ONE-DIMENSIONAL SCHRÖDINGER OPERATORS SOLVABLE IN TERMS OF THE CONFLUENT AND HYPERGEOMETRIC FUNCTION

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INTRODUCTION

Consider $-\infty \leq a < b \leq +\infty$ and a (possibly complex) locally integrable function $]a,b[\ni x \mapsto V(x)$. One-dimensional Schrödinger operators, for brevity during my talk called Hamiltonians, are operators of the form

$$L := -\partial_x^2 + V(x),$$

defined initially on $C_\mathrm{c}^\infty(]a,b[)$. Let $L_\bullet$ be a realization of L in the sense of $L^2]a,b[$ possessing a nonempty resolvent set. (Note that this implies that $L_\bullet$ is closed). For z in the resolvent set, the resolvent of $L_\bullet$, will be denoted $\frac{1}{L_\bullet - z}$ and its integral kernel $\frac{1}{L_\bullet - z}(x,y)$ will be called the Green function of $L_\bullet - z$.

I will describe 14 holomorphic families of Hamiltonians with Green functions expressible in hypergeometric class functions. These families on the formal level can be found in numerous papers in the physics literature, including collections of problems on Quantum Mechanics.

Unlike most of the old literature (including my old paper with Michał Wrochna) I will be interested in closed realizations of these Hamiltonians and their Green functions.

The Hamiltonians reducible to Bessel and Whittaker functions are discussed in Part I—joint work with Jinyeop Lee.

Those reducible to Gegenbauer and hypergeometric functions are discussed in Part II—joint work with Pedram Karimi.

1. Reducible to Bessel equation:

   (a) Bessel operator; (b) with exponential potential.

2. Reducible to Whittaker equation:

   (a) Whittaker; (b) Morse; (c) Isotonic oscillator.

3. Reducible to Gegenbauer equation:

   (a) spherical, (b) hyperbolic, (c) deSitterian.

4. Reducible to hypergeometric equation, 1st kind:

   (a) spherical, (b) hyperbolic, (c) deSitterian.

5. Reducible to hypergeometric equation, 2nd kind:

   (a) spherical, (b) hyperbolic, (c) deSitterian.

For a large set of parameters there will be a unique closed realization of L. However, for some values there will be a whole family of closed realizations, corresponding to various boundary conditions. In such cases our boundary conditions will be selected by analytic continuation from the region where we have the uniqueness.

In most applications one is interested in self-adjoint realizations of L. They are possible if V is real. It is however remarkable that the theory of closed realizations is very similar to that of self-adjoint realizations.

Part 1.
1-DIMENSIONAL SCHRÖDINGER OPERATORS SOLVABLE IN TERMS OF BESSEL AND WHITTAKER FUNCTION

in collaboration with JINYEOP LEE

Operators reducible to Bessel equation.

The modified Bessel equation (which is essentially a subclass of the Whittaker equation) is

$$\Big(-\partial_r^2-\frac{1}{r}\partial_r+\frac{m^2}{r^2}+1\Big)g=0.$$

Its standard solutions are the modified Bessel function $I_m(r)$ and the Macdonald function $K_m(r)$. They are characterized by

$$\begin{aligned} I_m(r)&\simeq\frac{1}{\Gamma(m+1)}\Big(\frac{r}{2}\Big)^m, \quad && r\to 0,\\ K_m(r)&\simeq\frac{\sqrt{2}\mathrm{e}^{-r}}{\sqrt{\pi r}}, && r\to+\infty. \end{aligned}$$

The so-called 1d modified Bessel equation

$$\Big(-\partial_r^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{r^2} + 1\Big) g = 0.$$

is equivalent to the usual modified Bessel equation by a simple gauge transformation. It is solved by

$$\mathcal{I}_m(r) := \sqrt{\frac{\pi r}{2}} I_m(r),$$
$$\mathcal{K}_m(r) := \sqrt{\frac{2r}{\pi}} K_m(r).$$

