Numerical ranges of real linear operators

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Real linear operators

Let X, Y be complex Banach spaces. Let $\Phi : X \to Y$ be continuous additive mapping, $\Phi(x + x') = \Phi(x) + \Phi(x')$ for all $x, x' \in X$.

1. If $\Phi(\lambda x) = \bar{\lambda}\Phi(x)$ for all $\lambda \in \mathbb{C}$, then Φ is antilinear operator.
2. If $\Phi(tx) = t\Phi(x)$ for all $t \in \mathbb{R}$, then Φ is real linear operator.

Theorem: Let $\Phi : X \to Y$ be a real linear operator. Then there exist a linear operator $T : X \to Y$ and an antilinear operator $A : X \to Y$ such that $\Phi = T + A$. The operators T and A are uniquely determined as

$$Tx = \frac{1}{2}(\Phi(x) - i\Phi(ix)),$$

$$Ax = \frac{1}{2}(\Phi(x) + i\Phi(ix)),$$

for all $x \in X$.

Adjoint of real linear operator

Denote by X^* and Y^* the dual spaces of X and Y, respectively. For a linear operator $T : X \to Y$ let $T^* : Y^* \to X^*$ be its dual, i.e., the linear operator satisfying $\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle$ for all $x \in X$ and $y^* \in Y^*$.

For an antilinear operator $A : X \to Y$, let $A^* : Y^* \to X^*$ be its antilinear dual, i.e., the antilinear operator satisfying $\langle Ax, y^* \rangle = \overline{\langle x, A^* y^* \rangle}$ for all $x \in X$ and $y^* \in Y^*$.

It is natural to define the dual of a real linear operator using its canonical linear/antilinear decomposition. If $\Phi : X \to Y$ is a real linear operator, $\Phi = T + A$ with $T, A : X \to Y$, T linear and A antilinear, then define $\Phi^* : Y^* \to X^*$ by $\Phi^* = T^* + A^*$.

Properties of adjoint of real linear operator

Proposition: Let $\Phi, \Psi : X \to Y$ *be bounded real linear operators. Then* $(\Phi + \Psi)^* = \Phi^* + \Psi^*$.

Proposition: Let X, Y, Z *be complex Banach spaces. Let* $\Phi : X \to Y$ *and* $\Psi : Y \to Z$ *be bounded real linear operators. Then* $(\Psi\Phi)^* = \Phi^*\Psi^*$.

Birkhoff-James orthogonality

Vector x in a Banach space Z is Birkhoff-James orthogonal to a vector $y \in Z$ (in symbol $x \perp_B y$) if

$$\|x + \lambda y\| \geqslant \|x\|$$

for all $\lambda \in \mathbb{C}$. In general, it is not a symmetrical relation.

Theorem: Let X, Y be Banach spaces, let $T : X \to Y$ be a linear operator, $A : X \to Y$ an antilinear operator, let $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$, $|\alpha| \leqslant |\alpha'|, |\beta| \leqslant |\beta'|$. Then

$$\|\alpha T + \beta A\| \leqslant \|\alpha' T + \beta' A\|.$$

Remark: Let X, Y be complex Banach spaces, let $T : X \to Y$ be a linear operator and $A : X \to Y$ an antilinear operator. Then

$$T \perp_B A \qquad \text{and} \qquad A \perp_B T$$

in $\mathcal{R}(X, Y)$.

Spectral theory of real linear operators in Banach space

Definition: Let $\Phi \in \mathcal{R}(X)$. The spectrum $\sigma(\Phi)$ is defined by

$$\sigma(\Phi) = \{\lambda \in \mathbb{C} : \Phi - \lambda I \text{ is not bijective}\},$$

where I denotes the identity operator on X.

Definition: Let $\Phi \in \mathcal{R}(X)$. Define:

(i) *point spectrum*

$$\sigma_p(\Phi) = \{\lambda \in \mathbb{C} : \Phi - \lambda I \text{ is not injective}\};$$

(ii) *approximate point spectrum*

$$\sigma_{ap}(\Phi) = \{\lambda \in \mathbb{C} : \Phi - \lambda I \text{ is not bounded below}\};$$

(iii) *surjective spectrum*

$$\sigma_{sur}(\Phi) = \{\lambda \in \mathbb{C} : \Phi - \lambda I \text{ is not surjective}\};$$

