Special values of spectral zeta functions of graphs and spaces

Based on joint works with D. Müller, M. Pallich, J. Jorgenson, and L. Smajlović.

OTAMP, Stockholm, 2026

Anders Karlsson, University of Geneva and Uppsala University

Some classical sums

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Madhava-Gregory-Leibniz: $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$$

Euler: $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

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These are the special values $L(1,\chi_{-4})$ and $\zeta(2)$ respectively.

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1} \qquad \mathrm{Re}(s) > 1$$

Special values: $\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \ldots$ $\qquad \zeta(2k) = \frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2(2k!)}$ Euler, 1730s

No formula for $\zeta(odd)$.

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We will see:

- Spectral interpretation, π enters trivially
- -> $\zeta(2k)$ is essentially a finite graph zeta value
- $\zeta(2k+1)$, no combinatorics

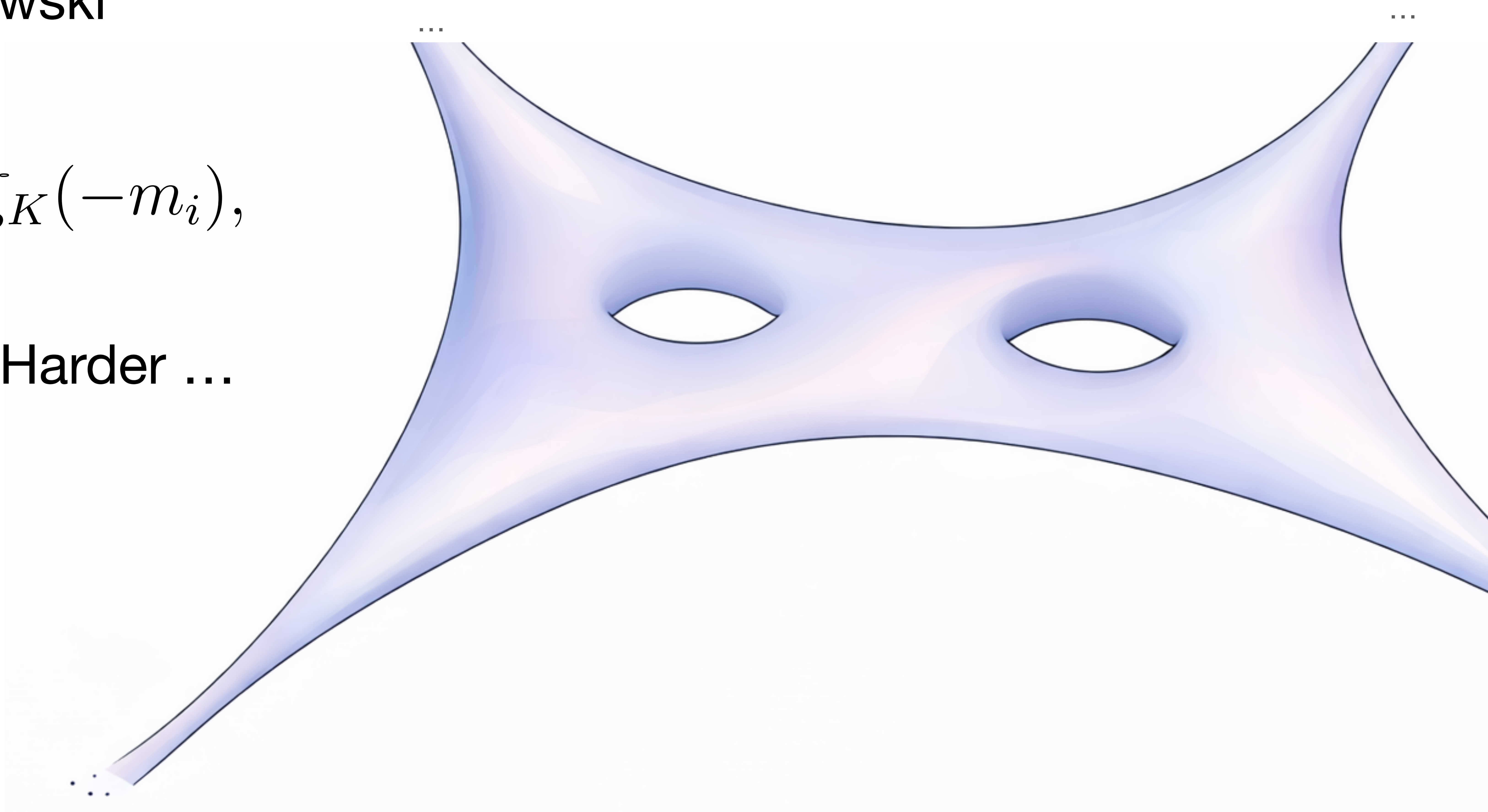
Special values in volume formulas

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \zeta(2)\zeta(3)\ldots\zeta(n)$$

Minkowski

$$\mathrm{vol}(G/\Gamma) = c^{[K:\mathbb{Q}]}\prod_{i=1}^{l}\zeta_K(-m_i),$$

Siegel, Weil, Langlands, Harder …



Spectral zeta functions

Spectral zeta functions

Carleman 1934; Minakshisundaraman-Pleijel 1949

Given a compact Riemannian manifold X with its Laplacian Δ, let

$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s}$$

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Given a compact Riemannian manifold X with its Laplacian Δ, let

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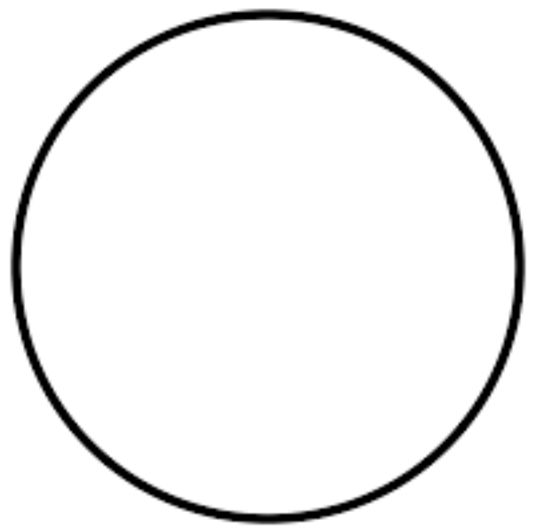
Example: $X = \mathbb{R}/\mathbb{Z}$, with $\Delta = -\frac{\partial^2}{\partial x^2}$



$$\zeta_{\mathbb{R}/\mathbb{Z}}(s) = \frac{2}{(2\pi)^{2s}} \zeta(2s)\,.$$

$\sin(2\pi n x)$ and $\cos(2\pi n x)$, $n \geq 0$ are the eigenfunctions of $-\partial^2/\partial x^2$.

$$\lambda = (2\pi n)^2$$

Spectral zeta functions from heat kernels

Given a space X with a Laplacian Δ, let

$$\zeta_X(s) = \text{``}\sum_{\lambda\neq 0} \frac{1}{\lambda^s}\text{''} = \int \lambda^{-s} d\mu(\lambda) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}^*(K_X)(t) t^s \frac{dt}{t}$$

where K_X is the heat kernel, i.e. $(\Delta_x + \frac{\partial}{\partial t}) K_X(t, x_0, x) = 0$ with $K_X(0, x_0, x) = \delta_{x_0}(x)$.

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Another interest comes from:

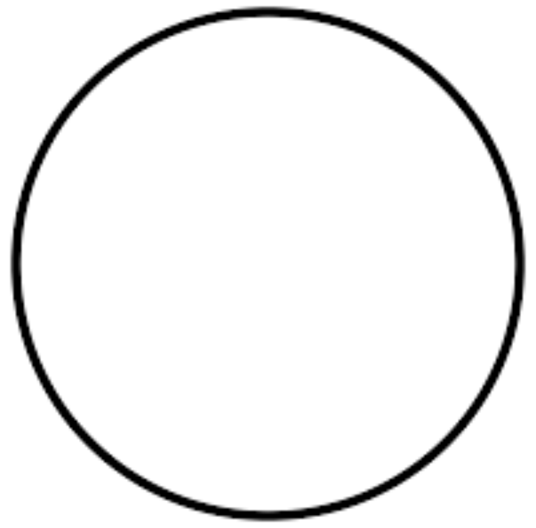
$$\det{}' \Delta_X := \exp(-\zeta_X'(0)) = \text{"}\lambda_1 \lambda_2 \lambda_3 \ldots\text{"}$$

-> talk by Julie Rowlett

1970s Ray-Singer, Dowker-Critchley, Hawking,…

Theta function or heat kernel on the circle

Riemann’s approach to his zeta function



$$\frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-(x+2\pi n)^2/4t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx}.$$

