Carl von Ossietzky Universität Oldenburg

Eigenvalue asymptotics for strong δ-interactions supported on curves with corners

Noah Körner, 22.06.2026

Carl von Ossietzky Universität Oldenburg

Definition of the main object

Definition

Let $\alpha > 0$ and $\Gamma \subset \mathbb{R}^2$ be a closed, injective, reasonably regular curve. Define the sesquilinear form

$$h_\alpha^\Gamma(u,v) = \int_{\mathbb{R}^2} \langle \nabla u, \nabla v \rangle \mathrm{d}x - \alpha \int_\Gamma \overline{u} v \, \mathrm{d}S, \quad D(h_\alpha^\Gamma) = H^1(\mathbb{R}^2)$$

and denote by H_α^Γ the self-adjoint operator in $L^2(\mathbb{R}^2)$ that is generated by h_α^Γ which is the Schrödinger operator with a delta interaction on Γ.

Some general properties of H_α^Γ are known (Behrndt, Brasche, Exner, Holzmann, Kuperin,...):

- H_α^Γ is shown to be the limit of Schrödinger operators with attractive regular potentials strongly localized near Γ
- $\mathrm{spec}_{ess}(H_\alpha^\Gamma) = [0, +\infty)$ for any $\alpha \in \mathbb{R}$
- $\mathrm{spec}(H_\alpha^\Gamma) = [0, +\infty)$ for any $\alpha \leq 0$
- there are finitely many discrete eigenvalues for any $\alpha > 0$

We are interested in the eigenvalue asymptotics of H_α^Γ as $\alpha \to +\infty$.

Carl von Ossietzky Universität Oldenburg

Motivation

Theorem (Exner, Yoshitomi, 2002)

Assume that Γ is an injective, closed C^4 curve and set

- $\ell :=$ *the length of* Γ
- $k :=$ *the curvature of* Γ
- $T = -\frac{\partial^2}{\partial s^2} - \frac{k^2}{4}$ *with domain* $D(T) = \{u \in H^2(0,\ell) \mid u(0) = u(\ell),\ u'(0) = u'(\ell)\}$

Then for any $n \in \mathbb{N}$ there exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha(n)$ the nth eigenvalue of H^Γ_α exists and has an asymptotic expansion of the form

$$E_n(H^\Gamma_\alpha) = -\frac{\alpha^2}{4} + E_n(T) + \mathcal{O}\left(\frac{\log \alpha}{\alpha}\right) \qquad \text{as } \alpha \to \infty$$

The proof uses a tubular neighbourhood of Γ that can only be constructed for sufficiently smooth Γ. What happens if Γ is not smooth?

Curves with corners

We call Γ a curve with corners if it is injective, closed, and piecewise smooth. To be more precise, Γ has a decomposition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_M$ where

- Γ_j are C^3-smooth with regular ends such that
- Γ_{j-1} and Γ_j meet at the corners A_j at an angle $2\theta_j$ with
- $\theta_j \in (0, \pi) \setminus \{\frac{\pi}{2}\}$.

We set

- $\ell_j := |\Gamma_j|$
- $k_j :=$ curvature of Γ_j

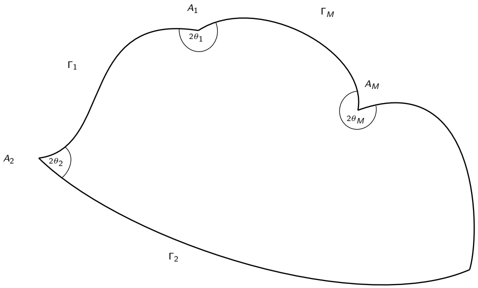

Figure: Curve with corners

Schrödinger Operator with a strong δ interaction on a half angle Γ_θ

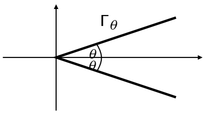

Figure: The curve $\Gamma_\theta \subset \mathbb{R}^2$, two straight lines meeting at an angle 2θ at the origin, $\theta \in (0, \pi)$.