Spectral theory of real linear operators in Banach space

Theorem: Let X be a complex Banach space and $\Phi \in \mathcal{R}(X)$. Then:

(i)
$$\sigma_p(\Phi) \subset \sigma_{ap}(\Phi) \subset \sigma(\Phi),$$
$$\sigma_{sur}(\Phi) \subset \sigma(\Phi),$$
$$\sigma(\Phi) = \sigma_p(\Phi) \cup \sigma_{sur}(\Phi) = \sigma_{ap}(\Phi) \cup \sigma_{sur}(\Phi);$$

(ii) $\sigma(\Phi) \subset \{z \in \mathbb{C} : |z| \leqslant \|\Phi\|\}$;

(iii) $\sigma(\Phi)$, $\sigma_{ap}(\Phi)$ *and* $\sigma_{sur}(\Phi)$ *are compact subsets of* $\mathbb{C}$;

(iv)
$$\sigma_{ap}(\Phi) = \sigma_{sur}(\Phi^*),$$
$$\sigma_{sur}(\Phi) = \sigma_{ap}(\Phi^*),$$
$$\sigma(\Phi) = \sigma(\Phi^*);$$

(v) $\partial\sigma(\Phi) \subset \sigma_{ap}(\Phi) \cap \sigma_{sur}(\Phi)$, *where* ∂ *denotes the topological boundary.*

Numerical range of real linear operators in Banach space

Let X be a complex Banach space. Define

$$\Pi(X) = \{(x, x^*) \in X \times X^* : \|x\| = \|x^*\| = \langle x, x^* \rangle = 1\}.$$

It is natural to define the numerical range of bounded real linear operators in the following way

$$W(\Phi) = \{\langle \Phi x, x^* \rangle : (x, x^*) \in \Pi(X)\}.$$

Theorem: Let $\Phi \in \mathcal{R}(X)$. Then

$$W(\Phi) \subset W(\Phi^*) \subset \overline{W(\Phi)}.$$

If X is reflexive, then $W(\Phi^*) = W(\Phi)$.

Circular symmetry of numerical range and spectrum of antilinear operators in Banach space

Let X be complex Banach space and $(x, x^*) \in \Pi(X)$. Let $A : X \to X$ be an antilinear operator. Then $(e^{i\theta}x, e^{-i\theta}x^*) \in \Pi(X)$ for every $\theta \in \mathbb{R}$ and we have

$$\left\langle Ae^{i\theta}x, e^{-i\theta}x^* \right\rangle = e^{-2i\theta} \langle Ax, x^* \rangle \in W(A).$$

When $\lambda \in \sigma_p(A)$, i.e. there is $x \in S_X$ such that $Ax = \lambda x$, then for every $\theta \in \mathbb{R}$ we have

$$A(e^{i\theta}x) = e^{-i\theta}Ax = e^{-i\theta}\lambda x = e^{-2i\theta}\lambda(e^{i\theta}x).$$

Circular symmetry also holds for $\sigma_{ap}(A), \sigma_{sur}(A)$ and $\sigma(A)$.

Numerical range of real linear operators in Banach space

Theorem: Let X be complex Banach space and $\Phi : X \to X$ be bounded real linear operator. Then:

(i) $\sigma_p(\Phi) \subset W(\Phi)$;

(ii) $\sigma(\Phi) \subset \overline{W(\Phi)}$.

Theorem: Let X be an at least two-dimensional complex Banach space, let $T : X \to X$ be a linear operator and $A : X \to X$ an antilinear operator. Let $x \in S_X$, $x^* \in S_{X^*}$ and $\langle x, x^* \rangle = 1$. Then

$$W(T + A) \supset \{z \in \mathbb{C} : |z - \langle Tx, x^* \rangle| \leqslant |\langle Ax, x^* \rangle|\}.$$

Numerical range of real linear operators in Banach space

Theorem: Let X be complex Banach space, $\dim X \geqslant 2$, *let $T : X \to X$ be a linear operator and $A : X \to X$ an antilinear operator. Then*

$$W(T + A) = \bigcup_{(x,x^*) \in \Pi(X)} \{z \in \mathbb{C} : |z - \langle Tx, x^* \rangle| \leqslant |\langle Ax, x^* \rangle|\}.$$

Theorem: Let X be complex Banach space, $\dim X \geqslant 2$, *let $T : X \to X$ be a linear operator and $A : X \to X$ an antilinear operator, let $\alpha, \beta \in \mathbb{C}$, $|\alpha| \leqslant |\beta|$. Then $W(T + \alpha A) \subset W(T + \beta A)$. In particular, $W(T) \subset W(T + A)$.*

Numerical ranges of antilinear operators on complex Banach spaces

Let X be complex Banach space, $\dim X \geqslant 2$ and let $A : X \to X$ be bounded antilinear operator. It was shown in [1] that $W(A)$ is always an annulus.