Let $m \in \mathbb{C}$. The Bessel operator is defined on $C_{\mathrm{c}}^{\infty}(]0,\infty[)$ by

$$H_m := -\partial_r^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{r^2}.$$

For $\mathrm{Re} m \geq 1$ there exists a unique closed realization of H_m. In particular, for real $m \geq 1$, H_m is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(]0,\infty[)$. It extends to a holomorphic family for $\mathrm{Re} m > -1$. For $\mathrm{Re} k > 0$ its Green function is

$$\frac{1}{H_m + k^2}(x,y) = \frac{1}{k}\begin{cases} \mathcal{I}_m(kx)\mathcal{K}_m(ky) & \text{if } 0 < x < y, \\ \mathcal{I}_m(ky)\mathcal{K}_m(kx) & \text{if } 0 < y < x. \end{cases}$$

The Schrödinger operator with the exponential potential is given on $C_{\rm c}^\infty(]0,\infty[)$ by

$$M_k := -\partial_x^2 + k^2 {\rm e}^{2x}.$$

For ${\rm Re} k > 0$ there exists a unique closed realization of M_k, depending holomorphically on k. Its Green function is

$$\frac{1}{M_k + m^2}(x,y) = \begin{cases} I_m(k{\rm e}^x)\, K_m(k{\rm e}^y) & \text{if } x < y, \\ I_m(k{\rm e}^y)\, K_m(k{\rm e}^x) & \text{if } y < x. \end{cases}$$

The following transmutation relation links Green functions of the two above families of operators:

$$\frac{1}{M_k + m^2}(x, y) = \mathrm{e}^{-\frac{x}{2}} \frac{1}{H_m + k^2}(\mathrm{e}^x, \mathrm{e}^y)\, \mathrm{e}^{-\frac{y}{2}}.$$

Note that the coupling constant becomes the spectral parameter and the other way around.

Operators reducible to Whittaker equation

The Whittaker equation

$$\Big(-\partial_r^2+\big(m^2-\frac14\big)\frac{1}{r^2}-\frac{\beta}{r}+\frac14\Big)v(r)=0.$$

is closely related to two kinds of the confluent equation:

1. (Kummer's) ${}_1F_1$ equation,
2. the ${}_2F_0$ equation.

The ${}_1F_1$ equation $\big(r\,\partial_r^2 + (c-r)\,\partial_r - a\big)f(r) = 0$ as its standard solution has the confluent function

$$ {}_1\mathbf{F}_1(a;c;r) := \sum_{k=0}^{\infty} \frac{(a)_k}{\Gamma(c+k)} \frac{r^k}{k!}. $$

The ${}_2F_0$ equation $\big(w^2\partial_w^2 - (1-(1+a+b)w)\partial_w + ab\big)v(w) = 0$ as its standard solution has the ${}_2F_0$ function with a branch point at 0, and the following asymptotic expansion:

$$ {}_2F_0(a,b;-;w) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} w^n, \quad |\arg(w)| < \pi - \varepsilon. $$

The Whittaker equation has the following standard solutions:

$$\mathcal{I}_{\beta,m}(r) := r^{\frac12+m}\mathrm{e}^{\mp\frac{r}{2}}{}_1\mathbf{F}_1\Big(\frac12+m\mp\beta;\ 1+2m;\ \pm r\Big),$$
$$\mathcal{K}_{\beta,m}(r) := r^{\beta}\mathrm{e}^{-\frac{r}{2}}{}_2F_0\Big(\frac12+m-\beta,\frac12-m-\beta;-;-r^{-1}\Big).$$

They are characterized by

$$\mathcal{I}_{\beta,m}(r) \simeq \frac{1}{\Gamma(2m+1)}r^{\frac12+m}, \quad r\to 0,$$
$$\mathcal{K}_{\beta,m}(r) \simeq r^{\beta}\mathrm{e}^{-r}, \qquad r\to +\infty.$$

