Zeros of Multiple Orthogonal Polynomials

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1. Zeros of Orthogonal Polynomials on the real line
2. Zeros of Multiple Orthogonal Polynomials on the real line
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4. Zeros of Multiple Orthogonal Polynomials on the unit circle

- Let μ be a measure on $\mathbb{R}$ with moments $\mu_k = \int_{\mathbb{R}} x^k d\mu(x)$
- Consider the Hankel matrix

$$H_n(\mu) = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{pmatrix}$$

- Asymptotics of $\det H_n(\mu)$ as $n \to \infty$ are governed by orthogonal polynomials on the real line (OPRL)
- Modern asymptotic analysis relies on the Riemann–Hilbert Problem *[Fokas–Its–Kitaev '92, Deift–Zhou '93]*

- Orthogonal polynomial $P_n = x^n + \kappa_{n-1}x^{n-1} + \cdots + \kappa_0$ with

$$\int_{\mathbb{R}} P_n(x) x^j \, d\mu(x) = 0, \qquad j = 0, \ldots, n-1$$

- This yields a linear system for κ_j with coefficient matrix H_n
- P_n exists uniquely $\iff$ $\det H_n(\mu) \neq 0$ ("n is normal for μ")
- If μ is a positive measure: each n is normal since

$$v^* H_n v = \int_{\mathbb{R}} \left(\sum_{j=0}^{n-1} v_j x^j \right)^2 d\mu(x) > 0$$

- First order Christoffel transform of μ:

$$d\tilde{\mu}(x) = (x - z_0)d\mu(x)$$

- The new MOPRL $\tilde{P}_n(x)$ is given by the Christoffel formula:

$$(x - z_0)\tilde{P}_n(x) = \frac{1}{P_n(z_0)} \det \begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_{n+1}(z_0) & P_n(z_0) \end{pmatrix}$$

- Existence condition:

$$P_n(z_0) \neq 0 \iff \tilde{P}_n(x) \text{ exists uniquely} \iff \det H_n(\tilde{\mu}) \neq 0$$

$$P_n(z_0) = 0 \iff \det H_n(\tilde\mu) = 0$$

- **Corollary**: if μ is positive, then zeros of P_n belong to $\mathrm{conv}(\operatorname{supp}\mu)$.

  Proof: Let $z_0 \notin \mathrm{conv}(\operatorname{supp}\mu)$. Then

$$v^* H_n(\tilde\mu) v = \int_{\mathbb{R}} \Big(\sum_{j=0}^{n-1} v_j x^j\Big)^2 (x - z_0) d\mu(x) \neq 0$$

  Since $\det H_n(\tilde\mu) \neq 0$, we have $P_n(z_0) \neq 0$.

- Second order Christoffel transform of μ:

$$d\tilde{\mu}(x) = (x - z_0)^2 d\mu(x)$$

- Define the Wronskian determinant:

$$W_n(z) = \det \begin{pmatrix} P_{n+1}(z) & P_n(z) \\ P'_{n+1}(z) & P'_n(z) \end{pmatrix}$$

- The new OPRL $\tilde{P}_n(x)$ is given by the Christoffel formula:

$$(x-z_0)^2 \tilde{P}_n(x) = \frac{1}{W_n(z_0)} \det \begin{pmatrix} P_{n+2}(x) & P_{n+1}(x) & P_n(x) \\ P_{n+2}(z_0) & P_{n+1}(z_0) & P_n(z_0) \\ P'_{n+2}(z_0) & P'_{n+1}(z_0) & P'_n(z_0) \end{pmatrix}$$

- Existence condition:

$$\det H_n(\tilde{\mu}) \neq 0 \iff \tilde{P}_n(x) \text{ exists uniquely} \iff W_n(z_0) \neq 0$$

- Existence condition:

$$\det H_n(\tilde\mu) \neq 0 \iff \tilde P_n(x) \text{ exists uniquely } \iff W_n(z_0) \neq 0$$

- **Corollary**: If μ is positive then zeros of P_n and P_{n+1} interlace on $\mathbb{R}$.
- *Proof:* $d\tilde\mu = (x-z_0)^2 d\mu$ is positive $\Rightarrow \det H_n(\tilde\mu) \neq 0 \Rightarrow W_n(z_0) \neq 0$ for all $z_0 \in \mathbb{R} \Rightarrow P_n$ and P_{n+1} interlace on $\mathbb{R}$.