Theorem: Let X be complex Banach space such that $\dim X \geqslant 2$. Let $A : X \to X$ be an antilinear operator. Then $0 \in W(A)$.

Corollary: Let A be an antilinear operator on a Banach space X with $\dim X \geqslant 2$. Then either $W(A) = \{z \in \mathbb{C} : |z| < w(A)\}$ or $W(A) = \{z \in \mathbb{C} : |z| \leqslant w(A)\}$. In particular, $W(A)$ is always a convex set.

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[1] Chø M., Hur I., Lee J.E. (2021) Numerical ranges of conjugations and antilinear operators on Banach spaces, *Filomat* **35**, 2715–2720.

Numerical ranges of real linear operators on Hilbert spaces

Let H be a complex Hilbert space. The numerical range of bounded real linear operator $\Phi \in \mathcal{R}(H)$ is defined as

$$W(T) = \{\langle \Phi x, x \rangle : x \in H, \|x\| = 1\}.$$

The numerical radius of Φ is defined by

$$w(\Phi) = \sup\{|\lambda| : \lambda \in W(\Phi)\} = \sup\{|\langle \Phi x, x \rangle| : x \in H, \|x\| = 1\}.$$

Theorem: Let H be complex Hilbert space such that $\dim H \geqslant 2$ and let $\Phi : H \to H$ be a real linear operator. Then $W(\Phi)$ is convex subset of $\mathbb{C}$.

Numerical ranges of real linear operators on Hilbert spaces

For linear operator T on complex Hilbert space H we have well known inequalities

$$\frac{\|T\|}{2} \leqslant w(T) \leqslant \|T\|.$$

The generalization for real linear operators has following form.

Theorem: Let H be complex Hilbert space and $T : H \to H$ be a linear operator, $A : H \to H$ an antilinear operator and let $A_1 = (A + A^*)/2$. Then $W(T + A) = W(T + A_1)$ and

$$\frac{1}{2}\|T + 2A_1\| \leqslant w(T + A_1) = w(T + A) \leqslant \|T + A_1\|.$$

Properties of numerical range of real linear operators on Hilbert spaces

Theorem: Let $\Phi : H \to H$ *be a real linear operator,* $\Phi^* = \Phi$*. Then* $w(\Phi) = \|\Phi\|$.

Theorem: Let Φ *be a real linear operator on a Hilbert space* H*. If* $w(\Phi) = \|\Phi\|$ *then*

$$r(\Phi) = w(\Phi) = \|\Phi\|.$$

Theorem: Let $\Phi : H \to H$ *be a real linear operator. Then*

$$W(\Phi^*) = \{\bar{z} : z \in W(\Phi)\}.$$

Numerical ranges of antilinear operators on complex Hilbert spaces

It was proved in [1] that for $A \in \mathcal{AL}(H)$, $W(A)$ is always an annulus. It follows from two observations:
(i) $W(A)$ is circularly symmetric. If $\lambda \in W(A)$ and $0 \leqslant t < 2\pi$ then there exists a unit vector $x \in H$ such that $\langle Ax, x\rangle = \lambda$. Let $y = e^{-it/2}x$. Then $\|y\| = 1$ and

$$\langle Ay, y\rangle = e^{it}\langle Ax, x\rangle = e^{it}\lambda.$$

So $e^{it}\lambda \in W(A)$ for all t, $0 \leqslant t < 2\pi$.

(ii) $W(A)$ is a connected set. Indeed, the unit sphere $S_H = \{x \in H : \|x\| = 1\}$ is connected in H and the mapping $S_H \ni x \mapsto \langle Ax, x\rangle$ is continuous. So $W(A)$ is also connected.

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[1] Hur I., Lee J.E. (2021) Numerical ranges of conjugations and antilinear operators, *Linear and Multilinear Algebra* **69**, 2990–2997.