We will also need the following closely related functions:

$$I_{\beta,m}(r) := \sqrt{\frac{2}{\pi r}}\,\mathcal{I}_{\beta,m}(r),$$
$$K_{\beta,m}(r) := \sqrt{\frac{\pi}{2r}}\,\mathcal{K}_{\beta,m}(r);$$

$$\mathbb{I}_{\beta,m}(v) := v^{-\frac{1}{2}}\mathcal{I}_{\frac{\beta}{2},\frac{m}{2}}(v^2),$$
$$\mathbb{K}_{\beta,m}(v) := v^{-\frac{1}{2}}\mathcal{K}_{\frac{\beta}{2},\frac{m}{2}}(v^2).$$

The Whittaker operator is defined on $C_{\mathrm{c}}^{\infty}(]0,\infty[)$ by

$$H_m := -\partial_r^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{r^2} - \frac{\beta}{r}.$$

For $\mathrm{Re} m \geq 1$ there exists a unique closed realization of $H_{\beta,m}$. It extends to a holomorphic family for $\mathrm{Re} m > -1$ with a singularity at $(\beta, m) = (0, -\frac{1}{2})$. Outside of spectrum, for $\mathrm{Re} k > 0$ its Green function is

$$\frac{1}{H_{\beta,m} + k^2}(x,y)$$

$$= \frac{1}{2k}\,\Gamma\big(\tfrac{1}{2} + m - \tfrac{\beta}{2k}\big) \begin{cases} \mathcal{I}_{\frac{\beta}{2k},m}(2kx)\mathcal{K}_{\frac{\beta}{2k},m}(2ky) & \text{for } 0 < x < y, \\ \mathcal{I}_{\frac{\beta}{2k},m}(2ky)\mathcal{K}_{\frac{\beta}{2k},m}(2kx) & \text{for } 0 < y < x. \end{cases}$$

Note a curious identity

$$H_{\beta,-\frac{1}{2}} = H_{\beta,\frac{1}{2}}, \quad \beta \neq 0.$$

This identity does not extend to $\beta = 0$. In fact, for all $\mathrm{Re}m > -1$, the Bessel operator is a special case of the Whittaker operator: $H_{0,m} = H_m$. Moreover

$$\begin{aligned} H_{-\frac{1}{2}} &= H_{\mathrm{N}}, \quad \text{Laplacian on halfline with Neumann b.c.} \\ H_{\frac{1}{2}} &= H_{\mathrm{D}}, \quad \text{Laplacian on halfline with Dirichlet b.c..} \end{aligned}$$

Clearly, $H_{\mathrm{N}} \neq H_{\mathrm{D}}$, which implies that the Whittaker operator has a singularity at $(\beta, m) = (0, -\frac{1}{2})$.

The Schrödinger operator with the Morse potential is formally

$$M_{\beta,k} := -\partial_x^2 + k^2 \mathrm{e}^{2x} - \beta \mathrm{e}^x.$$

For $\mathrm{Re}k > 0$ it possesses a unique closed realization in the sense of $L^2(\mathbb{R})$. It depends analytically on k. For $\mathrm{Re}m > 0$ its Green function is

$$\frac{1}{M_{\beta,k} + m^2}(x,y)$$

$$= \Gamma\big(\tfrac12 + m - \tfrac{\beta}{2k}\big) \begin{cases} I_{\frac{\beta}{2k},m}(2k\mathrm{e}^x) K_{\frac{\beta}{2k},m}(2k\mathrm{e}^y), & \text{if } x < y, \\ I_{\frac{\beta}{2k},m}(2k\mathrm{e}^y) K_{\frac{\beta}{2k},m}(2k\mathrm{e}^x), & \text{if } y < x. \end{cases}$$

The isotonic harmonic oscillator is formally defined by

$$N_{k,m} := -\partial_v^2 + \left(m^2 - \tfrac{1}{4}\right)\frac{1}{v^2} + k^2v^2.$$