- Consider the Cauchy transform of μ:

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{z-x} = \sum_{k=0}^{\infty} \frac{\mu_k}{z^{k+1}}, \qquad z \to \infty$$

- Padé approximation problem at ∞: find polynomials P_n of degree n and Q_n, so that

$$P_n(z)m(z) - Q_n(z) = \mathcal{O}(z^{-n-1}), \qquad z \to \infty$$

- **Claim**: this is equivalent to

$$\int_{\mathbb{R}} P_n(x) x^j \, d\mu(x) = 0, \qquad j = 0, \dots, n-1$$

- Normality of n is equivalent to existence and uniqueness of Padé approximant

- Let $\mu_1, \dots, \mu_r$ be r distinct measures on $\mathbb{R}$
- Consider their respective Cauchy transforms:

$$m_j(z) = \int_{\mathbb{R}} \frac{d\mu_j(x)}{z-x}, \qquad j = 1, \dots, r$$

- Let $\boldsymbol{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ be a multi-index with length $|\boldsymbol{n}| = n_1 + \dots + n_r$
- Type II Hermite-Padé problem: find a monic polynomial $P_{\boldsymbol{n}}$ of degree $|\boldsymbol{n}|$, and polynomials $Q_{\boldsymbol{n},j}$, such that

$$P_{\boldsymbol{n}}(z)m_j(z) - Q_{\boldsymbol{n},j}(z) = \mathcal{O}(z^{-n_j-1}), \qquad z \to \infty$$

for each $j = 1, \dots, r$.

- **Claim**: the Hermite–Padé problem is equivalent to a system of simultaneous orthogonality conditions
- $P_{\boldsymbol{n}}$ is a Multiple Orthogonal Polynomial (Type II) satisfying:

$$\int_{\mathbb{R}} P_{\boldsymbol{n}}(x) x^k d\mu_j(x) = 0, \qquad k = 0, \ldots, n_j - 1$$

for each $j = 1, \ldots, r$.

- We say that index $\boldsymbol{n} \in \mathbb{N}^r$ is normal if such $P_{\boldsymbol{n}}(x)$ exists and is unique.

- The orthogonality relations form a linear system for the coefficients of $P_{\boldsymbol{n}}(x) = x^{|\boldsymbol{n}|} + \kappa_{|\boldsymbol{n}|-1} x^{|\boldsymbol{n}|-1} + \ldots + \kappa_1 x + \kappa_0$
- The coefficient matrix is a stacked Hankel matrix:

$$H_{\boldsymbol{n}}(\boldsymbol{\mu}) = \left(\begin{array}{cccc} \mu_0^{(1)} & \mu_1^{(1)} & \cdots & \mu_{|\boldsymbol{n}|-1}^{(1)} \\ \mu_1^{(1)} & \mu_2^{(1)} & \cdots & \mu_{|\boldsymbol{n}|}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n_1-1}^{(1)} & \mu_{n_1}^{(1)} & \cdots & \mu_{|\boldsymbol{n}|+n_1-2}^{(1)} \\ \hline & & \vdots & \\ \hline \mu_0^{(r)} & \mu_1^{(r)} & \cdots & \mu_{|\boldsymbol{n}|-1}^{(r)} \\ \mu_1^{(r)} & \mu_2^{(r)} & \cdots & \mu_{|\boldsymbol{n}|}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n_r-1}^{(r)} & \mu_{n_r}^{(r)} & \cdots & \mu_{|\boldsymbol{n}|+n_r-2}^{(r)} \end{array} \right)$$

- $P_{\boldsymbol{n}}$ exists uniquely $\iff$ $\det H_{\boldsymbol{n}} \neq 0$

- Angelesco system: The measures $\mu_1, \ldots, \mu_r$ have pairwise disjoint convex hulls:

$$\mathrm{conv}(\mathrm{supp}\mu_i) \cap \mathrm{conv}(\mathrm{supp}\mu_j) = \varnothing, \qquad i \neq j$$

- Nikishin system ($r = 2$): The measures are linked via the Cauchy transform of a third measure σ:

$$d\mu_2(x) = m_\sigma(x) d\mu_1(x)$$

  where $\mathrm{supp}\sigma \cap \mathrm{supp}\mu_1 = \varnothing$.