The heat kernel on the circle, a.k.a. theta function

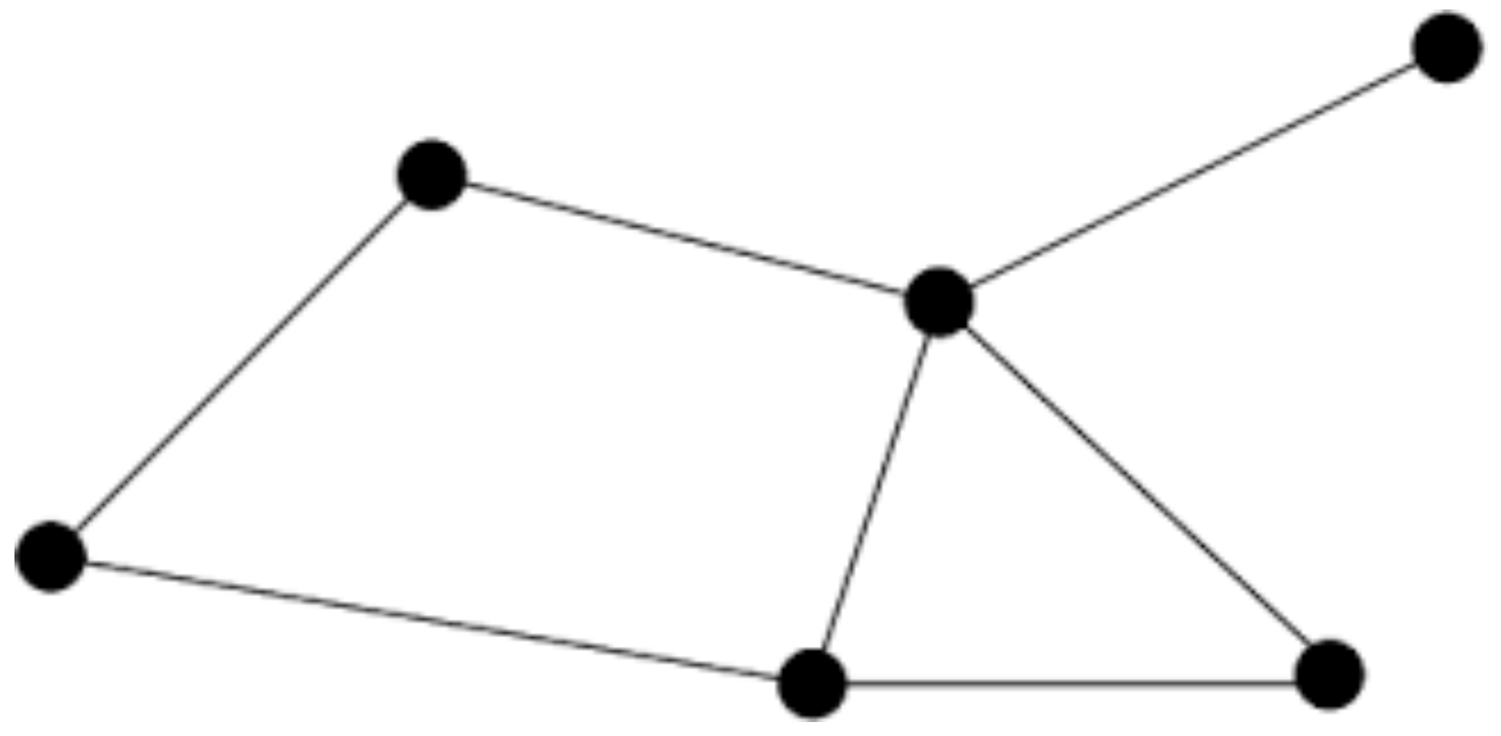
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty Heat(t) \cdot t^s \frac{dt}{t}$$

Graph Laplacian

Let $X = (VX, EX)$ be a countable graph with bounded vertex degree.

Its Laplacian on $L^2(VX)$ is

$$\Delta f(x) = d^* df(x) = \sum_{y:[x,y]\in EX} f(x) - f(y)\,.$$



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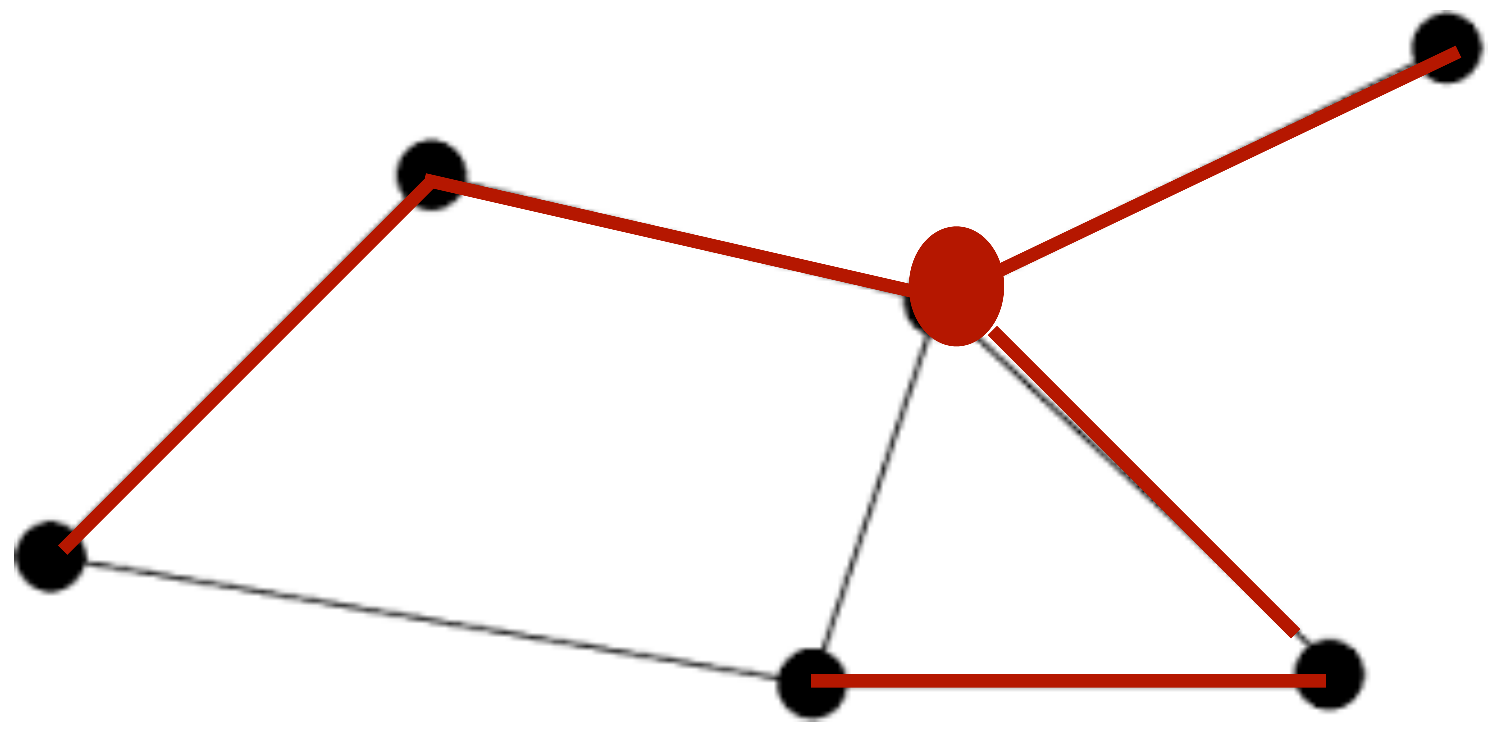

The Matrix-Tree theorem

Write $\det(\Delta + xI) = p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0$,

then p_n equals the number of rooted spanning forests of n components.