For $\mathrm{Re}m \geq 1$ it has a unique closed realization, which extends holomorphically to $\mathrm{Re}m > -1$. For $\mathrm{Re}k > 0$ it depends holomorphically also on k with the Green function

$$\frac{1}{N_{k,m} - 2\beta}(u,v)$$

$$= \tfrac{1}{2}\,\Gamma\Big(\tfrac{1}{2} + \tfrac{m}{2} - \tfrac{\beta}{2k}\Big)\begin{cases} \mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}u)\,\mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}v) & \text{if } 0 < u < v, \\ \mathbb{I}_{\frac{\beta}{k},m}(\sqrt{k}v)\,\mathbb{K}_{\frac{\beta}{k},m}(\sqrt{k}u) & \text{if } 0 < v < u. \end{cases}$$

We have transmutation relations permuting m, k and β:

$$\frac{1}{N_{k,m}-2\beta}(u,v)=2\Big(\frac{u^2}{2}\Big)^{-\frac14}\frac{1}{H_{\beta,\frac{m}{2}}+k^2}\Big(\frac{u^2}{2},\frac{v^2}{2}\Big)\Big(\frac{v^2}{2}\Big)^{-\frac14},$$

$$\frac{1}{M_{\beta,k}+m^2}(x,y)=\mathrm{e}^{-\frac{x}{2}}\frac{1}{H_{\beta,m}+k^2}(\mathrm{e}^{x},\mathrm{e}^{y})\mathrm{e}^{-\frac{y}{2}},$$

$$\frac{1}{N_{k,m}-2\beta}(u,v)=2\Big(\frac{u^2}{2}\Big)^{\frac14}\frac{1}{M_{\beta,k}+\big(\frac{m}{2}\big)^2}\Big(\log\frac{u^2}{2},\log\frac{v^2}{2}\Big)\Big(\frac{v^2}{2}\Big)^{\frac14}.$$

Part 2.

1-DIMENSIONAL SCHRÖDINGER OPERATORS SOLVABLE IN TERMS OF THE GEGENBAUER AND HYPERGEOMETRIC FUNCTION

in collaboration with PEDRAM KARIMI

The hypergeometric equation is given by

$$\mathcal{F}(a,b;c) := z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab.$$

The Gegenbauer equation (essentially, a special case of the hypergeometric equation) is given by

$$\mathcal{G}_{\alpha,\lambda} := (1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \left(\alpha+\tfrac{1}{2}\right)^2.$$

I will describe 3×3 families of 1-dim. Schrödinger operators reducible to the Gegenbauer or hypergeometric equation:

1. Gegenbauer Hamiltonians:

(a) spherical, (b) hyperbolic, (c) deSitterian.

2. Hypergeometric Hamiltonians of the first kind:

(a) spherical, (b) hyperbolic, (c) deSitterian.

3. Hypergeometric Hamiltonians of the second kind:

(a) spherical, (b) hyperbolic, (c) deSitterian.

In the literature these Hamiltonians have various names:

Our name	Name in DerWro	Name in Cotfas et al.
Spherical of 1st kind	Trig. Pöschl-Teller	Scarf I
Hyperbolic of 1st kind	Hyp. Pöschl-Teller	Scarf II
DeSitterian of 1st kind	Scarf	Gen. Pöschl-Teller
Spherical of 2nd kind	Rosen-Morse	Rosen-Morse I
Hyperbolic of 2nd kind	Eckart	Eckart
DeSitterian of 2nd kind	Manning-Rosen	Rosen-Morse II

Let us apply the Liouville transformation to the Gegenbauer operator:

$$-(1-w^2)^{\frac{\alpha}{2}+\frac{1}{4}}\mathcal{G}_{\alpha,\lambda}(w,\partial_w)(1-w^2)^{-\frac{\alpha}{2}-\frac{1}{4}}$$

$$=-\left((1-w^2)^{\frac{1}{2}}\partial_w\right)^2+\left(\alpha^2-\frac{1}{4}\right)\frac{1}{1-w^2}-\lambda^2.$$

Then we substitute

spherical case:	$w=\cos r,$	$w\in]-1,1[;$
hyperbolic case:	$w=-\cosh r,$	$w\in]-\infty,-1[;$
deSitterian case:	$w=-\mathrm{i}\sinh r,$	$w\in]-\mathrm{i}\infty,+\mathrm{i}\infty[.$

