Partial Isospectrality of Rank Two Singular Pertubations

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- We consider a semi bounded self adjoint operator A in Hilbert space H
- Assume A has discrete spectrum $\lambda_1 < \lambda_2 < \dots$ with corresponding e.f. $\psi_1, \psi_2, \dots$
- Let H_1 be the form domain and $H_{-1} = H_1^*$. Note $\mathrm{dom}(A) \subset H_1 \subset H \subset H_{-1}$
- Let $\varphi_1, \varphi_2 \in H_{-1}$ be H independent i.e. if $a\varphi_1 + b\varphi_2 \in H$ then $a = b = 0$

Let $\alpha_1, \alpha_2 \in \mathbb{R} \cup \{\infty\}$.
Define

$$Dom(A_{\alpha_1,\alpha_2}) = \{u \in H_1 : Au + \alpha_1\langle\varphi_1, u\rangle\varphi_1 + \alpha_2\langle\varphi_2, u\rangle\varphi_2 \in H\}$$
$$A_{\alpha_1,\alpha_2} u = Au + \alpha_1\langle\varphi_1, u\rangle\varphi_1 + \alpha_2\langle\varphi_2, u\rangle\varphi_2.$$

Then A_{α_1,α_2} is self adjoint and has discrete spectrum $\lambda_1^{\alpha_1,\alpha_2} \leq \lambda_2^{\alpha_1,\alpha_2} \leq \dots$

For rank one pertubations $A + \alpha\langle\varphi, \cdot\rangle\varphi$ we have **eigenvalue interlacing**: if $\alpha < \beta$ then $\lambda_k^\alpha \leq \lambda_k^\beta \leq \lambda_{k+1}^\alpha$ and we only have an equality if there is ψ eigenfunction of A_α, A_β corresponding to λ_k^β with $\langle\varphi, \psi\rangle = 0$
In particular we have $\lambda_1^\alpha = \lambda_1$ only when $\alpha\langle\varphi, \psi_1\rangle = 0$.

We want to know under what conditions $\lambda_1^{\alpha_1,\alpha_2} = \lambda_1$ and $\lambda_2^{\alpha_1,\alpha_2} = \lambda_2$.
We first characterize when λ_k is an eigenvalue of A_{α_1,α_2} then we look at the order of the eigenvalues

Let

$$Q(\lambda) = \begin{pmatrix} \langle \varphi_1, \frac{1}{A-\lambda}\varphi_1 \rangle & \langle \varphi_1, \frac{1}{A-\lambda}\varphi_2 \rangle \\ \langle \varphi_2, \frac{1}{A-\lambda}\varphi_1 \rangle & \langle \varphi_2, \frac{1}{A-\lambda}\varphi_2 \rangle \end{pmatrix}.$$

For λ not in the spectrum of A, λ is an eigenvalue of A_{α_1,α_2} iff

$$\det\left(Q(\lambda) + \begin{pmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{pmatrix} \right) = 0$$

We want to consider λ_k simple eigenvalue of A. We use a regularization of Q:

$$C_k(\lambda) = \begin{pmatrix} \langle \varphi_1, P_k^\perp \frac{1}{A-\lambda} P_k^\perp \varphi_1 \rangle & \langle \varphi_1, P_k^\perp \frac{1}{A-\lambda} P_k^\perp \varphi_2 \rangle \\ \langle \varphi_2, P_k^\perp \frac{1}{A-\lambda} P_k^\perp \varphi_1 \rangle & \langle \varphi_2, P_k^\perp \frac{1}{A-\lambda} P_k^\perp \varphi_2 \rangle \end{pmatrix}$$

where P_k is the projection on ψ_k and $P_k^\perp = 1 - P_k$.