- **Classical Result**: For Angelesco and Nikishin systems, every index $\boldsymbol{n}$ is normal.

- Given $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$, let $\tilde{\boldsymbol{\mu}}$ be

$$\big((x - z_0) d\mu_1, \dots, (x - z_0) d\mu_r\big)$$

- The new OPRL $\tilde{P}_{\boldsymbol{n}}(x)$ is given by

$$(x - z_0)\tilde{P}_{\boldsymbol{n}}(x) = \frac{1}{P_{\boldsymbol{n}}(z_0)} \det \begin{pmatrix} P_{\boldsymbol{n}+\boldsymbol{e}_j}(x) & P_{\boldsymbol{n}}(x) \\ P_{\boldsymbol{n}+\boldsymbol{e}_j}(z_0) & P_{\boldsymbol{n}}(z_0) \end{pmatrix}$$

- **Proposition**: Assume $\boldsymbol{n}$ is normal for $\boldsymbol{\mu}$. Then

$$P_{\boldsymbol{n}}(z_0) = 0 \iff \boldsymbol{n} \text{ is not normal for } \tilde{\boldsymbol{\mu}}$$

- $P_{\boldsymbol{n}}(z_0) = 0 \iff \boldsymbol{n}$ is not normal for $\tilde{\boldsymbol{\mu}}$:

$$\big((x - z_0)d\mu_1, \ldots, (x - z_0)d\mu_r\big)$$

- **Corollary (Angelesco)**: Zeros of $P_{\boldsymbol{n}}$ belong to $\bigcup_{j=1}^{r} \mathrm{conv}(\mathrm{supp}\mu_j)$.
- **Corollary (Nikishin)**: Zeros of $P_{\boldsymbol{n}}$ belong to $\mathrm{conv}(\mathrm{supp}\mu_1)$.
- *Proof*: If $z_0 \notin \mathrm{conv}(\mathrm{supp}\mu_j)$ for any j, then the system $\tilde{\boldsymbol{\mu}}$ remains an Angelesco (or Nikishin) system.
- Since these systems are perfect, $\boldsymbol{n}$ is normal for $\tilde{\boldsymbol{\mu}}$, implying $P_{\boldsymbol{n}}(z_0) \neq 0$.

- Type I Multiple Orthogonal Polynomials: a vector of polynomials $(A_{\boldsymbol{n}}^{(1)}, \dots, A_{\boldsymbol{n}}^{(r)})$ with $\deg A_{\boldsymbol{n}}^{(j)} \le n_j - 1$
- Summed orthogonality condition:

$$\sum_{j=1}^{r} \int_{\mathbb{R}} A_{\boldsymbol{n}}^{(j)}(x) x^k \, d\mu_j(x) = \begin{cases} 0, & k \le |\boldsymbol{n}| - 2, \\ 1, & k = |\boldsymbol{n}| - 1. \end{cases}$$

- The vector $\boldsymbol{A}_{\boldsymbol{n}} = (A_{\boldsymbol{n}}^{(1)}, \dots, A_{\boldsymbol{n}}^{(r)})$ exists uniquely $\iff$ $\det H_{\boldsymbol{n}}(\boldsymbol{\mu}) \ne 0 \iff P_{\boldsymbol{n}}$ exists uniquely.