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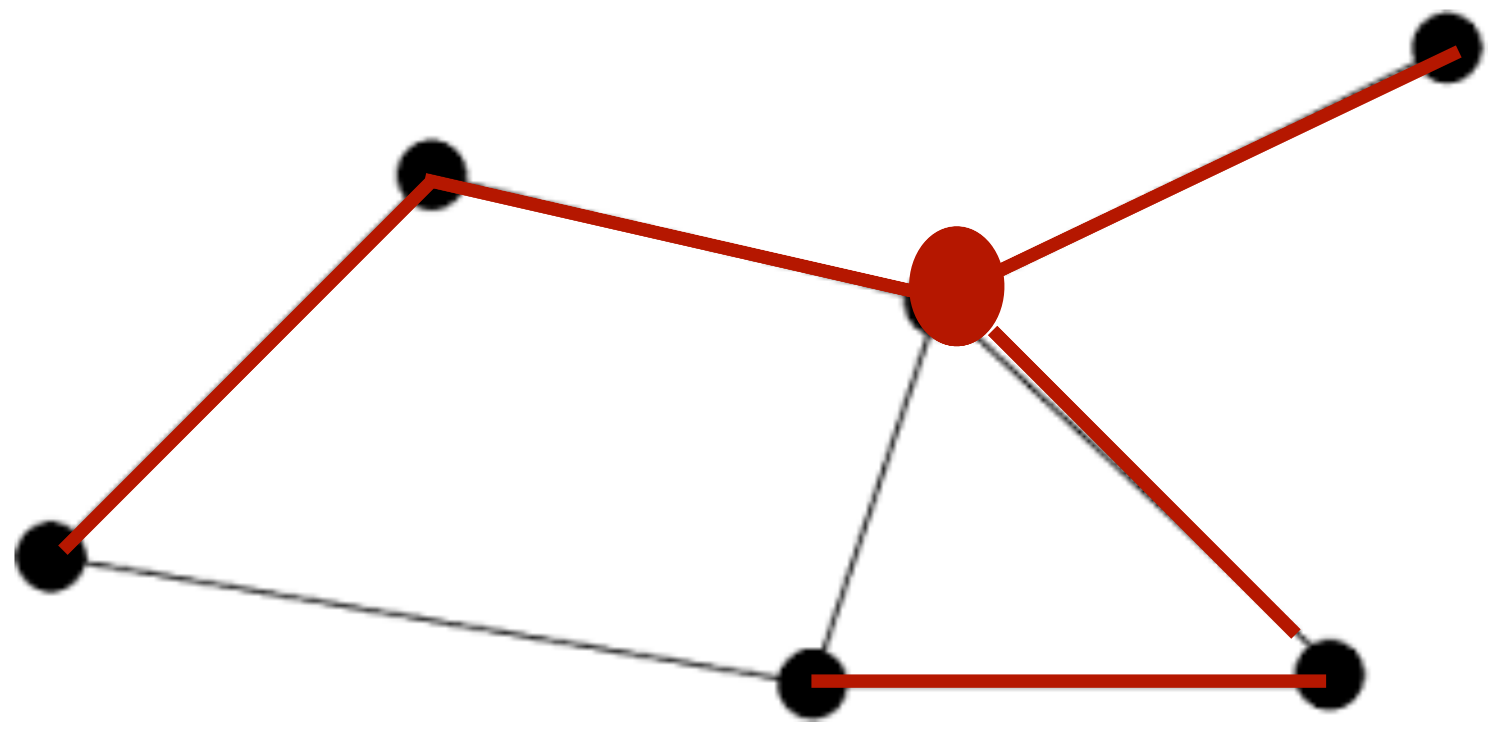

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Spectral zeta functions of graphs

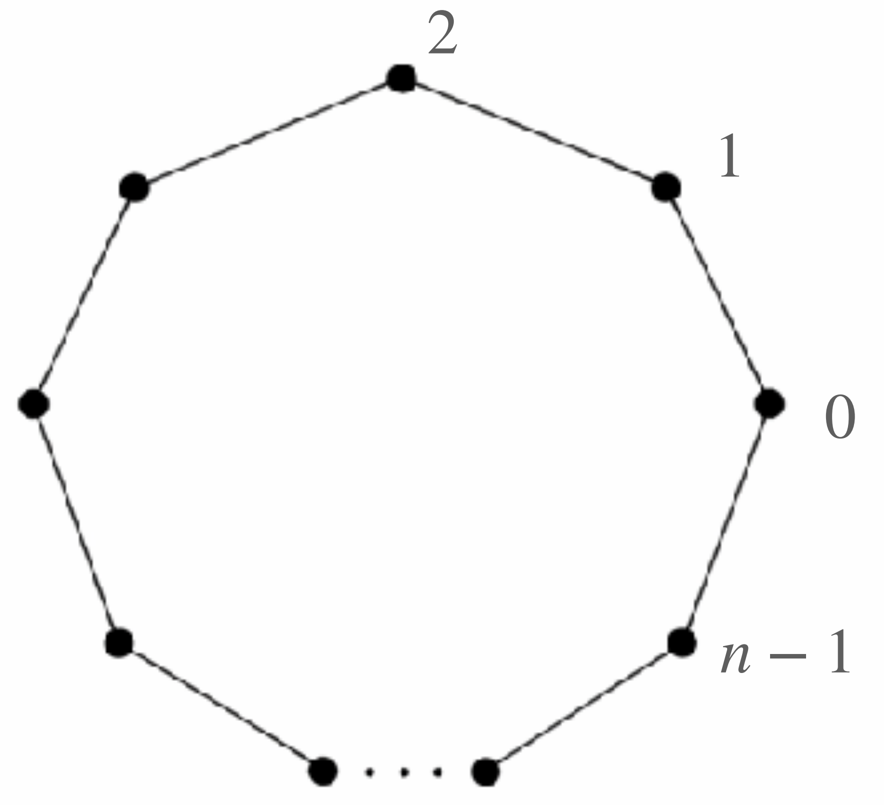
Let $X = (VX, EX)$ be a countable graph with bounded vertex degree .

$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} \qquad \text{or} \qquad \zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}^*(K_X)(t) t^s \frac{dt}{t}$$

In contrast to the Ihara zeta function (Ihara, Sunada, Bass, Terras, Kottos-Smilansky,…)
there are much fewer studies of this topic: Dowker, Chinta-Jorgenson-K. 2010, Kirsten ,Harrison, Weynand, Knill, Friedli-K., Meiners-Vertman, Lehman, Xie-Zhao-Zhao, Müller, …

Example: discrete circles

For $\mathbb{Z}/n\mathbb{Z}$ the Laplacian is: $\Delta f(x) = 2f(x) - f(x-1) - f(x+1)$



$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \sum_{k=1}^{n-1} \frac{1}{(4\sin^2(\pi k/n))^s}$$

1990s Dowker

Special values appear in several physics contexts, for example Verlinde formulas:

$$\dim_{\mathbb{C}} H^0(\mathcal{N}_{g,2,0}, \mathcal{L}^n) = (n+2)^{g-1} 2^{g-1} \zeta_{\mathbb{Z}/(n+2)\mathbb{Z}}(g-1)$$

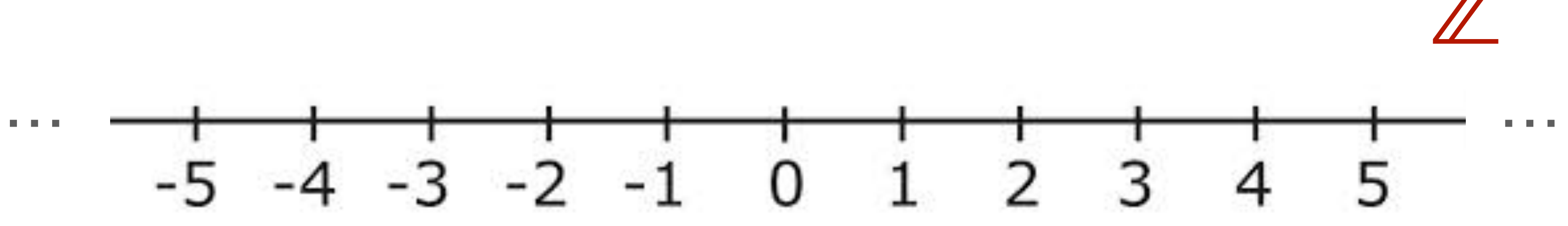
The case of $\zeta_{\mathbb{Z}}(s)$

The infinite discrete line

K_X is the heat kernel, i.e. $(\Delta_x + \frac{\partial}{\partial t})K_X(t, x_0, x) = 0$ with $K_X(0,x_0,x) = \delta_{x_0}(x)$.

Example:

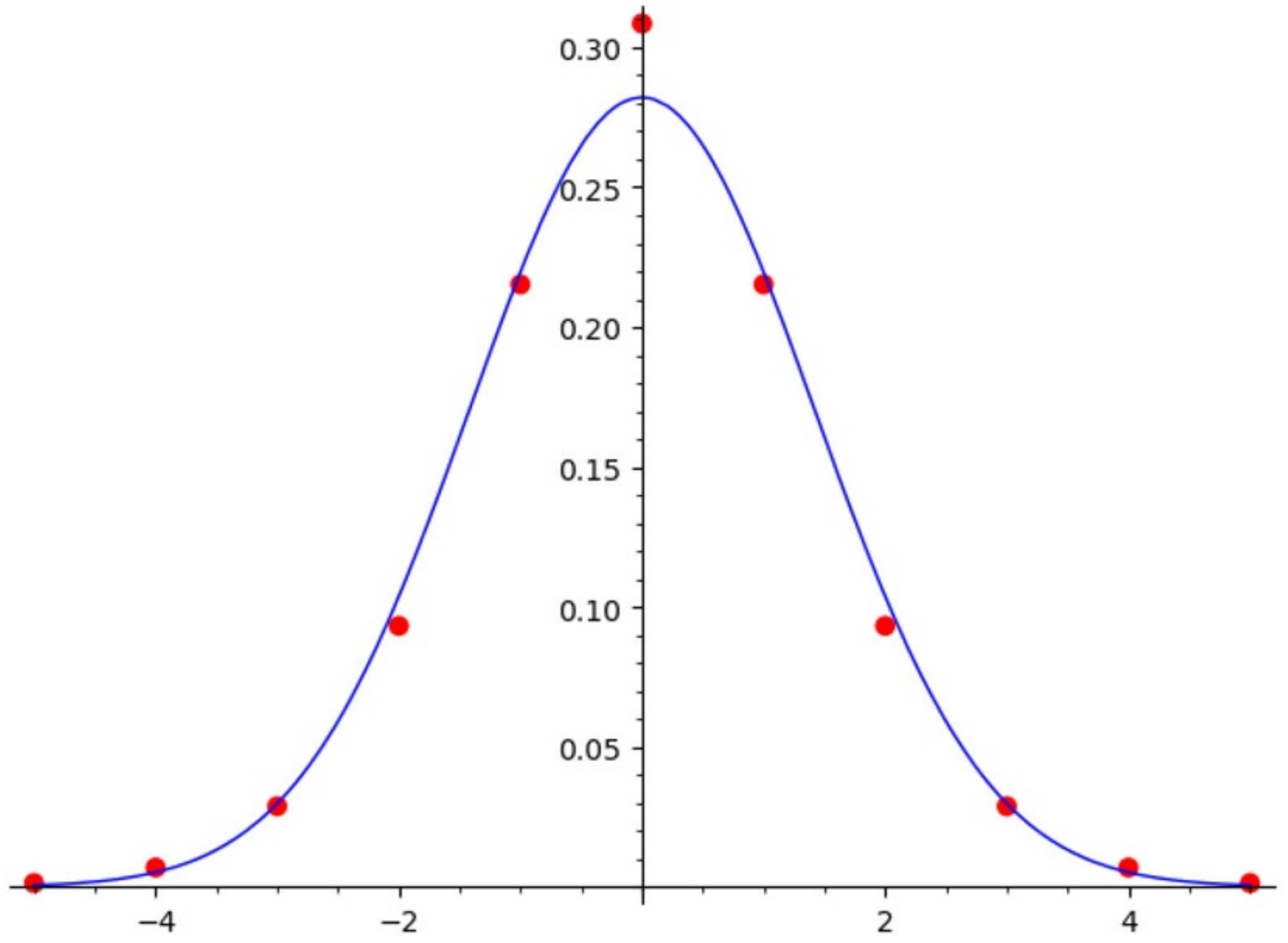
$\Delta f(x) = 2f(x) - f(x-1) - f(x+1)$



The discrete Gaussian:

$$K_{\mathbb{Z}}(t,0,x) = e^{-2t} I_x(2t)$$

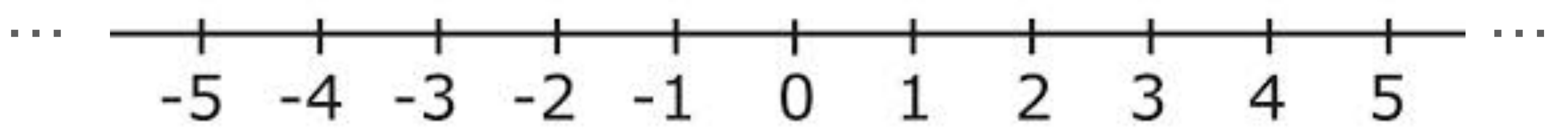
I-Bessel function



Chinta, Jorgenson, K, Smajlović, J.Physics A, 2025

Ther zeta function of $\mathbb{Z}$

$\mathbb{Z}$



$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t} = \frac{1}{2^{2s}\pi^{1/2}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)} = \binom{-2s}{-s} = \prod_{k=1}^{\infty} \frac{(k-s)^2}{k(k-2s)}.$$

Friedli-K. 2017

Dubout, 2019 K.-Pallich, 2023

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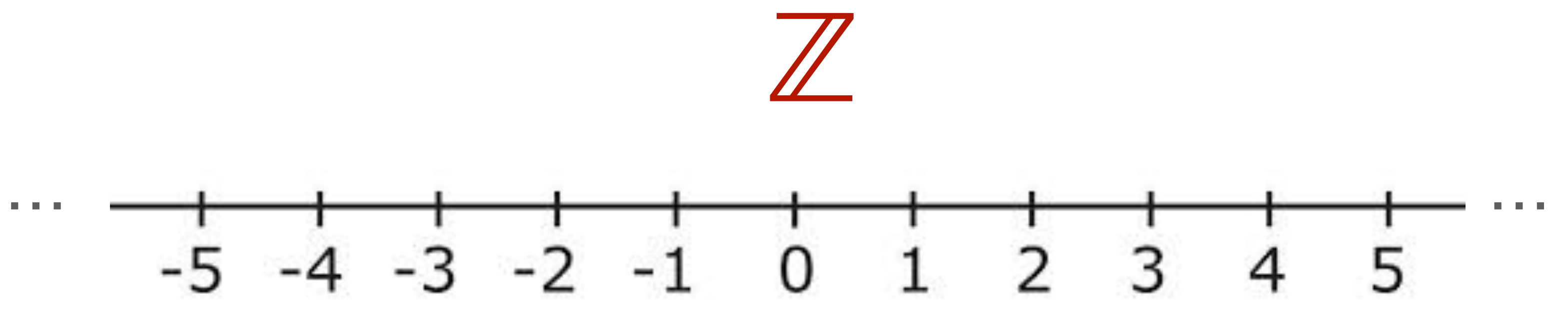

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$\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s)$ for all s .

Appears in Eisenstein series, Selberg zeta function, Gindikin-Karpelevich,… and special values s=-n in Chebyshev.

The volume of spheres

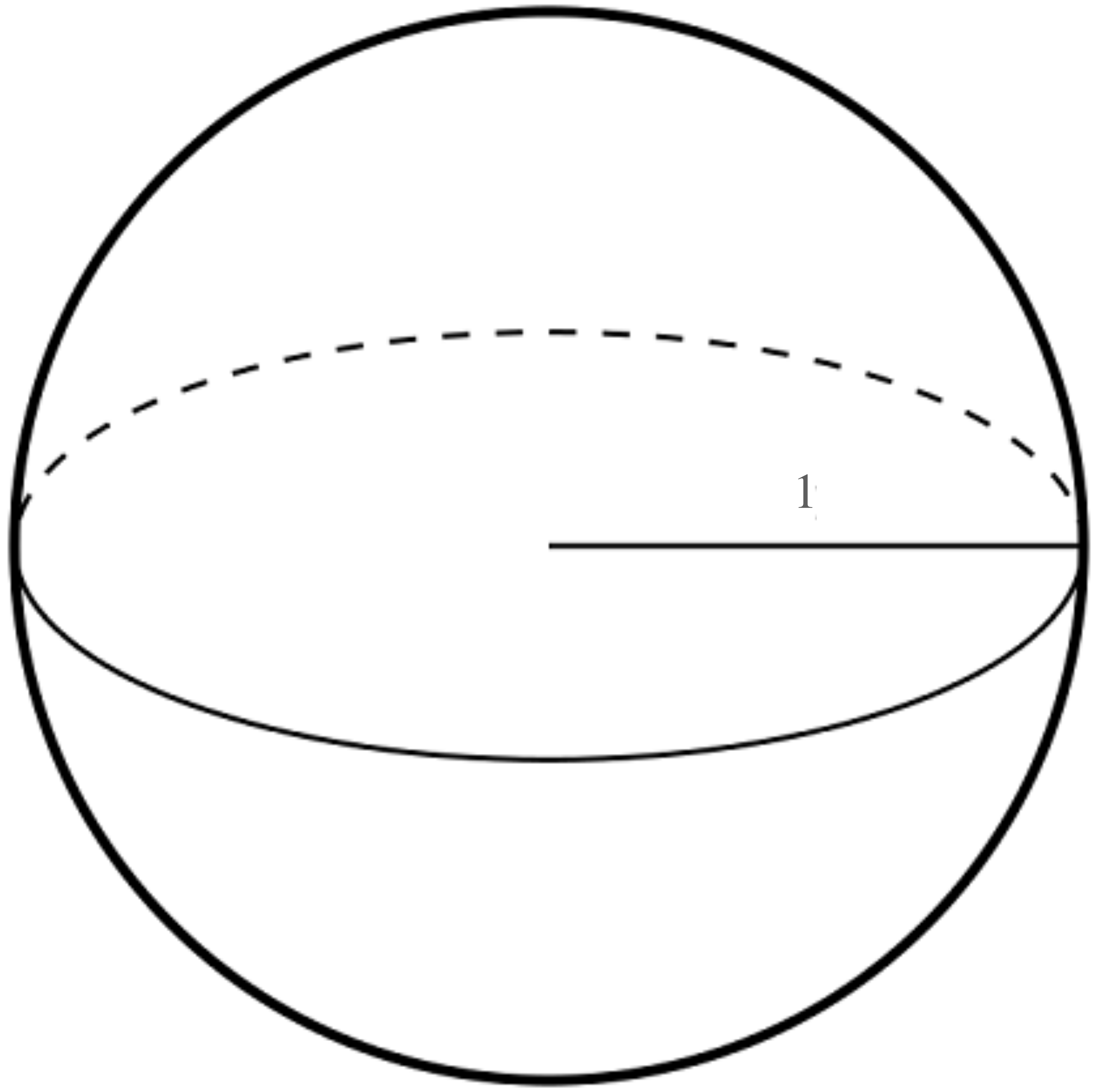
$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t} = \frac{1}{2^{2s}\pi^{1/2}}\frac{\Gamma(1/2-s)}{\Gamma(1-s)} = \binom{-2s}{-s} = \prod_{k=1}^{\infty}\frac{(k-s)^2}{k(k-2s)}.$$