Proposition (Exner, Nemcova, 2003; Benhellal, Körner, Pankrashkin, 2026)

- $H_\alpha^{\Gamma_\theta}$ is unitary equivalent to $\alpha^2 H_1^{\Gamma_\theta}$
- The essential spectrum of $H_\alpha^{\Gamma_\theta}$ is $[-\frac{\alpha^2}{4}, \infty)$
- Set $\kappa(\theta)$ as the number of discrete Eigenvalues of $H_\alpha^{\Gamma_\theta}$ then
  - $1 \leq \kappa(\theta) < \infty$ for any $\theta \in (0, \pi) \setminus \{\frac{\pi}{2}\}$
  - $(0, \frac{\pi}{2}) \ni \theta \mapsto \kappa(\theta)$ is non-increasing
- For any Eigenfunction ψ_α of $H_\alpha^{\Gamma_\theta}$, $\theta \in (0, \frac{\pi}{2})$ there exist $B, b > 0$ such that

$$\int_{\mathbb{R}^2} e^{b\alpha|x|}\left(\frac{1}{\alpha^2}|\nabla\psi_\alpha|^2 + |\psi_\alpha|^2\right)dx \leq B\|\psi_\alpha\|^2_{L^2(\mathbb{R}^2)}$$

Carl von Ossietzky Universität Oldenburg

Main result (simple version)

- $\mathcal{E} :=$ disjoint union of $\{E_n(H_1^{\Gamma_{\theta_j}}), \quad n = 1, \dots, \kappa(\theta_j)\}$ for $j = 1, \dots, M$
- $\mathcal{K} := \kappa(\theta_1) + \dots + \kappa(\theta_M)$
- $\mathcal{E}_n :=$ nth Element of $\mathcal{E}$ numbered in non-decreasing order

Carl von Ossietzky Universität Oldenburg

Main result (simple version)

- $\mathcal{E} :=$ disjoint union of $\{E_n(H_1^{\Gamma_{\theta_j}}), \quad n = 1, \ldots, \kappa(\theta_j)\}$ for $j = 1, \ldots, M$
- $\mathcal{K} := \kappa(\theta_1) + \cdots + \kappa(\theta_M)$
- $\mathcal{E}_n :=$ nth Element of $\mathcal{E}$ numbered in non-decreasing order

Theorem ((Benhellal, Körner, Pankrashkin, 2026))

Let $n \in \{1, \ldots, \mathcal{K}\}$. Then there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ the nth discrete eigenvalue of H_α^Γ exists and has the following asymptotic expansion:

$$E_n(H_\alpha^\Gamma) = \alpha^2(\mathcal{E}_n + \alpha^{-\frac{2}{3}}), \qquad \text{as } \alpha \to \infty.$$

Assuming that all angles θ_j lie in $[\frac{\pi}{4}, \frac{3\pi}{4}] \setminus \{\frac{\pi}{2}\}$ then for any $n \in \mathbb{N}$ there exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha(n)$ the $\mathcal{K} + n$th eigenvalue of H_α^Γ exists and has an asymptotic expansion of the form

$$E_{\mathcal{K}+n}(H_\alpha^\Gamma) = -\frac{\alpha^2}{4} + E_n\Big(\bigoplus_{j=1}^{M}\Big(D_j - \frac{k_j^2}{4}\Big)\Big) + \mathcal{O}\Big(\frac{\log \alpha}{\sqrt{\alpha}}\Big) \qquad \text{as } \alpha \to \infty$$

where D_j is the Dirichlet-Laplacian on the interval $(0, \ell_j)$.

Carl von Ossietzky Universität Oldenburg

General proof idea

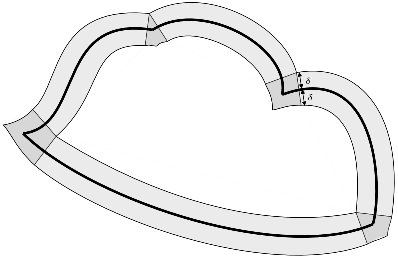

Figure: Decomposition of a neighbourhood of Γ

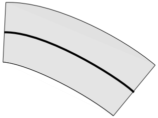

Figure: Tubular neighbourhood of an edge

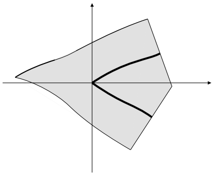

Figure: Neighbourhood of a corner

Truncation of an infinite half angle

Denote by $D^R_{\theta,\alpha}/N^R_{\theta,\alpha}$ the operators generated by ($x \in \{d, n\}$)

$$x^D_{\theta,\alpha}(u) = \int\limits_{K^R_\theta} |\nabla u|^2 \mathrm{d}x - \alpha \int\limits_{\Gamma^R_\theta} |u|^2 \mathrm{d}S$$

$$D(n^D_{\theta,\alpha}) = H^1(K^R_\theta), \quad D(d^D_{\theta,\alpha}) = H^1_0(K^R_\theta)$$