Fixing one eigenvalue

Theorem

A simple eigenvalue λ_k of A is an eigenvalue of A_{α_1,α_2} if and only if

$$|\langle\varphi_1,\psi_k\rangle|^2\alpha_1 + |\langle\varphi_2,\psi_k\rangle|^2\alpha_2 + c_k\alpha_1\alpha_2 = 0$$

where

$$c_k = \left\langle \begin{pmatrix} -\langle\psi_k,\varphi_2\rangle \\ \langle\psi_k,\varphi_1\rangle \end{pmatrix}, C_k(\lambda_k)\begin{pmatrix} -\langle\psi_k,\varphi_2\rangle \\ \langle\psi_k,\varphi_1\rangle \end{pmatrix} \right\rangle$$

Proof

Every element $u \in \mathrm{dom}(A_{\alpha_1,\alpha_2})$ can be written as

$$u = \hat{u} + P_k^{\perp} \frac{1}{A - \lambda_k} P_k^{\perp} (a_1 \varphi_1 + a_2 \varphi_2)$$

where $\hat{u} \in \mathrm{dom}(A)$ and $a_1, a_2 \in \mathbb{C}$ satisfy

$$\left(C_k(\lambda_k) + \begin{pmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{pmatrix} \right) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \langle \varphi_1, \hat{u} \rangle \\ \langle \varphi_2, \hat{u} \rangle \end{pmatrix}$$

Proof

We want u to be an eigenfunction of A_{α_1,α_2}:

$$0 = (A_{\alpha_1,\alpha_2} - \lambda_k)u = (A - \lambda_k)\hat{u} + P_k(a_1\varphi_1 + a_2\varphi_2)$$

$(A - \lambda_k)\hat{u}$ and $P_k(a_1\varphi_1 + a_2\varphi_2)$ are orthogonal so both terms are zero. Hence $\hat{u}$ and $a_1\varphi_1 + a_2\varphi_2$ are orthogonal
So we get

$$\begin{aligned}\left\langle \left(C_k(\lambda_k) + \begin{pmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{pmatrix} \right) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} \langle \varphi_1, \hat{u} \rangle \\ \langle \varphi_2, \hat{u} \rangle \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\rangle \\ &= \langle \hat{u}, a_1\varphi_1 + a_2\varphi_2 \rangle \\ &= 0\end{aligned}$$

Since $P_k(a_1\varphi_1 + a_2\varphi_2) = 0$, $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a multiple of $\begin{pmatrix} \frac{-\overline{\langle \varphi_2, \psi_k \rangle}}{\overline{\langle \varphi_1, \psi_k \rangle}} \end{pmatrix}$

M-function

We can also write the condition using the M-function $M(\lambda) = -Q(\lambda)^{-1}$. This function is defined for all λ not in the spectrum of $A_{\infty,\infty}$. We have $c_k = -\frac{|\langle\varphi_1,\psi_k\rangle|^2+|\langle\varphi_2,\psi_k\rangle|^2}{M_{11}(\lambda_k))+M_{22}(\lambda_k))}$

Theorem

A simple eigenvalue λ_k of A, which is not in the spectrum of $A_{\infty,\infty}$ is an eigenvalue of A_{α_1,α_2} if and only if

$$-M_{22}(\lambda_k)\alpha_1 - M_{11}(\lambda_k)\alpha_2 + \alpha_1\alpha_2 = 0$$

Fixing two eigenvalues

Requiring both λ_1 and λ_2 to be eigenvalues of A_{α_1,α_2} gives:

$$\begin{cases} |\langle\varphi_1,\psi_1\rangle|^2\alpha_1 + |\langle\varphi_2,\psi_1\rangle|^2\alpha_2 + c_1\alpha_1\alpha_2 = 0 \\ |\langle\varphi_1,\psi_2\rangle|^2\alpha_1 + |\langle\varphi_2,\psi_2\rangle|^2\alpha_2 + c_2\alpha_1\alpha_2 = 0. \end{cases}$$

Let us denote

$$\begin{aligned} \Delta &:= \det \begin{bmatrix} |\langle\varphi_1,\psi_1\rangle|^2 & |\langle\varphi_2,\psi_1\rangle|^2 \\ |\langle\varphi_1,\psi_2\rangle|^2 & |\langle\varphi_2,\psi_2\rangle|^2 \end{bmatrix} \\ \Delta_1 &:= \det \begin{bmatrix} c_1 & |\langle\varphi_1,\psi_1\rangle|^2 \\ c_2 & |\langle\varphi_1,\psi_2\rangle|^2 \end{bmatrix} \\ \Delta_2 &:= \det \begin{bmatrix} c_1 & |\langle\varphi_2,\psi_1\rangle|^2 \\ c_2 & |\langle\varphi_2,\psi_2\rangle|^2 \end{bmatrix}. \end{aligned}$$