K.-Pallich, 2023:

$$Vol(S^n) = 2\tilde{\zeta}_{\mathbb{Z}}(0)\tilde{\zeta}_{\mathbb{Z}}(-1)\ldots\tilde{\zeta}_{\mathbb{Z}}(-n+1)$$

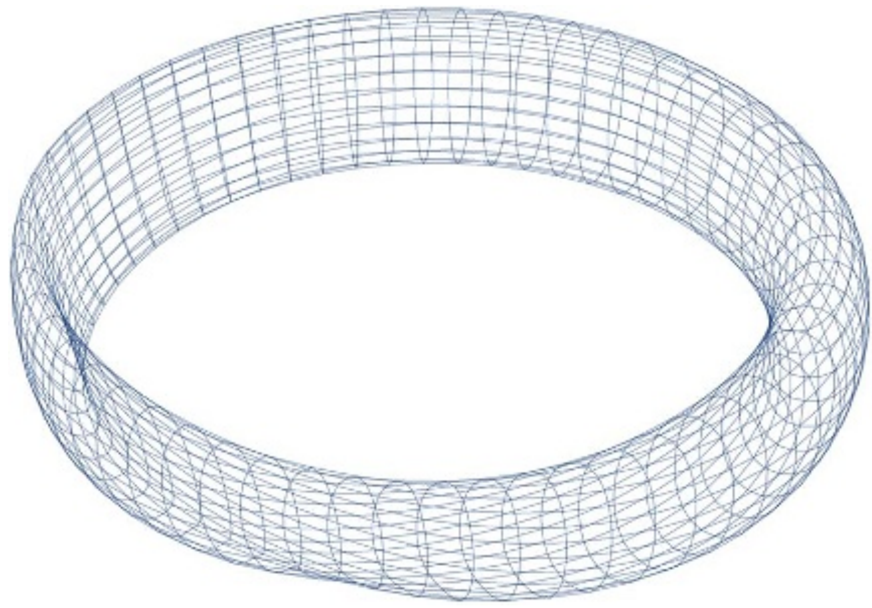
$$2, 2\pi, 4\pi, 2\pi^2, \tfrac{8}{3}\pi^2, \ldots$$

$$\tilde{\zeta}_{\mathbb{Z}}(s) := \pi 2^s \zeta_{\mathbb{Z}}(s/2)$$



Asymptotics:
discrete -> continuous

Asymptotics of determinant of Laplacians



Theorem (Chinta-Jorgenson-K. 2010)

Let A_n be matrices s.t. $A_n/(\det A_n)^{1/d} \to A \in SL_d(\mathbb{R})$. Then as $n \to \infty$

$$\log \det{}^* \Delta_{\mathbb{Z}^d/A_n\mathbb{Z}^d} = \log \det \Delta_{\mathbb{Z}^d} + \frac{2}{d} \log \det A_n + \log \det{}^* \Delta_{\mathbb{R}^d/A\mathbb{Z}^d} + o(1)\,.$$

Studied in physics, previously known: d=2, and lead term all d
(Kasteleyn, Barber, David-Duplantier, Sokal-Starinets…)
In QFT, conjecture of Reshetikhin-Vertman.

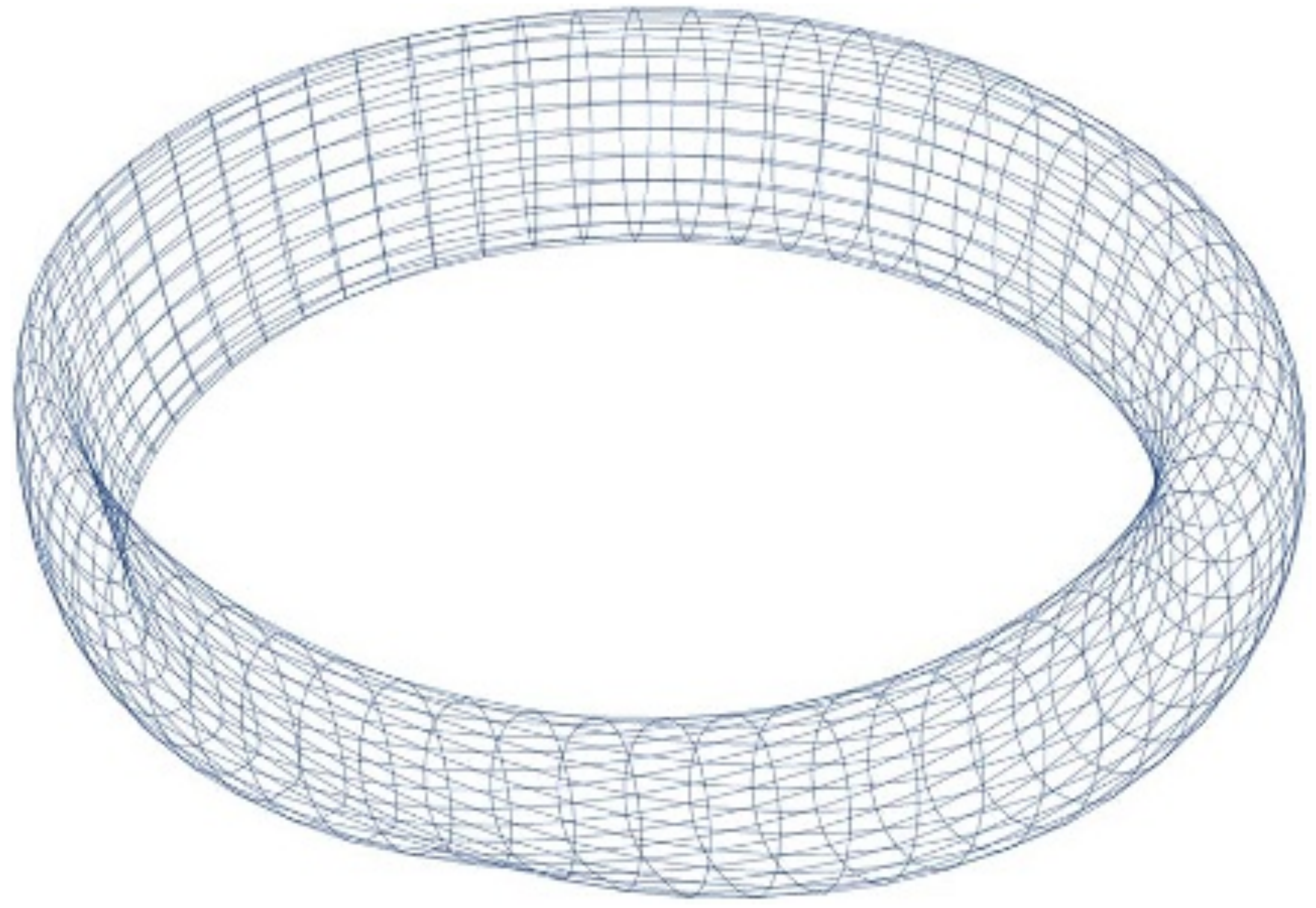
Gives also link between maximal complexity of torus graph and maximal height of torus. -> talk by Julie Rowlett.

