of LSB-Property

Let $A \geqslant 0$ be a non-negative symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* such that $A_0 = \widehat{A}_F$. We say that A satisfies LSB-property [a] if for every $B = B^* \in \mathcal{C}(\mathcal{H})$ the following equivalence holds:

$$A_B = A_B^* \quad \text{is lower semi-bounded} \quad \Longleftrightarrow \quad B = B^* \quad \text{is lower semi-b-ded.} \tag{21}$$

[a] abbreviation of lower semi-boundedness

Grubb's sufficient result ensuring the LSB-property

First suff-nt cond-n for a sym. oper. $A \geqslant m_A > 0$ to meet LSB-prop-ty was discovered by G. Grubb [Gru68]. Her result reads as follows.

Grubb's LSB-Theorem

Let $A \geqslant m_A > 0$ be a posit. definite sym. oper in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a b-ry triple for A^* with $A_0 = \widehat{A}_F$ and $A_1 = \widehat{A}_K$. Then the equiv-ce (21) holds provided that the op-r $(\widehat{A}_{\mathrm{F}})^{-1}$ is compact.

Another proof of this result was obtained by Gorbachuk and Mikhailets in [GorMik,DAN76] and later also in [DM,JFA91].
Note that due to Grubb's Theorem the equivalence (21) holds for non-negative elliptic boundary value problem in bounded domain with sufficiently smooth boundary. Indeed, in this case the Friedrichs' realization $\widehat{A}_{\mathrm{F}}$ of the corresponding minimal operator A is just the Dirichlet realization and has compact inverse. The validity of equivalence (21) for certain elliptic boundary value problems in unbounded domains was investigated in [Gru12].

Improvement of the Grubb result

Main LSB-Theorem (that improves Grubb's result)

Let A be a symmetric non-negative operator in $\mathfrak{H}$, $A \geqslant m_A \geqslant 0$, and let $\Pi_F = \{\mathcal{H}, \Gamma_0^F, \Gamma_1^F\}$ be a boundary triple for A^* such that $A_0 = \widehat{A}_F$. Let as above P_{-1} be the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{N}_{-1} = \mathfrak{H}_2$. Then A meets the LSB-property, i.e. the equivalence (21) holds, provided that

$$P_{-1}(I_{\mathfrak{H}} + \widehat{A}_{\mathrm{F}})^{-1} \upharpoonright \mathfrak{N}_{-1} \in \mathfrak{S}_\infty(\mathfrak{N}_{-1}) \qquad (22)$$

Does the Main LSB-Th. stronger than Grubb's result? I.

Th-m: Oper-s satisf. cond. (22) while the inverse $(\widehat{A}_{\mathrm{F}})^{-1}$ is not compact

Let $A \geqslant 0$ be a symmetric oper-r in $\mathfrak{H}$ with $n_{\pm}(A) = \infty$ and such that its Friedrichs' exten-n $\widehat{A}_F$ meets cond-n (22). Then:

(i) The ess-l spectra of oper-s $(I + \widehat{A}_F)^{-1}$ and $P_1(I + A)^{-1}$ are related as

$$\sigma_{\mathrm{ess}}((I + \widehat{A}_F)^{-1}) = \sigma_{ess}(P_1(I + A)^{-1}) \cup \{0\}. \tag{23}$$

(ii) Assume that, in place of (22), the foll-ing stronger cond-n holds

$$P_{-1}(I_{\mathfrak{H}} + \widehat{A}_{\mathrm{F}})^{-1} \upharpoonright \mathfrak{N}_{-1} \in \mathfrak{S}_1(\mathfrak{N}_{-1}). \tag{24}$$

Then the ac-parts of oper-s $(I + \widehat{A}_F)^{-1}$ and $P_1(I + A)^{-1}$ are unit. equi-t:

$$((I+\widehat{A}_F)^{-1})^{ac} \sim (P_1(I+A)^{-1})^{ac} (\Longrightarrow \sigma_{ac}((I+\widehat{A}_F)^{-1}) = \sigma_{ac}(P_1(I+A)^{-1})). \tag{25}$$

Does the Main LSB-Th. stronger than Grubb's result? II.