Theorem

1. *If $\Delta \neq 0$, then there exists a unique (non-zero) solution*

$$\alpha_1 = \frac{\Delta}{\Delta_1} = \frac{|\langle\varphi_1,\psi_1\rangle|^2|\langle\varphi_2,\psi_2\rangle|^2 - |\langle\varphi_1,\psi_2\rangle|^2|\langle\varphi_2,\psi_1\rangle|^2}{c_1|\langle\varphi_1,\psi_2\rangle|^2 - c_2|\langle\varphi_1,\psi_1\rangle|^2},$$

$$\alpha_2 = -\frac{\Delta}{\Delta_2} = \frac{|\langle\varphi_1,\psi_1\rangle|^2|\langle\varphi_2,\psi_2\rangle|^2 - |\langle\varphi_1,\psi_2\rangle|^2|\langle\varphi_2,\psi_1\rangle|^2}{c_1|\langle\varphi_2,\psi_2\rangle|^2 - c_2|\langle\varphi_2,\psi_1\rangle|^2}.$$

   for which the spectrum of the perturbed operator A_{α_1,α_2} contains the eigenvalues λ_1 and λ_2.

2. *If $\Delta = 0$ and at least one of the numbers Δ_1 or Δ_2 is 0, the system has infinitely many solutions.*
3. *If $\Delta = 0, \Delta_1 \neq 0, \Delta_2 \neq 0$ then there is no non-trivial solution.*

Preserving the order of the eigenvalues

We have picked α_1, α_2 such that λ_1, λ_2 are eigenvalues of A_{α_1,α_2}. However we want to see when λ_1 and λ_2 are actually the first two eigenvalues of A_{α_1,α_2}.

Theorem

Assume λ_k, the k:th eigenvalue of A, is simple and λ_k is also an eigenvalue of A_{α_1,α_2}. If at least one of $\alpha_1\langle\varphi_1, \psi_k\rangle$, $\alpha_2\langle\varphi_2, \psi_k\rangle$ is non-zero, then λ_k is the k:th eigenvalue of A_{α_1,α_2} iff $\alpha_1\alpha_2 < 0$.

Metric graphs

A compact **metric graph** $\Gamma = (\mathbf{E}, \mathbf{V})$ is given by:

- **edges** $e_n = [x_{2n-1}, x_{2n}]$, $n = 1, ..., N$
- **vertices** $v_1, ...v_M$ that form a partition of the end points $x_1, ..., x_{2N}$.

An edge connects two vertices if it has an end point in each vertex

$L^2(\Gamma) = \bigoplus_{n=1}^{N} L^2(e_n)$ forms a Hillbert space of functions on Γ. For $u \in W^{2,2}(\Gamma)$ define: $\partial u(x_j) := (-1)^{j+1} u'(x_j)$ normal derivative at x_j.

Vertex Conditions

We consider three types of vertex conditions.
Standard conditions at vertex v_m:

$$\begin{cases} u(x_i) = u(x_j) =: u(v_m), x_j \in v_m \\ \sum_{x_j \in v_m} \partial u(x_j) = 0 \end{cases}$$

Delta conditions at vertex v_m, with coupling parameter α:

$$\begin{cases} u(x_i) = u(x_j) =: u(v_m), x_j \in v_m \\ \sum_{x_j \in v_m} \partial u(x_j) = \alpha u(v_m) \end{cases}$$

Dirichlet conditions at vertex v_m:

$$u(x_i) = 0, x_j \in v_m$$

Delta potential

- Consider the Laplacian $L = -\frac{d^2}{dx^2}$ on Γ with standard conditions.
- Let δ_{v_m} be the delta potential supported at the vertex v_m.
- Then $L_{\alpha_1,\alpha_2} = L + \alpha_1\delta_{v_1} + \alpha_2\delta_{v_2}$ is the Laplacian on Γ with delta couplings at v_1 and v_2.
- If $\alpha_m = \infty$ we instead have Dirichlet conditions at that vertex.