- *[Van Assche, '11]*: Type II polynomials $P_{\boldsymbol{n}}$ satisfy nearest-neighbor recurrence relations (NNRR):

$$xP_{\boldsymbol{n}}(x) = P_{\boldsymbol{n}+\boldsymbol{e}_k}(x) + b_{\boldsymbol{n},k}P_{\boldsymbol{n}}(x) + \sum_{j=1}^{r} a_{\boldsymbol{n},j}P_{\boldsymbol{n}-\boldsymbol{e}_j}(x)$$

- Type I polynomial vectors $\boldsymbol{A}_{\boldsymbol{n}}(x)$ satisfy the dual recurrence:

$$x\boldsymbol{A}_{\boldsymbol{n}}(x) = \boldsymbol{A}_{\boldsymbol{n}-\boldsymbol{e}_k}(x) + b_{\boldsymbol{n}-\boldsymbol{e}_k,k}\boldsymbol{A}_{\boldsymbol{n}}(x) + \sum_{j=1}^{r} a_{\boldsymbol{n},j}\boldsymbol{A}_{\boldsymbol{n}+\boldsymbol{e}_j}(x)$$

- **The Operator Connection**: The NNRR relations define a Jacobi operator on an infinite $(r+1)$-homogeneous rooted tree, *[Aptekarev–Denisov–Yattselev, '20, '22]*.

- Given $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$, let $\tilde{\boldsymbol{\mu}}$ be

$$\big((x - z_0)d\mu_1, d\mu_2, \dots, d\mu_r\big)$$

- The new type I MOPRL $\tilde{A}^{(1)}_{\boldsymbol{n}}(x)$ is given by

$$(x - z_0)\tilde{A}^{(1)}_{\boldsymbol{n}}(x) = \frac{1}{A^{(1)}_{\boldsymbol{n}}(z_0)} \det \begin{pmatrix} A^{(1)}_{\boldsymbol{n}+\boldsymbol{e}_j}(x) & A^{(1)}_{\boldsymbol{n}}(x) \\ A^{(1)}_{\boldsymbol{n}+\boldsymbol{e}_j}(z_0) & A^{(1)}_{\boldsymbol{n}}(z_0) \end{pmatrix}$$

- **Proposition**: Assume $\boldsymbol{n}$ is normal for $\boldsymbol{\mu}$. Then

$$A^{(1)}_{\boldsymbol{n}}(z_0) = 0 \iff \boldsymbol{n} \text{ is not normal for } \tilde{\boldsymbol{\mu}}$$

- **Proposition**: $A_{\boldsymbol{n}}^{(1)}(z_0) = 0 \iff \boldsymbol{n}$ is not normal for $\tilde{\boldsymbol{\mu}}$:

$$((x - z_0)d\mu_1, d\mu_2, \ldots, d\mu_r)$$

- **Angelesco systems**: If $z_0 \notin \operatorname{conv}(\operatorname{supp}\mu_1)$, then $\tilde{\boldsymbol{\mu}}$ is also Angelesco.
- Thus, the zeros of $A_{\boldsymbol{n}}^{(1)}$ belong to $\operatorname{conv}(\operatorname{supp}\mu_1)$.
- **Nikishin systems**: The system $((x - z_0)d\mu_1, d\mu_2)$ is "almost" Nikishin with σ modified to $(x - z_0)d\sigma$.
- If $n_1 \leq n_2 - 1$, then the zeros of $A_{\boldsymbol{n}}^{(1)}$ belong to $\operatorname{conv}(\operatorname{supp}\sigma)$. *[K.–Vaktnäs '26]*

- **Type II Interlacing** ($P_{\boldsymbol{n}}$ and $P_{\boldsymbol{n}+\boldsymbol{e}_k}$):
  - Transform: Christoffel transform

$$((x-z_0)^2 d\mu_1, (x-z_0)^2 d\mu_2, \dots, (x-z_0)^2 d\mu_r)$$

  - Normality of $\tilde{\boldsymbol{\mu}}$ implies interlacing for Angelesco and Nikishin *[Haneczok–Van Assche, '12]*

- **Type I Interlacing** ($A_{\boldsymbol{n}}^{(1)}$ and $A_{\boldsymbol{n}+\boldsymbol{e}_k}^{(1)}$):
  - Transform: partial Christoffel transform

$$((x-z_0)^2 d\mu_1, d\mu_2, \dots, d\mu_r)$$

  - Normality of $\tilde{\boldsymbol{\mu}}$ implies interlacing for **Angelesco**: *[Aptekarev–Denisov–Yattselev, '20]*
  - for **Nikishin**: *[Kozhan–Vaktnäs, '26]*