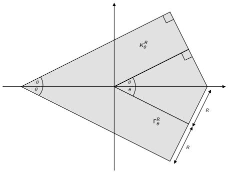

Figure: The Kite K^R_θ, a truncation of Γ_θ around the origin

Lemma (Benhellal, Körner, Pankrashkin, 2026)

- $E_n(D^R_{\theta,\alpha}) \sim E_n(H^{\Gamma_\theta}_\alpha), \quad n \in \{1, \dots, \kappa(\theta)\}$
- $E_n(N^R_{\theta,\alpha}) \sim E_n(H^{\Gamma_\theta}_\alpha), \quad n \in \{1, \dots, \kappa(\theta)\}$
- $E_{\kappa(\theta)+1}(D^R_{\theta,\alpha}) \geq -\frac{\alpha^2}{4}$
- $E_{\kappa(\theta)+1}(N^R_{\theta,\alpha}) \geq -\frac{\alpha^2}{4} + o(\alpha^2)$

as $\alpha \to +\infty$, $R \to 0^+$ and $\alpha R \to +\infty$.

Non-resonance condition

Definition

A half-angle $\theta \in (0, \frac{\pi}{2})$ is called non-resonant, if there exists $C > 0$ such that as $\alpha R \to +\infty$ it holds:

$$E_{\kappa(\theta)+1}(N^R_{\theta,\alpha}) \geq -\frac{\alpha^2}{4} + \frac{C}{R^2}.$$

We call $\theta \in (\frac{\pi}{2}, \pi)$ non-resonant, if $\pi - \theta$ is non-resonant.

Non-resonance condition

Definition

A half-angle $\theta \in (0, \frac{\pi}{2})$ is called non-resonant, if there exists $C > 0$ such that as $\alpha R \to +\infty$ it holds:

$$E_{\kappa(\theta)+1}(N^R_{\theta,\alpha}) \geq -\frac{\alpha^2}{4} + \frac{C}{R^2}.$$

We call $\theta \in (\frac{\pi}{2}, \pi)$ non-resonant, if $\pi - \theta$ is non-resonant.

Lemma

Let $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \setminus \{\frac{\pi}{2}\}$. Then it holds $\kappa(\theta) = 1$ and θ is non-resonant.

Main result (complete version)

Theorem (Benhellal, Körner, Pankrashkin, 2026)

Let $n \in \mathcal{K}$. Then there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ the nth discrete eigenvalue of H^{Γ}_{α} exists and has the following asymptotic expansion:

$$E_n(H^{\Gamma}_{\alpha}) = \alpha^2(\mathcal{E}_n + \alpha^{-\frac{2}{3}}), \quad n \in \{1, \dots, \mathcal{K}\} \quad \text{as } \alpha \to \infty.$$

Assuming that all angles θ_j are non-resonant then for any $n \in \mathbb{N}$ there exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha(n)$ the $\mathcal{K} + n$th eigenvalue of H^{Γ}_{α} exists and has an asymptotic expansion of the form

$$E_{\mathcal{K}+n}(H^{\Gamma}_{\alpha}) = -\frac{\alpha^2}{4} + E_n\Big(\bigoplus_{j=1}^{M}\big(D_j - \frac{k_j^2}{4}\big)\Big) + \mathcal{O}\Big(\frac{\log\alpha}{\sqrt{\alpha}}\Big) \quad \text{as } \alpha \to \infty$$

where D_j is the Dirichlet-Laplacian on the interval $(0, \ell_j)$.

Closing remarks

- The general approach was adapted from a paper on eigenvalue asymptotics of the Robin-Laplacian on a curved polygon (Khalile, Oumieres-Bonafos, Pankrashkin, 2020). Let Γ be a curve with corners and Ω the interior of Γ. Then the Robin-Laplacian on Ω is generated by
$$r_\alpha^\Omega(u) = \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_\Gamma |u|^2 \mathrm{dS}, \quad D(r_\alpha^\Omega) = H^1(\Omega)$$
In the case of the Robin-Laplacian, there are corner- and edge-induced eigenvalues as well (for constant curvature and under a similar non-resonance condition)
- The non-resonance condition for the Schrödinger operator remains underexplored
  - Are there more non-resonant angles?
  - Are there 'resonant' angles?