M-function for a metric graph

Let $u \in W^{2,2}$ be a continuous function satisfying standard conditions at all vertices except v_1, v_2.

For any λ not in the spectrum of the Dirichlet operator $L_{\infty,\infty}$ the $u(v_m)$ uniquely determine a solution to $-\frac{d^2}{dx^2}u = \lambda u$.

The M-function is the matrix valued function s.t.

$$\mathbf{M}(\lambda) : \begin{pmatrix} u(v_1) \\ u(v_2) \end{pmatrix} \mapsto \begin{pmatrix} \sum_{x_j \in v_1} \partial u(v_j) \\ \sum_{x_j \in v_2} \partial u(v_j) \end{pmatrix}$$

Interval

Consider just the interval $[0, \ell]$. We put delta couplings at the two endpoints. The M-function is given by

$$M(\lambda) = \begin{pmatrix} -\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}\ell)}{\sin(\sqrt{\lambda}\ell)} & \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\ell)} \\ \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\ell)} & -\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}\ell)}{\sin(\sqrt{\lambda}\ell)} \end{pmatrix}$$

The condition on α_1, α_2 is

$$\frac{1}{\ell}\alpha_1 + \frac{1}{\ell}\alpha_2 + \alpha_1\alpha_2 = 0$$
$$\alpha_1 + \alpha_2 = 0$$

We are in the case $\Delta = 0$, $\Delta_1, \Delta_2 \neq 0$ so there is no non-trivial solution.

Lasso Graph

Let Γ be the lasso graph with:

- loop of length ℓ_1 parametrized by $e_1 = [x_1, x_2]$
- outgrowth of length ℓ_2 parameterized by $e_2 = [x_3, x_4]$
- vertex $v_1 = \{x_1, x_2, x_3\}$ where loop and outgrowth meet
- vertex $v_2 = \{x_4\}$ at the end of the outgrowth

At the vertices v_1, v_2 we add δ-couplings of strength α_1 and α_2 respectively.

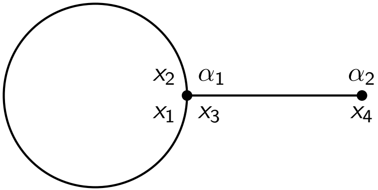

Figure: Compact lasso graph

Lasso Graph - Dirichlet spectrum

If λ_k is an eigenvalue of L as well as the Dirichlet operator $L_{\infty,\infty}$ then $\sqrt{\lambda_k}\ell_1$ is a multiple of π.

If $\sqrt{\lambda_k}\ell_1$ is an even multiple of π: then $\sin(\sqrt{\lambda_k}x)$ is an eigenfunction on the loop and we will get that λ_k is an eigenvalue of A_{α_1,α_2} for all α_1, α_2.
Note: The lowest two eigenvalues $\lambda_1 = 0$ and λ_2 are smaller than $(\frac{2\pi}{\ell_1})^2$.

If $\sqrt{\lambda_k}\ell_1$ is an odd multiple of π: then we get that λ_k is an eigenvalue of A_{α_1,α_2} if $\alpha_2 = 0$ or $\alpha_1 = \infty$

Lasso Graph - Non-Dirichlet spectrum

For the Non-Dirichlet spectrum we can use the M-function:

$$M(\lambda) = \begin{pmatrix} 2\sqrt{\lambda}\frac{1-\cos(\sqrt{\lambda}\ell_1)}{\sin(\sqrt{\lambda}\ell_1)} - \sqrt{\lambda}\frac{\cos(\sqrt{\lambda}\ell_2)}{\sin(\sqrt{\lambda}\ell_2)} & \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\ell_2)} \\ \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\ell_2)} & -\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}\ell_2)}{\sin(\sqrt{\lambda}\ell_2)} \end{pmatrix}$$