- Let μ be a measure on $\partial\mathbb{D}$ with moments $\mu_k = \int_{\partial\mathbb{D}} z^{-k} d\mu(z)$
- Consider the Toeplitz matrix

$$T_n(\mu) = \begin{pmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-(n-1)} \\ \mu_1 & \mu_0 & \cdots & \mu_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_0 \end{pmatrix}$$

- Asymptotics of $\det T_n(\mu)$ as $n \to \infty$ are governed by orthogonal polynomials on the unit circle (OPUC)
- Modern asymptotic analysis relies on the Riemann–Hilbert Problem *[Fokas–Its–Kitaev '92, Baik–Deift–Johansson '99]*

- Orthogonal polynomial $\Phi_n(z) = z^n + \kappa_{n-1} z^{n-1} + \cdots + \kappa_0$ with

$$\int_{\partial\mathbb{D}} \Phi_n(z) z^{-j} d\mu(z) = 0, \qquad j = 0, \ldots, n-1$$

- This yields a linear system for κ_j with coefficient matrix T_n
- Φ_n exists uniquely iff $\det T_n(\mu) \neq 0$ ("n is normal for μ")
- If μ is a positive measure: each n is normal since

$$v^* T_n v = \int_{\partial\mathbb{D}} \Big| \sum_{j=0}^{n-1} v_j z^j \Big|^2 d\mu(z) > 0$$

- First order Christoffel transform of μ:

$$d\tilde{\mu}(z) = (z - z_0)d\mu(z)$$

- The new OPUC $\tilde{\Phi}_n(z)$ is given by the Christoffel formula:

$$(z - z_0)\tilde{\Phi}_n(z) = \frac{1}{\Phi_n(z_0)} \det \begin{pmatrix} \Phi_{n+1}(z) & \Phi_n(z) \\ \Phi_{n+1}(z_0) & \Phi_n(z_0) \end{pmatrix}$$

- Existence condition:

$$\det T_n(\tilde{\mu}) \neq 0 \iff \tilde{\Phi}_n(z) \text{ exists uniquely} \iff \Phi_n(z_0) \neq 0$$

$$\det T_n(\tilde\mu) \neq 0 \iff \tilde\Phi_n(z) \text{ exists uniquely} \iff \Phi_n(z_0) \neq 0$$

- **Corollary 1**: if μ is positive, then zeros of Φ_n belong to the open unit disk $\mathbb{D}$.

  Proof: Let $|z_0| \geq 1$. Then $(z - z_0)$ belongs to a half-plane for all $z \in \partial\mathbb{D}$. Thus:

$$v^* T_n(\tilde\mu) v = \int_{\partial\mathbb{D}} \Big|\sum_{j=0}^{n-1} v_j z^j\Big|^2 (z - z_0) d\mu(z) \neq 0$$

  Since $\det T_n(\tilde\mu) \neq 0$, we have $\Phi_n(z_0) \neq 0$.

- **Corollary 2**: For any $\tau \in \mathbb{C}$ with $|\tau| = 1$, the para-orthogonal polynomials (POPUC)

$$\Phi_n(z, \tau) = \Phi_n(z) + \tau\Phi_n^*(z)$$

  have all their zeros on $\partial\mathbb{D}$.

- In OPRL, P_n arises from Padé approximation of the Cauchy transform at ∞.