Spherical Gegenbauer Hamiltonian, $L^2]0,\pi[$:

$$L_\alpha^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sin^2 r}.$$

Hyperbolic Gegenbauer Hamiltonian, $L^2(\mathbb{R}_+)$:

$$L_\alpha^{\mathrm{h}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sinh^2 r}.$$

DeSitterian Gegenbauer Hamiltonian, $L^2(\mathbb{R})$:

$$L_\alpha^{\mathrm{dS}} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}.$$

All Gegenbauer Hamiltonians have a natural geometric interpretation:

1. The Laplacian on the sphere $\mathbb{S}^d$ in spherical coordinates becomes the spherical Gegenbauer Hamiltonian.
2. The Laplacian on the hyperbolic space $\mathbb{H}^d$ in spherical coordinates becomes the hyperbolic Gegenbauer Hamiltonian.
3. The d'Alembertian on the deSitter space dS^d in the so-called global coordinates becomes the deSitterian Gegebauer Hamiltonian.

Hypergeometric Hamiltonians of the 1st kind.

$$-z^{\frac{\alpha}{2}+\frac14}(1-z)^{\frac{\beta}{2}+\frac14}\mathcal{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha+\beta-\mu+1}{2};1+\alpha\right)z^{-\frac{\alpha}{2}-\frac14}(1-z)^{-\frac{\beta}{2}-\frac14}$$

$$=-\Big(\big(z(1-z)\big)^{\frac12}\partial_z\Big)^2+\left(\alpha^2-\frac14\right)\frac{1}{4z}+\left(\beta^2-\frac14\right)\frac{1}{4(1-z)}-\frac{\mu^2}{4}.$$

spherical case: $\quad z=\sin^2 r, \qquad z\in]0,1[;$

hyperbolic case: $\quad z=-\sinh^2 r, \qquad z\in]-\infty,0[;$

deSitterian case: $\quad z=\frac12-\frac{\mathrm{i}}{2}\sinh 2r, \qquad z\in\left]\frac12-\mathrm{i}\infty,\frac12+\mathrm{i}\infty\right[.$

Spherical hypergeometric Hamiltonian of the 1st kind, $L^2]0,\pi[$:

$$L_{\alpha,\beta}^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sin^2\frac{r}{2}} + \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cos^2\frac{r}{2}}.$$

Hyperbolic hypergeometric Hamiltonian of the 1st kind, $L^2(]0,\infty[)$:

$$L_{\alpha,\beta}^{\mathrm{h}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sinh^2\frac{r}{2}} - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cosh^2\frac{r}{2}}.$$

DeSitterian hypergeometric Hamiltonian of the 1st kind, $L^2(\mathbb{R})$:

$$\begin{aligned} L_{\alpha,\beta}^{\mathrm{dS}} :=& -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} + \frac{\mathrm{i}\sinh r}{2}\right) \\ & - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} - \frac{\mathrm{i}\sinh r}{2}\right). \end{aligned}$$

All hypergeometric Hamiltonians of the 1st kind have a natural geometric interpretation:

1. The Laplacian on the sphere $\mathbb{S}^{p+q}$ in double spherical coordinates becomes the spherical Gegenbauer Hamiltonian.
2. The pseudoLaplacian on the hyperboloid $\mathbb{H}^{p,q}$ in double spherical coordinates becomes the hyperbolic Gegenbauer Hamiltonian.
3. The pseudoLaplacian on the hyperboloid $\mathbb{H}^{p-1,p}$ in the complex coordinates becomes the deSitterian Gegebauer Hamiltonian.