Asymptotics of spectral zeta functions

Friedli-K. 2017

Theorem (Friedli-K. 2017) Let $A = diag(a_1, \ldots, a_d)$ and $A_n := nA$. Then:

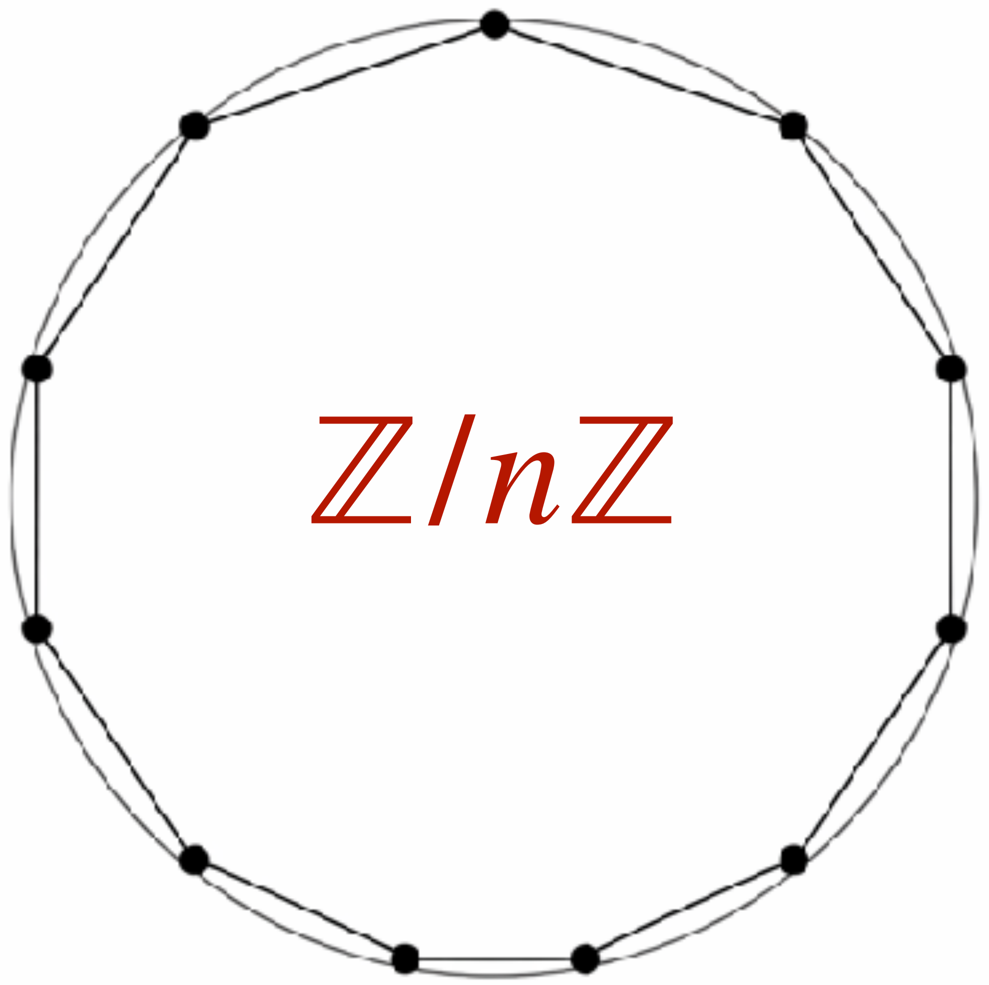
$$\zeta_{\mathbb{Z}^d/A_n\mathbb{Z}^d}(s) = \zeta_{\mathbb{Z}^d}(s) \det A\, n^d + \zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s) n^{2s} + o(n^{2s})\,. \quad n \to \infty$$



Asymptotics in dimension 1

$$2^{-s}\sum_{k=1}^{n-1}\frac{1}{\sin^s(\pi k/n)} = \zeta_{\mathbb{Z}}(s/2)n + 2(2\pi)^{-s}\zeta(s)n^s + \frac{s}{12}(2\pi)^{2-s}\zeta(s-2)n^{s-2} + o(n^{s-2}) \qquad n \to \infty\,.$$

Also for Dirichlet L-functions (Friedli 2016, K.-Müller 2025)



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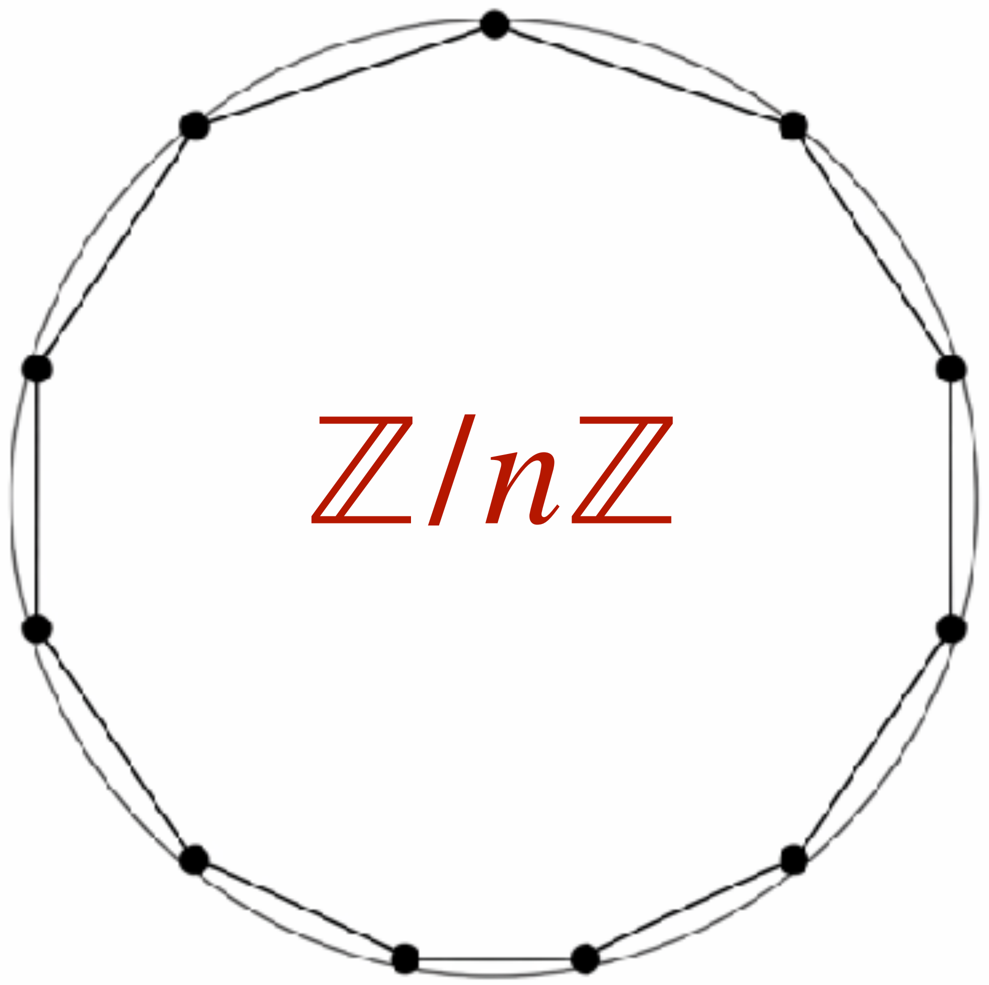


Theorem (Friedli-K, Friedli , K-Müller)

GRH holds iff $\lim_{n\to\infty}\left|\frac{\xi_n(s,\chi)}{\xi_n(1-s,\overline{\chi})}\right| = 1.$

Theorem (K-Müller) Let χ be a real, even, non-principal Dirichlet character.

If $L_n(-m,\chi) \geq 0$, then the corresponding Dirichlet L-function has no real zero.

New recursive identities

Identities among special values

K.-Müller, 2025 (Zagier 1996, Xie-Zhao-Zhao 2024)

And analogous for Dirichlet L-functions

Corollary 1.3. *For any integers $n, m \geq 1$, the following holds:*

$$\boxed{\zeta_{\mathbb{Z}/n\mathbb{Z}}(m) = \sum_{k=0}^{m} a_{m-k}(m)\zeta_{\mathbb{R}/\mathbb{Z}}(k)n^{2k}.}$$