- For OPUC, consider the Carathéodory function:

$$F(z) = \int_{\partial\mathbb{D}} \frac{w+z}{w-z} d\mu(w)$$

- Two-point Padé approximation problem: given Carathéodory $F(z)$, find polynomials Φ_n and Ψ_n of degree n, so that

$$\Phi_n(z)F(z) + \Psi_n(z) = \mathcal{O}(z^n), \qquad z \to 0,$$
$$\Phi_n(z)F(z) + \Psi_n(z) = \mathcal{O}(z^{-1}), \qquad z \to \infty$$

- Φ_n solves the problem $\iff$ Φ_n is orthogonal polynomial for μ

- The generalized two-point Hermite-Padé problem: Let $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^r$. Find Laurent polynomials

$$\Phi_{\boldsymbol{n};\boldsymbol{m}}, \Psi_{\boldsymbol{n};\boldsymbol{m},1}, \ldots, \Psi_{\boldsymbol{n};\boldsymbol{m},r} \in \operatorname{span}\left\{z^k\right\}_{k=-|\boldsymbol{m}|}^{|\boldsymbol{n}|}$$

that simultaneously satisfy for each $j = 1, \ldots, r$:

$$\begin{aligned} \Phi_{\boldsymbol{n};\boldsymbol{m}}(z)F_j(z) + \Psi_{\boldsymbol{n};\boldsymbol{m},j}(z) &= \mathcal{O}(z^{n_j}), \qquad z \to 0 \\ \Phi_{\boldsymbol{n};\boldsymbol{m}}(z)F_j(z) + \Psi_{\boldsymbol{n};\boldsymbol{m},j}(z) &= \mathcal{O}(z^{-m_j-1}), \quad z \to \infty \end{aligned}$$

- **Theorem** *[Kozhan–Vaktnäs, '26]*: $\Phi_{\boldsymbol{n};\boldsymbol{m}}$ solves the Hermite–Padé problem $\iff$ it satisfies

$$\int_{\partial\mathbb{D}} \Phi_{\boldsymbol{n};\boldsymbol{m}}(z) z^{-k}\, d\mu_j(z) = 0, \qquad k = -m_j, \ldots, n_j - 1$$

- We say that the index pair $(\boldsymbol{n}, \boldsymbol{m})$ is normal if the normalized polynomial $\Phi_{\boldsymbol{n};\boldsymbol{m}}(z)$ exists and is unique.
- This is equivalent to $\det T_{\boldsymbol{n};\boldsymbol{m}}(\boldsymbol{\mu}) \neq 0$, where $T_{\boldsymbol{n};\boldsymbol{m}}$ is the stacked Toeplitz matrix:

$$T_{\boldsymbol{n};\boldsymbol{m}}(\boldsymbol{\mu}) = \left(\begin{array}{cccc}
\mu^{(1)}_{|\boldsymbol{m}|-m_1} & \mu^{(1)}_{|\boldsymbol{m}|-m_1-1} & \cdots & \mu^{(1)}_{1-m_1-|\boldsymbol{n}|} \\
\mu^{(1)}_{|\boldsymbol{m}|-m_1+1} & \mu^{(1)}_{|\boldsymbol{m}|-m_1} & \cdots & \mu^{(1)}_{2-m_1-|\boldsymbol{n}|} \\
\vdots & \vdots & \ddots & \vdots \\
\mu^{(1)}_{|\boldsymbol{m}|+n_1-1} & \mu^{(1)}_{|\boldsymbol{m}|+n_1-2} & \cdots & \mu^{(1)}_{n_1-|\boldsymbol{n}|} \\
\hline
 & \vdots & & \\
\hline
\mu^{(r)}_{|\boldsymbol{m}|-m_r} & \mu^{(r)}_{|\boldsymbol{m}|-m_r-1} & \cdots & \mu^{(r)}_{1-m_r-|\boldsymbol{n}|} \\
\mu^{(r)}_{|\boldsymbol{m}|-m_r+1} & \mu^{(r)}_{|\boldsymbol{m}|-m_r} & \cdots & \mu^{(r)}_{2-m_r-|\boldsymbol{n}|} \\
\vdots & \vdots & \ddots & \vdots \\
\mu^{(r)}_{|\boldsymbol{m}|+n_r-1} & \mu^{(r)}_{|\boldsymbol{m}|+n_r-2} & \cdots & \mu^{(r)}_{n_r-|\boldsymbol{n}|}
\end{array}\right)$$

- Angelesco system *[Kozhan–Vaktnäs, '26]*: The measures $\mu_1, \dots, \mu_r$ on $\partial\mathbb{D}$ have pairwise disjoint convex hulls:

$$\operatorname{conv}(\operatorname{supp}\mu_i) \cap \operatorname{conv}(\operatorname{supp}\mu_j) = \varnothing, \qquad i \neq j$$

- Nikishin system *[Kozhan, '26]* ($r = 2$): The measures are linked via the Carathéodory function of a third measure σ on $\partial\mathbb{D}$:

$$d\mu_2(z) = F_\sigma(z) d\mu_1(z)$$

where $\operatorname{supp}\sigma \cap \operatorname{supp}\mu_1 = \varnothing$.