From this we can derive a secular equation

$$\tan(\sqrt{\lambda}\ell_2) + 2\tan(\sqrt{\lambda}\ell_1/2) = 0$$

Moreover for λ_k that is an eigenvalue of L:

$$M_{11}(\lambda_k) = -\sqrt{\lambda}\frac{1}{\cos(\sqrt{\lambda}\ell_2)\sin(\sqrt{\lambda}\ell_2)}$$

$$M_{22}(\lambda_k) = -\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}\ell_2)}{\sin(\sqrt{\lambda}\ell_2)}$$

Lasso Graph - Fixing two eigenvalues

Let $k_2 = \sqrt{\lambda_2}$. The conditions on α_1, α_2 are

$$0 = \frac{1}{\ell_2}\alpha_1 + \frac{1}{\ell_2}\alpha_2 + \alpha_1\alpha_2$$
$$0 = \frac{\cos(k_2\ell_2)}{\sin(k_2\ell_2)}\alpha_1 + \frac{1}{\cos(k_2\ell_2)\sin(k_2\ell_2)}\alpha_2 + \alpha_1\alpha_2$$

The solution to this system is

$$\alpha_1 = -\frac{k_2 \sin^2(k_2\ell_2)}{\sin(k_2\ell_2)\cos(k_2\ell_2) - k_2\ell_2\cos^2(k_2\ell_2)},$$
$$\alpha_2 = -\frac{k_2 \sin^2(k_2\ell_2)}{k_2\ell_2 - \sin(k_2\ell_2)\cos(k_2\ell_2)}.$$

Lasso Graph - When is order preserved?

We can also ask the question of whether the two eigenvalues are the lowest two eigenvalues of the operator L_{α_1,α_2}.
λ_1 and λ_2 are the first and second eigenvalues of L_{α_1,α_2} if and only if $\alpha_1\alpha_2 < 0$.
From the formulas for α_1 and α_2 we see

- $\alpha_2 < 0$
- $\alpha_1 < 0$ if and only if $k_2\ell_2 < \frac{\pi}{2}$
- $\alpha_1 > 0$ if and only if $k_2\ell_2 > \frac{\pi}{2}$

Note: the case $k_2\ell_2 = \frac{\pi}{2}$ corresponds to $\alpha_1 = \infty$, $\alpha_2 = -\frac{1}{\ell_2}$. In this case $\ell_1 = 2\ell_2$ and the order is preserved.

From the secular equation $\tan(\sqrt{\lambda}\ell_2) + 2\tan(\sqrt{\lambda}\ell_1/2) = 0$, we see that k_2 always lies in between $\frac{\pi}{\ell_1}$ and $\frac{\pi}{2\ell_2}$

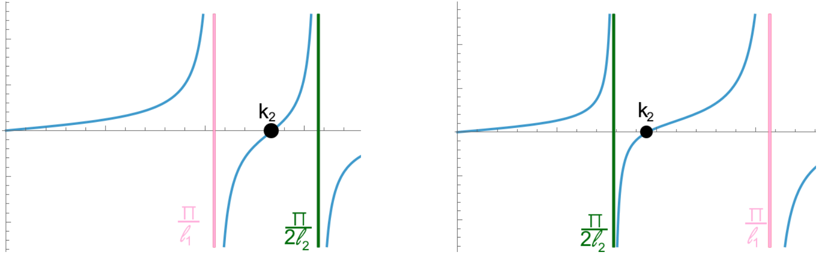

Figure: $y = \tan(x\ell_2) + 2\tan(x\frac{\ell_1}{2})$ when $\ell_1 > 2\ell_2$ (left) and $\ell_1 < 2\ell_2$ right

If $2\ell_2 < \ell_1$ then $k_2\ell_2 < \frac{\pi}{2}$ and if $2\ell_2 > \ell_1$ then $k_2\ell_2 > \frac{\pi}{2}$
Hence there is a unique non-zero choice of α_1, α_2 such that $\lambda_1^{\alpha_1,\alpha_2} = \lambda_1$, $\lambda_2^{\alpha_1,\alpha_2} = \lambda_2$ iff $2\ell_2 \geq \ell_1$

Thank You!