Numerical ranges of antilinear operators on complex Hilbert spaces

Corollary: Let H be a complex Hilbert space, $\dim H \geqslant 2$ *and* $A \in \mathcal{AL}(H)$*. Then* $0 \in W(A)$*.*

Proof: Let $x, y \in H$ be linearly independent unit vectors. If $\langle Ay, y\rangle = 0$, then $0 \in W(A)$.

Suppose that $\langle Ay, y\rangle \neq 0$. For $\lambda \in \mathbb{C}$ consider the unit vector

$$u(\lambda) = \frac{x + \lambda y}{\|x + \lambda y\|}.$$

Then

$$\langle Au(\lambda), u(\lambda)\rangle = \frac{\langle Ax, x\rangle + \bar{\lambda}(\langle Ax, y\rangle + \langle Ay, x\rangle) + \bar{\lambda}^2\langle Ay, y\rangle}{\|x + \lambda y\|^2}.$$

The quadratic equation $\langle Ax, x\rangle + \bar{\lambda}(\langle Ax, y\rangle + \langle Ay, x\rangle) + \bar{\lambda}^2\langle Ay, y\rangle = 0$ in $\bar{\lambda}$ has a root. So there exists $\lambda_1 \in \mathbb{C}$ such that $0 = \langle Au(\lambda_1), u(\lambda_1)\rangle \in W(A)$.

Numerical ranges of antilinear operators on complex Hilbert spaces

Theorem: Let A be an antilinear operator on a complex Hilbert space H. The following statements are equivalent:

(i) $W(A) = \{0\}$;

(ii) $\langle Ax, y\rangle = -\langle Ay, x\rangle$ *for all* $x, y \in H$;

(iii) $A^* = -A$.

Corollary: If A is a bounded antilinear operator on a complex separable Hilbert space H, then

$$0 \leqslant w(A) \leqslant \|A\|, \tag{1}$$

and the bounds are sharp.

Theorem: Let A be an antilinear operator on a Hilbert space H. Let $w(A) = 0$ and $n \in \mathbb{N}$. Then $w(A^{2n+1}) = 0$.

Properties of numerical range of antilinear operators on Hilbert spaces

If $A : H \to H$ is an antilinear operator, then one can write $A = A_1 + A_2$, where

$$A_1 = \frac{A + A^*}{2}, \qquad A_2 = \frac{A - A^*}{2},$$

and $A_1^* = A_1$, $A_2^* = -A_2$.

Corollary: Let $A : H \to H$ be an antilinear operator. Then

$$W(A) = W\left(\frac{A + A^*}{2}\right) \quad \text{and} \quad w(A) = \frac{\|A + A^*\|}{2}.$$

Numerical index of complex Banach space

Numerical index of a complex Banach space X is defined as follows

$$
\begin{aligned}
n(X) &= \inf \{w(T) : T \in B(X), \|T\| = 1\} \\
&= \inf \left\{ \frac{w(T)}{\|T\|} : T \in B(X), \|T\| > 0 \right\},
\end{aligned}
$$

It is well known that

$$
\frac{1}{e} \leqslant n(X) \leqslant 1
$$

for all complex Banach spaces and $n(X) = \frac{1}{2}$ if X is complex Hilbert space such that $\dim X \geqslant 2$.
For Banach spaces such as c_0, ℓ^1, ℓ^∞ the numerical index is equal to 1.

Antilinear numerical index of complex Banach space

Definition: Let X be a complex Banach space. The antilinear numerical index of X is defined by

$$\begin{aligned} n_{\mathcal{AL}}(X) &= \inf\left\{w(A) : A \in \mathcal{AL}(X), \|A\| = 1\right\} \\ &= \inf\left\{\frac{w(A)}{\|A\|} : A \in \mathcal{AL}(X), \|A\| > 0\right\}, \end{aligned}$$

Remark: It is obvious that for all complex Hilbert spaces H such that $\dim H \geqslant 2$ we have $n_{\mathcal{AL}}(H) = 0$.

Theorem: Let A be an antilinear operator on c_0, ℓ^1 or ℓ^∞. Then $w(A) = \|A\|$.

Remark: For complex Banach spaces c_0, ℓ^1, ℓ^∞ the antilinear numerical index coincide with standard numerical index.

Properties of numerical range of antilinear operators on Hilbert spaces Part 2

Theorem: Let $A : H \to H$ be an antilinear operator and $A_1 = (A + A^*)/2$ its self-adjoint part. Then the following conditions are equivalent:

(i) $w(A) = \|A\|$;

(ii) $\|A\| = \|A_1\|$;

(iii) $\{z \in \mathbb{C} : |z| = \|A\|\} \subset \sigma_{ap}(A)$;

(iv) $\{z \in \mathbb{C} : |z| = \|A\|\} \subset \sigma_{ap}(A_1)$.