Explicitly, the case $n = 1$ is the formula (1) above, and in general for $n \geq 1$,

$$\frac{(-1)^{m+1}}{4}\sum_{a=0}^{2m} n^{-a}\binom{2m+1}{a+1}\sum_{j=0}^{a+1}(-1)^j\binom{a+1}{j}\frac{a+1-2j}{a+1}\binom{m+jn+(a-1)/2}{2m+a}$$

$$= \sum_{k=0}^{m} a_{m-k}(m)\frac{\zeta(2k)}{(2\pi)^{2k}}n^{2k}.$$

The case n=1

$$\sum_{k=0}^{m} a_{m-k}(m)\frac{\zeta(2k)}{(2\pi)^{2k}} = 0.$$

$$\left(\frac{z/2}{\sin(z/2)}\right)^{2s} = \sum_{k\geq 0} a_k(s)z^{2k}.$$

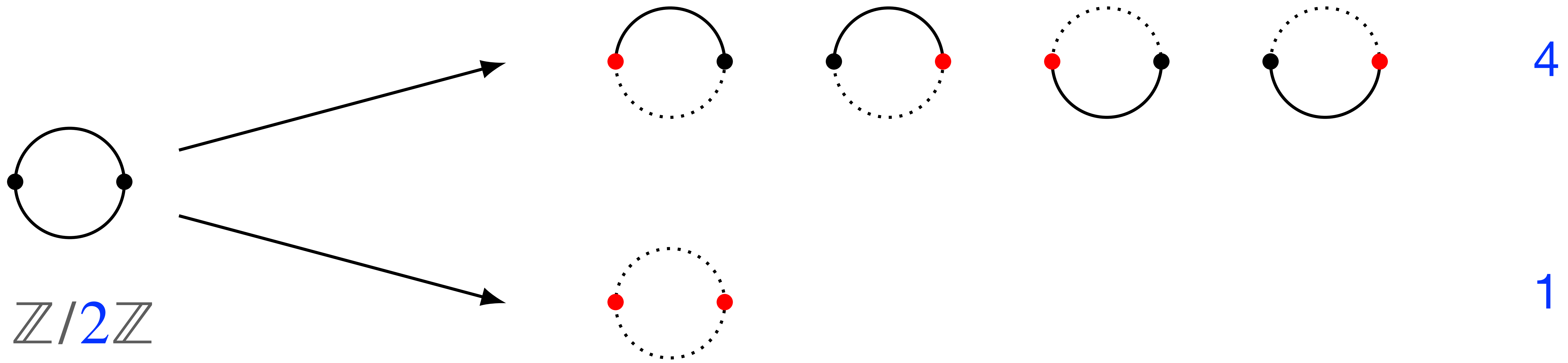
Combinatorics to Euler

Spanning forests to Euler

Special case $m = 1$: $$\zeta_{\mathbb{Z}/n\mathbb{Z}}(1) = \sum_{k=0}^{1} a_{1-k}(1)\zeta_{\mathbb{R}/\mathbb{Z}}(1)n^{2k}$$

n=2

Spanning forests to Euler



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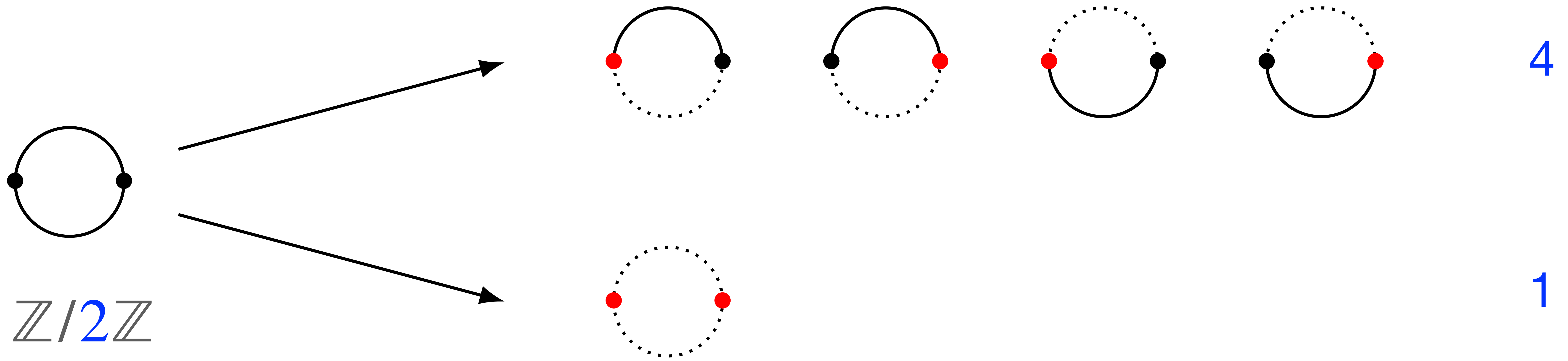

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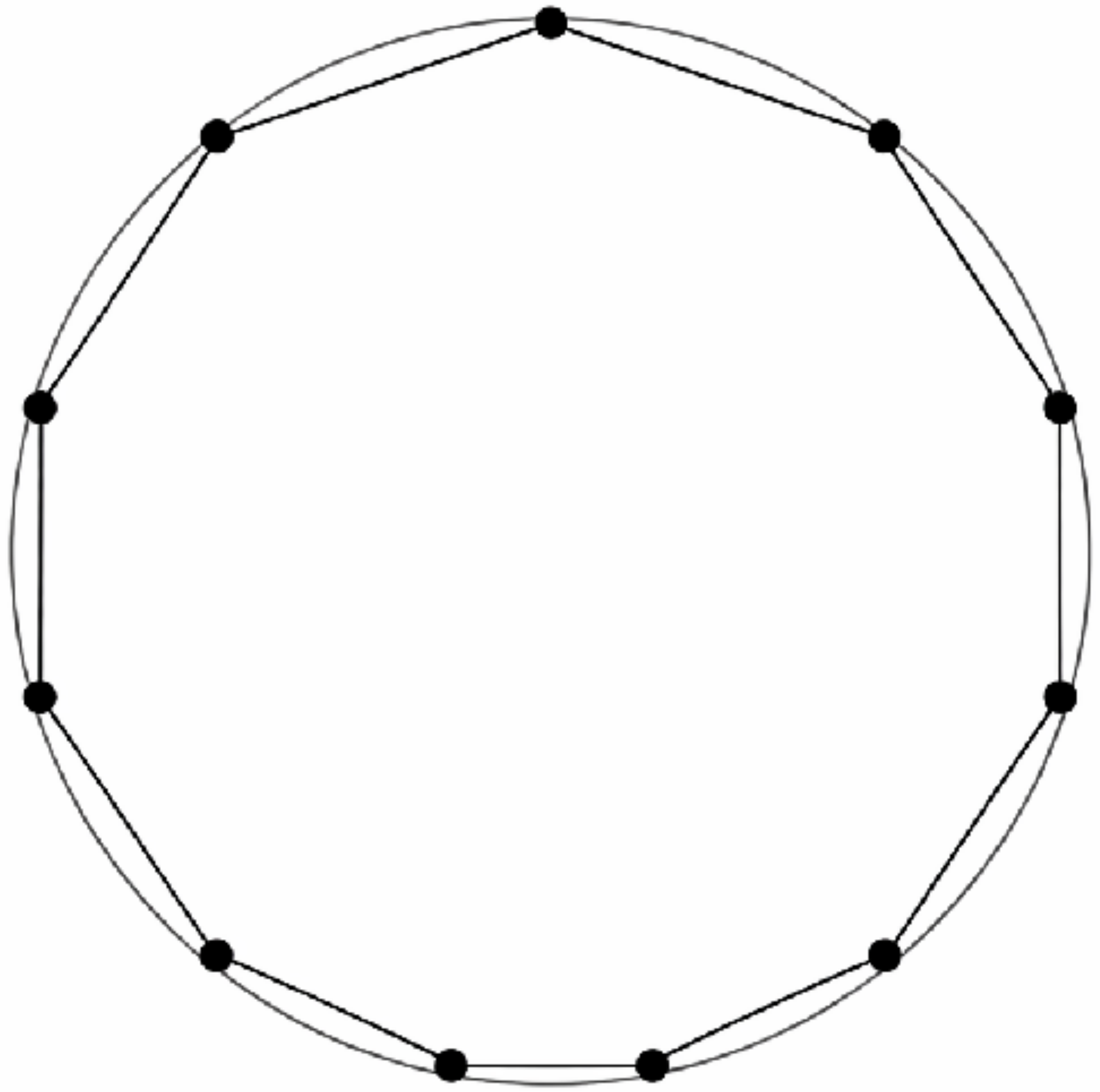
n=2

$$\zeta(2) = \left(\frac{1}{4} - \frac{2}{12}\left(-\frac{1}{2}\right)\right)\frac{(2\pi)^2}{2\cdot 2^2} = \frac{\pi^2}{6}.$$

Scheme for these special values



1. Pass to the circle and spectral zeta
2. Define analogue for discrete circle
3. Study asymptotics $n \to \infty$
4. For integral s, -> symmetric power-sums
5. Relate to elementary symmetric polynomials — Kirchhoff
6. Polynomiality in n
7. Conclude: Exact, finite-terms relations. Combinatorics -> Euler

New identities among special values

$$L_{\dots}(s,\chi) = \sum_{k\geq 0} \frac{\chi(k)}{\lambda^s}.$$

Theorem (K.-Müller, 2025)

Let χ be an odd Drichlet character mod q. Then for any $n \geq 1, m \geq 0$,

$$L_n(m,\chi) = 2\sum_{k=0}^{m} b_{m-k}(m)\, L(2k+1,\chi) \left(\frac{qn}{2\pi}\right)^{2k+1}.$$

$$\left(\frac{z/2}{\sin(z/2)}\right)^{2s} \cot(z/2) = \sum_{k\geq 0} b_k(s) z^{2k-1}.$$

$$L_n(s,\chi) = \sum_{j=1}^{qn-1} \chi(j) \frac{\cot\left(\frac{\pi j}{qn}\right)}{\left[4\sin^2\left(\frac{\pi j}{qn}\right)\right]^s}.$$

n=1 due to Xie-Zhao-Zhao, 2024

Example:

$$L(1,\chi) = \frac{\pi}{2}\frac{1}{qn}\sum_{j=1}^{qn-1}\chi(j)\cot\left(\frac{\pi j}{qn}\right),$$

$$L(1,\chi_{-4}) = \frac{\pi}{2}\frac{1}{4}\sum_{j=1}^{3}\chi_{-4}(j)\cot(\pi j/4) = \ldots = \frac{\pi}{4}$$

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- Everything also for Dirichlet L-functions

Summary

- Spectral zeta functions for graphs; special values in many contexts

- Asymptotics link discrete spectral zetas to classical ones (physics; analytic number theory)

- This gives a combinatorial structure to the special values; and structure to volumes of spheres

- Infinite families of recursive identities between special values of Dirichlet L-functions

- What about spectral zeta funcitons of metric graphs and special values?