Next we show that for each sym-c oper. $A \geqslant 0$ whose Friedrichs' ext-n $\widehat{A}_F$ meets cond. (22) there exists a sym. oper. $S \geqslant 0$ close (in a sense) to A (see formula (26) below) and with $\widehat{S}_F$ also satisfying (22) and having arbitrary prescribed essential (or ac) spectrum.

Th-m: Oper-s satisf. cond. (22) while the op-r $(\widehat{A}_{\mathrm{F}})^{-1}$ is not compact. II

Let $A \geqslant m_A > 0$ be a sym-c oper. with $n_\pm(A) = \infty$ and its Friedrichs' ext-n $\widehat{A}_F$ meets cond-n (22). Let $R = R^* \in \mathcal{B}(\mathfrak{H}_1)$ be any op-r with

$$0 \leqslant R \leqslant 2(1 + m_A)^{-1} < 2 \quad \text{and} \quad ran(R) = ran(P_1(I_{\mathfrak{H}} + A)^{-1}).$$

Then: (i) There exists a sym-c op-r $S \geqslant m_A$ such that:

$$\left[(I + \widehat{S}_F)^{-1} - (I + \widehat{S}_K)^{-1}\right] \upharpoonright \mathfrak{N}_{-1}(S) = \left[(I + \widehat{A}_F)^{-1} - (I + \widehat{A}_K)^{-1}\right] \upharpoonright \mathfrak{N}_{-1}(\tag{26}$$

Besides, $\widehat{S}_{\mathrm{F}}$ satisfies condition (22) and

$$\sigma_{ess}((I_{\mathfrak{H}} + \widehat{S}_F)^{-1}) = 2^{-1}\sigma_{ess}(R) \cup \{0\}. \tag{27}$$

Continuation

Continuation

(ii) If, in addition, cond-n (24) holds, i.e.

$$P_{-1}(I_{\mathfrak{H}} + \widehat{A}_{\rm F})^{-1} \upharpoonright \mathfrak{N}_{-1} \in \mathfrak{S}_1(\mathfrak{N}_{-1}), \tag{28}$$

then the operators S and $\widehat{S}_F$ in (i) meet the following conditions

$$P_{-1}(I_{\mathfrak{H}} + \widehat{S}_{\rm F})^{-1} \upharpoonright \mathfrak{N}_{-1} \in \mathfrak{S}_1(\mathfrak{N}_{-1}(S)) \quad \text{and} \quad (26), \tag{29}$$

and

$$\left((I_{\mathfrak{H}} + \widehat{S}_F)^{-1}\right)^{ac} \sim 2^{-1} R^{ac}. \tag{30}$$

In particular, $\sigma_{ac}((I_{\mathfrak{H}} + \widehat{S}_F)^{-1}) = 2^{-1}\sigma_{ac}(R)$.

Explicit Solution to Birman Prob-m

Denote by $-\Delta$ the Laplace operator in $L^2(\mathbb{R}^n)$,
The operator $-\Delta$ is selfadjoint and $-\Delta \geqslant 0$. In the case $n \in \{2, 3\}$, for every closed subset Y in $\mathbb{R}^n$, the restriction $-\Delta_Y$ of $-\Delta$ to the domain

$$\mathrm{dom}(-\Delta_Y) = W_Y^{2,2}(\mathbb{R}^2) := \{u \in W^{2,2}(\mathbb{R}^n) : u(y) = 0,\ y \in Y\} \tag{31}$$

is well defined. The operator Δ_Y is closed and symmetric, and its selfadjoint ext-s are called the Laplace oper-s with point interactions. Spectral proper-s of these op-s are inves-d in numerous papers.

Theorem 5. Sol-n to the Birman Problem

Let $\{\Pi_k\}_{k\in\mathbb{N}}$ be a partition of $\mathbb{R}^2$ by squares $\Pi_k = \Pi(y_k; d_k)$ centered at y_k and with sides of length d_k parallel to the coordinate axes. Assume also that $q \geqslant 0$, cond-n (??) is met, and the set of centers $Y := \{y_k\}_1^\infty$ is discrete and

$$\lim_{k\to\infty} d_k = 0. \tag{32}$$

(i) $A := I - \Delta_Y + q$ $(\geqslant I)$ is a closed densely defined symmetric op-r with $n_\pm(A) = \infty$. Moreover, the set $C_0^\infty(\mathbb{R}^2 \setminus Y)$ is a core for A;
(ii) The inverse operator A^{-1} is compact;
(iii) The reduced Krein extension $\widehat{A}'_K = (\widehat{A}'_K)^*$ is positive definite, $m_{\widehat{A}'_K} \geqslant m_A = 1$, and its spectrum is discrete, i. e., $(\widehat{A}'_K)^{-1}$ is compact;
(iv) The Friedrichs extension $\widehat{A}_F$ of A is $\widehat{A}_F = I - \Delta + q$. In particular,

$$\sigma(\widehat{A}_F) = \sigma_{\rm ess}(\widehat{A}_F) = [1, \infty). \tag{33}$$

Corollary. Ac-spectrum

Let the conditions of Theorem 5 be met, and let $|q(x)| = O(|x|^{-(1+\varepsilon)})$ for some $\varepsilon > 0$. Then the spectrum of the operator $\widehat{A}_F$ is purely absolutely continuous,

$$\sigma(\widehat{A}_F) \cap [1, \infty) = \sigma_{\mathrm{ac}}(\widehat{A}_F) = [1, \infty) \qquad \text{and} \qquad \sigma_s(\widehat{A}_F) = \emptyset. \tag{34}$$

Remark on n-dimensional Schrödinger operators.

The results is extended to the case of Schrödinger operators in $L^2(\mathbb{R}^n)$. The proof involves certain properties of the (m, p)-capacity of sets defined by means of the Sobolev space $W^{m,p}(\mathbb{R}^n)$.

New effects for extensions A^2.

Examples of operators $A = I - \Delta_Y + q$, where $-\Delta_Y$ is defined by (31), make it possible to complement Theorem 5 by considering curious examples arising as extensions $(\widehat{A}^2)_F$ and $(\widehat{A}_F)^2$ of A^2. They demonstrate the significantly different nature of their spectra.

Extensions $(\widehat{A}^2)_F$ and $(\widehat{A}_F)^2$ of A^2 with strange properties

A symme-c oper-r A^2 does not give a solution to the Birman Problem!

Proposition on spec. prop-s of $(\widehat{A}^2)_F$ and $(\widehat{A}_F)^2$.

Under the assumptions of Theorem 5 the following assertions hold:
(i) A^2 is a closed densely defined and positive definite symmetric operator with $n_{\pm(A)} = \infty$ and with $C_0^\infty(\mathbb{R}^2 \setminus Y)$ being its core.
(ii) The operator A^{-2} is compact.
(iii) The Friedrichs extension $(\widehat{A}^2)_F$ of A^2 is $A^*A (\in \mathrm{Ext}_{A^2}(0,\infty))$. Moreover, its spectrum $\sigma((\widehat{A}^2)_F) = \sigma(A^*A)$ is discrete, i. e., $((\widehat{A}^2)_F)^{-1}$ is compact. So, A^2 does not give a solution to the Birman Prob.
(iv) The operator $(\widehat{A}_F)^2 (\in \mathrm{Ext}_{A^2}(0,\infty))$ is not discrete. Moreover,

$$\sigma((\widehat{A}_F)^2) = \sigma_{\mathrm{ess}}((\widehat{A}_F)^2) = [1,\infty).$$

Thank you for your patience!