Theorem: Let $A : H \to H$ be an antilinear operator and $A_1 = (A + A^*)/2$ be its self-adjoint part. Then $\|A_1\| \in \sigma_p(A_1)$ if and only if $W(A) = \{z \in \mathbb{C} : |z| \leqslant \|A_1\|\}$.

Shapes of numerical range on Hilbert space $\mathbb{C}^2$

For linear operators on Hilbert space $\mathbb{C}^2$ numerical range is always elliptic disc (possibly degenerated to a point or line segment).

For antilinear operators on Hilbert space $\mathbb{C}^2$ numerical range is always a disc (possibly degenerated to a point) centered at zero.

For real linear operators on Hilbert space $\mathbb{C}^2$ more there are much more possible shapes of numerical range.

Example:Let $H = \mathbb{C}^2$ be a Hilbert space and $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ a real linear operator defined by

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 + \eta_1 \bar{x}_1 \\ \lambda_2 x_2 + \eta_2 \bar{x}_2 \end{pmatrix}, \quad x_1, x_2 \in \mathbb{C},$$

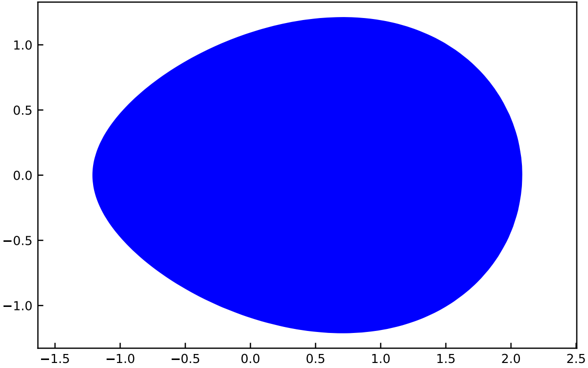
where $\lambda_1, \lambda_2, \eta_1, \eta_2 \in \mathbb{C}$. Then
$W(\Phi) = \operatorname{conv}\left(\left(\lambda_1 + \overline{\mathbb{D}}_{|\eta_1|}\right) \cup \left(\lambda_2 + \overline{\mathbb{D}}_{|\eta_2|}\right)\right)$

Numerical example 1

Let $H = \mathbb{C}^2$ be a Hilbert space and $\Phi_2 : \mathbb{C}^2 \to \mathbb{C}^2$ be the real linear operator

$$\Phi_1 \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left(\begin{array}{c} x_1 + \bar{x}_1 \\ -x_2 + \bar{x}_1 \end{array} \right), \quad x_1, x_2 \in \mathbb{C}.$$

$W(\Phi_1)$ is presented below

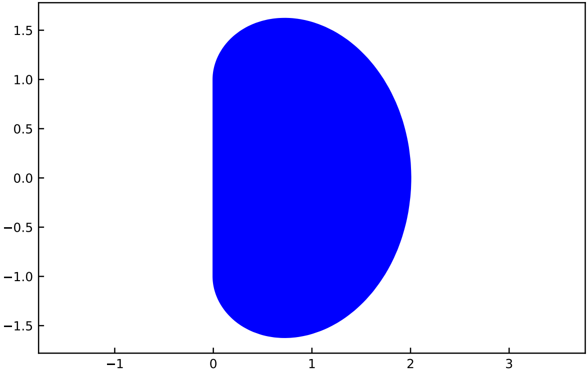


Numerical example 2

Let $H = \mathbb{C}^2$ be a Hilbert space and $\Phi_2 : \mathbb{C}^2 \to \mathbb{C}^2$ be the real linear operator

$$\Phi_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + x_2 + \bar{x}_2 \end{pmatrix}, \quad x_1, x_2 \in \mathbb{C},$$

$W(\Phi_2)$ is presented below



1. D. Kołaczek, V. Müller, *Numerical ranges of antilinear operators*, Integral Equations and Operator theory **96**:Paper No. 17, 2024.
2. D. Kołaczek, V. Müller, *Real linear operators and numerical ranges*, preprint: arXiv:2508.04347, 2025 (accepted in Integral Equations and Operator Theory).

Thank you for your attention