- **Theorem** *[Kozhan–Vaktnäs, '26]*: For Angelesco systems, every index $(\boldsymbol{n}; \boldsymbol{n})$ and $(\boldsymbol{n}; \boldsymbol{n} \pm \boldsymbol{e}_j)$ is normal.

- **Theorem (Angelesco)** *[Kozhan–Vaktnäs, '26]*: Zeros of $\Phi_{\boldsymbol{n};\boldsymbol{m}}$ belong to $\bigcup_{j=1}^{r} \operatorname{conv}(\operatorname{supp} \mu_j)$ for every index $(\boldsymbol{n};\boldsymbol{n})$ and $(\boldsymbol{n};\boldsymbol{n} \pm \boldsymbol{e}_j)$.
- **Theorem (Angelesco)** *[Kozhan–Vaktnäs, '26]*: For any $\tau \in \mathbb{C}$ with $|\tau| = 1$, the para-orthogonal polynomials (POPUC)

$$\Phi_{\boldsymbol{n};\boldsymbol{m}}(z,\tau) = \Phi_{\boldsymbol{n};\boldsymbol{m}}(z) + \tau \Phi^*_{\boldsymbol{n};\boldsymbol{m}}(z)$$

have all their zeros on $\partial\mathbb{D}$.

- **Theorem** *[Kozhan–Vaktnäs, '24]*: Type II polynomials satisfy Szegő's recurrences:

$$\Phi_{\boldsymbol{n};\boldsymbol{m}}(z) = \alpha_{\boldsymbol{n};\boldsymbol{m}} \Phi^*_{\boldsymbol{n};\boldsymbol{m}}(z) + \sum_{j=1}^{r} \rho_{\boldsymbol{n};\boldsymbol{m},j}\, z \Phi_{\boldsymbol{n}-\boldsymbol{e}_j;\boldsymbol{m}}(z)$$

$$\Phi^*_{\boldsymbol{n};\boldsymbol{m}}(z) = \Phi^*_{\boldsymbol{n}-\boldsymbol{e}_k;\boldsymbol{m}}(z) + \beta_{\boldsymbol{n};\boldsymbol{m}} z \Phi_{\boldsymbol{n}-\boldsymbol{e}_k;\boldsymbol{m}}(z)$$

- **Theorem** *[Kozhan–Vaktnäs, '24]*: Type I polynomials satisfy the dual recurrences:

$$\boldsymbol{A}^*_{\boldsymbol{n};\boldsymbol{m}}(z) = \boldsymbol{A}^*_{\boldsymbol{n}+\boldsymbol{e}_k;\boldsymbol{m}}(z) - \alpha_{\boldsymbol{n};\boldsymbol{m}} z \boldsymbol{A}_{\boldsymbol{n}+\boldsymbol{e}_k;\boldsymbol{m}}(z)$$

$$\boldsymbol{A}_{\boldsymbol{n};\boldsymbol{m}}(z) = -\beta_{\boldsymbol{n};\boldsymbol{m}} \boldsymbol{A}^*_{\boldsymbol{n};\boldsymbol{m}}(z) + \sum_{j=1}^{r} \rho_{\boldsymbol{n};\boldsymbol{m},j}\, z \boldsymbol{A}_{\boldsymbol{n}+\boldsymbol{e}_j;\boldsymbol{m}}(z)$$

- **Open Problem**: On the real line, multi-index recurrences yield Jacobi operators on trees. Do these Szegő relations define a CMV/unitary operator on an r-ary tree?

Thank you for your attention

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