Hypergeometric Hamiltonians of the 2nd kind.

$$-4z^{1+\frac{\alpha}{2}}(1-z)^{1+\frac{\beta}{2}}\mathcal{F}(\tfrac{\alpha+\beta+\mu+1}{2},\tfrac{\alpha+\beta-\mu+1}{2};1+\alpha)z^{-\frac{\alpha}{2}}(1-z)^{-\frac{\beta}{2}}$$

$$=-\Big(2z(1-z)\partial_z\Big)^2+\alpha^2(1-z)+\beta^2z-(\mu^2-1)z(1-z).$$

spherical case: $\quad z=\dfrac{1}{1-\mathrm{e}^{2\mathrm{i}u}},\qquad z\in\Big]\dfrac{1}{2}-\mathrm{i}\infty,\dfrac{1}{2}+\mathrm{i}\infty\Big[;$

hyperbolic case: $\quad z=\dfrac{1}{1-\mathrm{e}^{2u}},\qquad z\in]-\infty,0[;$

deSitterian case: $\quad z=\dfrac{1}{1+\mathrm{e}^{2u}},\qquad z\in]0,1[.$

Spherical hypergeometric Hamiltonian of the 2nd kind, $L^2]0,\pi[$:

$$K^{\mathrm{s}}_{\tau,\mu} := -\partial_u^2 + \tau\frac{\cos u}{\sin u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sin^2 u}.$$

Hyperbolic hypergeometric Hamiltonian of the 2nd kind, $L^2(]0,\infty[)$:

$$K^{\mathrm{h}}_{\kappa,\mu} := -\partial_u^2 + \kappa\frac{\cosh u}{\sinh u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sinh^2 u}.$$

DeSitterian hypergeometric Hamiltonian of the 2nd kind, $L^2(\mathbb{R})$:

$$K^{\mathrm{dS}}_{\kappa,\mu} := -\partial_u^2 - \kappa\frac{\sinh w}{\cosh w} - \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\cosh^2 w}.$$

Open Question: Find a geometric interpretation of hypergeometric Hamiltonians of the 2nd kind!

Gegenbauer Hamiltonians are special cases of hypergeometric Hamiltonians of both the first and second kind. In fact, we have the following coincidencies:

$$
\begin{aligned}
L_\alpha^{\mathrm{s}} =& L_{\alpha,\alpha}^{\mathrm{s}} = K_{0,2\alpha}^{\mathrm{s}};\\
L_\alpha^{\mathrm{h}} =& L_{\alpha,\alpha}^{\mathrm{h}} = K_{0,2\alpha}^{\mathrm{h}};\\
L_\alpha^{\mathrm{dS}} =& L_{\alpha,\alpha}^{\mathrm{dS}} = K_{0,2\alpha}^{\mathrm{dS}}.
\end{aligned}
$$

Moreover,

$$K^{\mathrm{s}}_{\tau,-1} = K^{\mathrm{s}}_{\tau,1}, \quad \tau \neq 0,$$
$$K^{\mathrm{h}}_{\kappa,-1} = K^{\mathrm{h}}_{\kappa,1}. \quad \kappa \neq 0.$$

But

$$K^{\mathrm{s}}_{-1} = L^{\mathrm{s}}_{-\frac{1}{2}} = -\Delta_{\mathrm{NN}} \neq K^{\mathrm{s}}_{1} = L^{\mathrm{s}}_{\frac{1}{2}} = -\Delta_{\mathrm{DD}},$$
$$K^{\mathrm{h}}_{-1} = L^{\mathrm{h}}_{-\frac{1}{2}} = -\Delta_{\mathrm{N}} \neq K^{\mathrm{h}}_{1} = L^{\mathrm{h}}_{\frac{1}{2}} = -\Delta_{\mathrm{D}}.$$

Therefore, $(0,-1)$ is a singularity of the functions $(\tau,\mu) \mapsto K^{\mathrm{s}}_{\tau,\mu}$ and $(\kappa,\mu) \mapsto K^{\mathrm{h}}_{\kappa,\mu}$.

There are many transmutation relations:

1. Gegenbauer spherical — Gegenbauer deSitterian;
2. 1st kind spherical — 1st kind hyperbolic;
3. 2nd kind spherical — 2nd kind hyperbolic;
4. 1st kind spherical — 2nd kind deSitterian;
5. 1st kind hyperbolic — 2nd kind hyperbolic;
6. 1st kind deSitterian — 2nd kind spherical.

THANK YOU FOR YOUR ATTENTION.