Thank you for your attention!

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}}\frac{\Gamma(1/2-s)}{\Gamma(1-s)} = \begin{pmatrix} -2s \\ -s \end{pmatrix}$$

Appears in Selberg's and Langlands' theory of Eisenstein series E(z,s)

$$E(z,s) = \phi(s)E(z,1-s)$$

Gindikin-Karpelevich formula

$$\int_{\mathbb{Q}_p} a(u_\ell)^{\lambda+\rho+w(\gamma_1+\ldots+\gamma_{\ell-1})}du_\ell = \sqrt{\pi}\frac{\Gamma(z_\ell/2)}{\Gamma\left((z_\ell+1)/2\right)}.$$

p=infinity

Spectral zeta vs Ihara zeta function

$$\zeta(s) := \sum_{1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

Spectral zeta = Mellin (trace of heat kernel)
Ihara zeta <-> Laplace (trace of heat kernel) (see Chinta-Jorgenson-K. 2014)

$$\zeta_X^{Ih}(u) = \prod_{[P]} (1 - u^{l(P)})^{-1}$$

$$K_X(t, x_0, x_0) = K_{q+1}(t, x_0, x_0) + e^{-(q+1)t} \sum_{m=1}^{\infty} N_m^0 q^{-m/2} I_m(2\sqrt{q}t)$$

X is (q+1)-regular, vertex-transitive graph

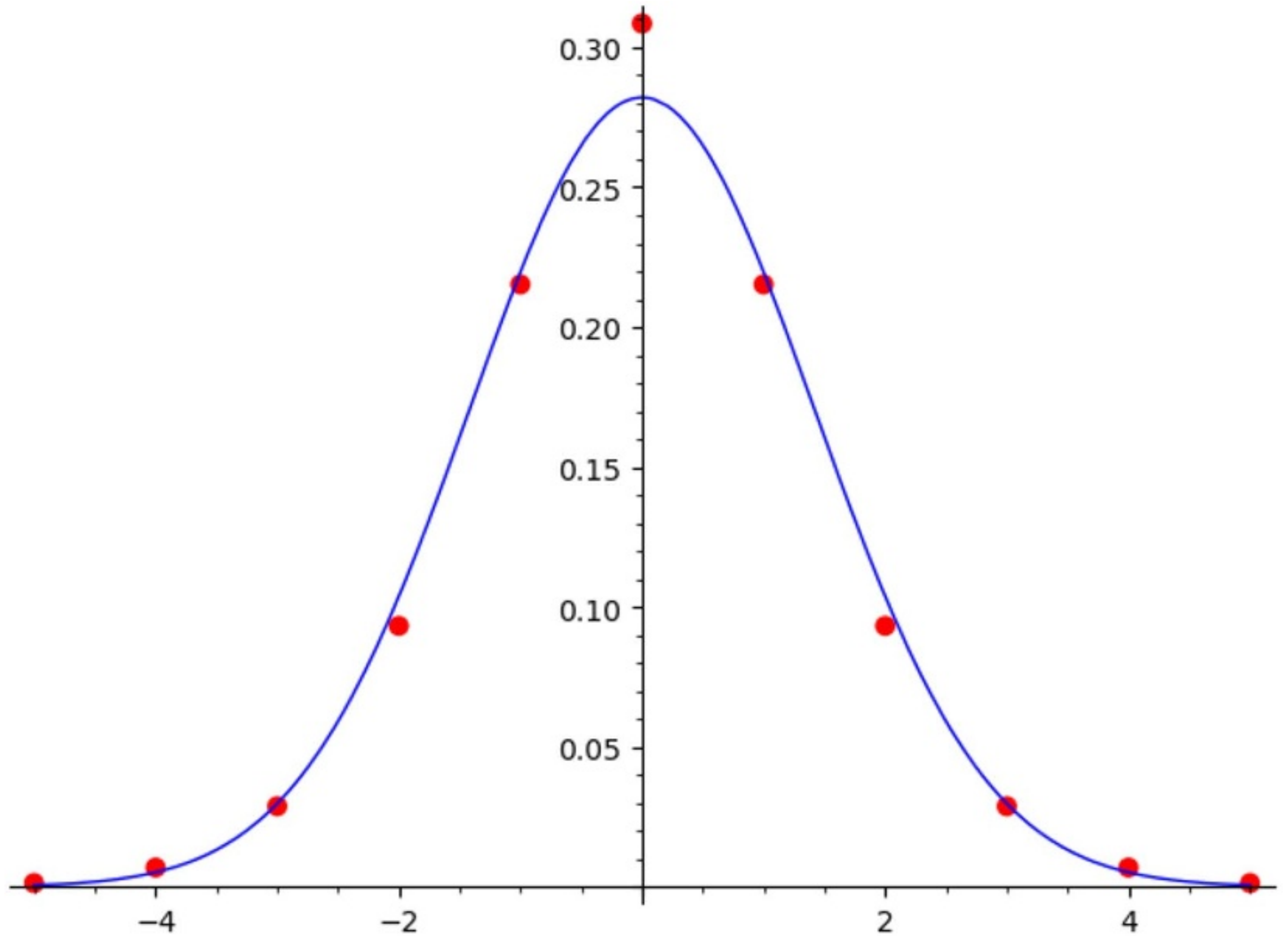
Recall that the Bernoulli numbers B_k are defined via

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{1}{2}x + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}.$$

$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} = \int \lambda^{-s} d\mu(\lambda) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}^*(K_X)(t) t^s \frac{dt}{t}$$

where K_X is the heat kernel, i.e. $(\Delta_x + \frac{\partial}{\partial t})K_X(t, x_0, x) = 0$ with $K_X(0, x_0, x) = \delta_{x_0}(x)$.

$$K_{\mathbb{Z}}(t,0,x) = e^{-2t} I_x(2t)$$



where the functions $b_k(s)$ are polynomials of degree k, given by the Laurent expansion

$$\left(\frac{z/2}{\sin(z/2)}\right)^{2s}\cot(z/2)=\sum_{k\geq 0} b_k(s) z^{2k-1}.$$

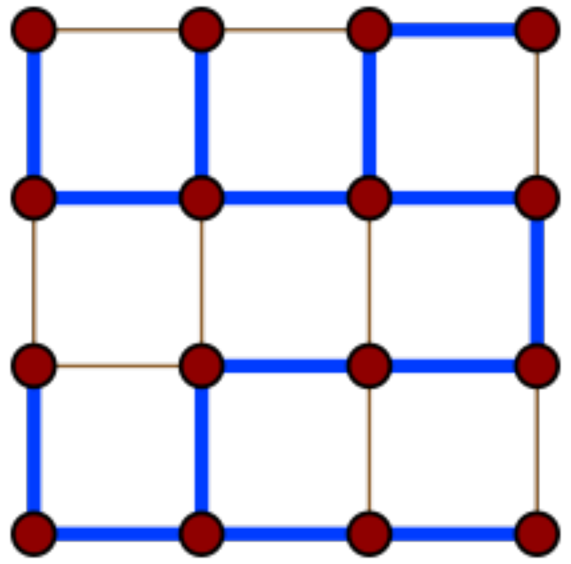
The first coefficients are $b_0(s)=2$, $b_1(s)=(s-1)/6$ and $b_2(s)=(5s^2-9s-2)/720$.

$$L_n(s,\chi)=\sum_{j=1}^{qn-1}\chi(j)\frac{\cot\left(\frac{\pi j}{qn}\right)}{\left[4\sin^2\left(\frac{\pi j}{qn}\right)\right]^s}.$$

$$L_n(m,\chi)=2\sum_{k=0}^{m} b_{m-k}(m)\,L(2k+1,\chi)\left(\frac{qn}{2\pi}\right)^{2k+1}.$$

Counting Molecules and Electrical networks

Spanning trees in graphs



Counting structural isomers: