The Point Spectrum of Periodic Quantum Trees

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The Hebrew University of Jerusalem

Plan

Part I: The Discrete Case

- Periodic Jacobi matrices on trees
- The classical Aomoto Index
- The mass of eigenvalues and the structure of eigenfunctions

Part II: The Continuum Setting

- Quantum graphs and metric covering trees
- The density of states measure
- A surprising difference between discrete and continuous

Part III: Main Results

- Construction of the Q-Aomoto Set
- **The mass of eigenvalues - a surprising similarity**
- **Eigenvalues and the graph below**
- **Generic absence of point spectrum**

Part IV: Proof Ingredients

- A central tree lemma
- Translation to a discrete graph problem

Part I: Periodic Jacobi Matrices on Trees

The Discrete Setting: Periodic Jacobi Matrices on Trees

Let $G = (V, E)$ be a finite connected graph (with at least one cycle), and let $T = (\mathcal{V}, \mathcal{E})$ be its unique universal covering tree. Let $\Xi : T \to G$ be the covering map.

- We lift edge-weights $a_e > 0$ and potentials $b_v \in \mathbb{R}$ from G to T by setting $a_{\tilde{e}} = a_{\Xi(\tilde{e})}$ and $b_{\tilde{v}} = b_{\Xi(\tilde{v})}$.

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- This defines a bounded, self-adjoint **periodic Jacobi matrix** J_T acting on $\ell^2(\mathcal{V})$:

$$(J_T\phi)(\tilde{v}) = b_{\tilde{v}}\phi(\tilde{v}) + \sum_{\tilde{u}\sim\tilde{v}} a_{\tilde{u}\tilde{v}}\phi(\tilde{u})$$

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- In the case that G is 2-regular, $T = \mathbb{Z}$, and J_T is a one-dimensional periodic Jacobi matrix. These operators have a rich spectral theory, with connections to condensed matter physics, potential theory, orthogonal polynomials, and integrable systems.

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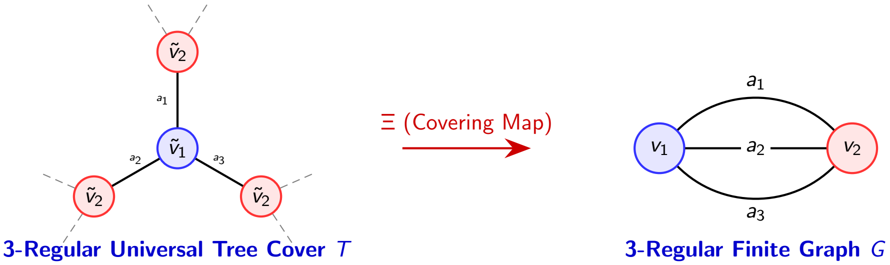

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- Even more recent: Garza-Vargas and Kulkarni ('19), Banks, Garza-Vargas and Mukherjee ('20), Avni-B-Simon ('20), Christiansen-Simon-Zinchenko ('21), Avni-B-Kalai-Simon ('22), Christiansen-Simon-Zinchenko ('21), Banks-B-Garza-Vargas-Seelig-Simon ('24).

The Discrete Setting: Periodic Jacobi Matrices on Trees

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- The operator J_T has no singular continuous spectrum, namely all spectral measures for J_T are absolutely continuous w.r.t. the Lebesgue measure with perhaps some atoms (Avni-B-Simon '20). This actually follows immediately from algebraicity of the Green's functions, a result going back at least to Chomsky-Schützenberger ('63).

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- Eigenvalues are well understood: In particular, if $\mathcal{G}$ is regular then J_T has no eigenvalues, but eigenvalues are possible otherwise (Aomoto '91, Banks, Garza Vargas and Mukherjee '20).

The Discrete Setting: Periodic Jacobi Matrices on Trees

The density of states (DOS), $\mathrm{d}\nu$, is

$$\nu_d = \frac{1}{|V(G)|} \sum_{v \in \mathcal{D}} \mu_v$$

where the sum is of spectral measures over a fundamental domain, $\mathcal{D}$, of the action of the group of deck transformations acting on T (as a cover of G).

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Theorem (Avni-B-Kalai-Simon, '22)

Let $(\mathcal{G}_n)_{n=1}^{\infty}$ be a tower of normal finite covers (with $\mathcal{G}_0 = \mathcal{G}$) such that $|V(\mathcal{G}_n)| \to \infty$. Let J_n be the lift of J to $\mathcal{G}_n$. Then the normalized eigenvalue counting measure of J_n converges weakly to ν_d.

Point Spectrum in the Discrete Case

Aomoto (1991) proved a formula for the mass assigned by ν_d to each eigenvalue:

Definition (The Discrete Aomoto Set)

For a fixed $\lambda \in \mathbb{R}$, the associated **Aomoto set** $\mathcal{A}(\lambda) \subseteq V(G)$ *is the set of vertices v, such that J_T has a λ-eigenfunction ψ with $\psi(\widetilde{v})$ for some $\widetilde{v} \in \Xi^{-1}(v)$.*

Equivalently, λ is an atom of $\mu_{\widetilde{v}}$ for every $\widetilde{v} \in \Xi^{-1}(v)$.

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Fix a finite graph, G, and $\lambda \in \mathbb{R}$. Let $\partial\mathcal{A}(\lambda)$ denote the set of vertices outside $\mathcal{A}(\lambda)$ and connected to it by an edge. Let $C\mathcal{A}(\lambda)$ denote the number of connected components of the subgraph induced by $\mathcal{A}(\lambda)$, and define the index $I_\lambda(G) = C\mathcal{A}(\lambda) - |\partial\mathcal{A}(\lambda)|$.

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Moreover, the subgraph induced by $\mathcal{A}(\lambda)$ is a forest.

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$$\nu_d(\{\lambda\}) = \frac{I_\lambda(G)}{|V(G)|}.$$

Moreover, the subgraph induced by $\mathcal{A}(\lambda)$ is a forest. In particular, if G is regular, the point spectrum of J_T is empty.

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- Under the same condition, if $\widetilde{G}$ is an m-cover of G then λ is also an eigenvalue of the lift of the Jacobi matrix to $\widetilde{G}$ with mutliplicity at least $mI_\lambda(G)$.

Point Spectrum in the Discrete Case

- IF X is an acyclic subset of G, with positive Aomoto index, and such that $\lambda \in \mathbb{R}$ is an eigenvalue for each connected compoent of X. Then λ is an eigenvalue of J_T and $\nu_d\left(\{\lambda\}\right) \geq \frac{I_\lambda(X)}{|V(G)|}$.

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- For any connected finite graph with at least one cycle, G, the set of edge weights and vertex potentials for which the point spectrum of J_T is non-empty, is a semialgebraic closed subset of $\mathbb{R}^{|E|+|V|}$ of codimension at least $\max(d_{min} - 1, 1)$.
  In particular, for a generic choice of weights and potentials on a finite graph G, the point spectrum of its infinite universal covering tree is empty.

Part II: The Continuum Setting — Quantum Trees

Transitioning to the Continuum: Metric Graphs

To move from discrete matrices to differential operators, we define a **metric graph** $\Gamma = (V, E, \ell)$.

- Each edge $e \in E$ is assigned a positive length $\ell_e > 0$ and is identified with the real interval $[0, \ell_e]$. We assume a global positive lower bound on ℓ_e.

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- We consider the standard self-adjoint **Schrödinger operator** on each edge segment:

$$H_\Gamma f = -\frac{d^2 f}{dx_e^2} + q_e(x_e) f$$

where q_e is a real (piecewise) continuous potential. .

Vertex Matching Conditions: δ-Type Boundary Conditions

To make H_Γ a well-defined self-adjoint operator, functions within its domain $\mathcal{D}(H_\Gamma) \subset \bigoplus_e H^2(0, \ell_e)$ must satisfy specific matching conditions at each vertex $v \in V$.

Generalized δ-Type Vertex Conditions

For each vertex $v \in V$, the functions must be continuous across all incident edge endpoints and satisfy a flux constraint:

$$f_e(v) = f_{e'}(v) \equiv f(v) \quad \forall e, e' \sim v \quad \textbf{(Continuity)}$$

$$\sum_{e \sim v} \frac{df}{dx_e}(v) = \alpha_v f(v) \qquad \textbf{(Flux Balance)}$$

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- Setting $\alpha_v = 0$ for all vertices yields standard **Kirchhoff conditions**.

Periodic Quantum Trees

Let $\Gamma = (V, E, \ell)$ be a compact metric graph. Its universal cover is an infinite metric tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \tilde{\ell})$, where edge lengths and vertex coupling strengths are lifted directly from their base values:

- If $\Xi(\tilde{e}) = e$, then $\tilde{\ell}_{\tilde{e}} = \ell_e$.
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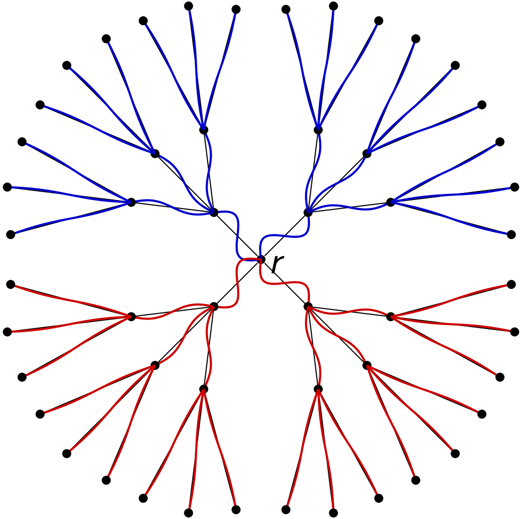
What can be said about the spectral theory of such operators? We start with eigenvalues.

Periodic Quantum Trees

An observation - 1 is an eigenvalue for this regular quantum tree ($q \equiv 0$, Kirchhoff BC)..



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We define the **density of states** measure, ν_c, by

$$\nu_c = \frac{1}{\mathcal{L}_E} \sum_{k=1}^{\infty} \mu_{g_k} \tag{1}$$

where $\mathcal{L}_E = \sum_{e \in E(\Gamma)} \ell_e$.

The density of states

Theorem

- *The density of states is well defined (Levi).*
- ν_c *is locally finite (B-Levi)..*
- *For a sequence of quantum covers of* Γ *with girth* $\to \infty$, *the properly normalized eigenvalue counting measure converges vaguely to* ν_c *(Anatharaman-Ingremeau-Sabri-Winn '21, Levi '27?).*

Anantharaman, Ingremeau, Sabri, and Winn (21) use Benjamini-Schramm limits of quantum graphs to define the density of states measure in a more general context. In the context of periodic quantum trees, it is not hard to see that their definition coincides with the one given above.

Part III: Main Results — The Q-Aomoto Set & Point Spectrum

The Q-Aomoto Set

In the continuum setting, eigenfunctions are supported on edges and may or may not vanish on vertices, so the relevant Aomoto set needs to be defined in a different way.

Definition (The Q-Aomoto Set)

Let $\lambda \in \mathbb{R}$ be an eigenvalue of $H_{\mathcal{T}}$. The Q-Aomoto set $X_\lambda(\Gamma) = V_\lambda(\Gamma) \cup E_\lambda(\Gamma)$ is a subset of the graph Γ defined as follows:

- *$V_\lambda(\Gamma)$ is the set of all vertices $\upsilon \in V$ such that there exists $\tilde{\upsilon} \in \Xi^{-1}(\upsilon)$ and a function $h \in \ker(\lambda - H_{\mathcal{T}})$ satisfying $h(\tilde{\upsilon}) \neq 0$.*

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- *$E_\lambda(\Gamma)$ is the set of all edges $e \in E$ such that there exist $\tilde{e} \in \Xi^{-1}(e)$ and a function $h \in \ker(\lambda - H_{\mathcal{T}})$ for which $h|_{\tilde{e}}$ is not identically zero.*

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- The Q-Aomoto set $X_\lambda(\Gamma)$ is not necessarily a subgraph of Γ!
- The boundary of $X_\lambda(\Gamma)$, $\partial X_\lambda(\Gamma)$, is defined to be the set of vertices $\notin V_\lambda(\Gamma)$ connected to an edge $\in E_\lambda(\Gamma)$.

Main Theorem 1

As in the discrete case, we define the index of the Q-Aomoto set to be

$$I_\lambda(\Gamma) = CX_\lambda(\Gamma) - |\partial X_\lambda(\Gamma)|.$$

Theorem (B-Levi)

Let $\Gamma = (V, E, \ell, \mathcal{H}_\Gamma)$ be a compact quantum graph with δ-type vertex conditions and a Schrödinger-type differential operator $\mathcal{H}_\Gamma$. Let $\mathcal{T}$ be the universal cover of Γ with covering map $\Xi : \mathcal{T} \to \Gamma$, and let $\lambda \in \sigma_{\mathfrak{p}}(\mathcal{H}_\mathcal{T})$. Then

- *The subgraph of Γ induced by the vertices in $V_\lambda(\Gamma)$ is acyclic.*
- *The density of states satisfies:*

$$\nu_c(\{\lambda\}) = \frac{I_\lambda(\Gamma)}{\mathcal{L}_E}.$$

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Eigenvalues for Regular Periodic Quantum Trees

Theorem (B-Levi)

If $\mathcal{T}$ is regular then for every $\lambda \in \sigma_{\mathfrak{p}}(H_{\mathcal{T}})$ there exists an edge $e \in E$ such that the following differential equation admits a nontrivial solution on $[0, \ell_e]$:

$$\begin{aligned} &\forall x \in [0, \ell_e], \quad -\frac{d^2 f}{dx^2}(x) + q|_e(x) f(x) = \lambda f(x) \\ &f(0) = f(\ell_e) = 0\,. \end{aligned} \tag{2}$$

In the case $q = 0$, this means that

$$\sigma_{\mathfrak{p}}(H_{\mathcal{T}}) \subseteq \left\{ \frac{n^2 \pi^2}{\ell_e^2} \mid e \in E,\ n \in \mathbb{N} \right\}.$$

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$$\sigma_{\mathfrak{p}}(H_{\mathcal{T}}) \subseteq \left\{ \frac{n^2 \pi^2}{\ell_e^2} \mid e \in E,\ n \in \mathbb{N} \right\}.$$

- In general, as in the discrete case, it is possible to compute the eigenvalues of $H_{\mathcal{T}}$ by computing the spectra of Q-Aomoto type sets of Γ.

Main Theorem 2

Theorem (B-Levi)

For any $\lambda \in \sigma_{\mathfrak{p}}(H_{\mathcal{T}})$, $\lambda \in \sigma_{\mathfrak{p}}(H_{\Gamma})$ (namely λ is in the point spectrum of the graph below).

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- *There exists a subspace of the eigenspace of H_Γ and λ of dimension at least*

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  whose elements vanish outside the Q-Aomoto set $X_{\lambda}(\Gamma)$.

- *Let $\widetilde{\Gamma}$ be an n-lift of Γ with covering map $\xi : \widetilde{\Gamma} \to \Gamma$. Then $\lambda \in \sigma_{\mathfrak{p}}(H_{\widetilde{\Gamma}})$. Moreover, there exists a subspace of the eigenspace of λ in $L^2\left(\widetilde{\Gamma}\right)$, of dimension at least*

  $$n \cdot I_{\lambda}(\Gamma) = n \cdot \mathcal{L}_E \cdot \nu_c\left(\{\lambda\}\right)$$

  whose elements vanish outside the lifted Q-Aomoto set $\xi^{-1}(X_{\lambda}(\Gamma))$.

Main Theorem 3

Finally, and again as in the discrete case, we can show that in the case of $q = 0$, having non-empty point spectrum is a topologically rare event in the space of possible edge lengths.

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Theorem (B-Levi)

Let $\Gamma = (V, E, \ell)$ be a compact quantum graph with δ-type vertex conditions, where the Dirichlet condition is allowed only at vertices of degree 1, with at least one cycle and with the standard Schrödinger operator $H_\Gamma = -d^2/dx^2$, and let $\mathcal{T}$ be its universal cover.

Then for a residual set of lengths in $\mathbb{R}_+^{|E|}$, $\sigma_{\mathfrak{p}}(\mathcal{H}_{\mathcal{T}}) = \emptyset$.

Part IV: Ingredients of the Proof

Main Idea

- The main idea behind the proof is to translate the metric graph problem to a Jacobi matrix on a combinatorial graph. Namely, given an eigenvalue, λ, on a quantum graph we want to construct a discrete graph with a Jacobi matrix defined on it and a map between λ-eigenfunctions on the quantum graph and 0-eigenfunctions on the discrete one.

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- This idea goes back at least to Exner (1997) and has many applications and variations (see, e.g., Brüning-Geyler-Pankrashkin 2007, Lledó-Post 2008, Exner-Kostenko-Malamud-Neidhardt 2018, and others...)

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- For our purposes, we need an extension in two directions: first – non-zero q; and second – the consideration of eigenfunctions that vanish at both end points for some of the edges.

A Central Lemma

This is particularly important in the proof of the following crucial lemma

Lemma

Let $T = (V, E, \ell, \mathcal{H})$ be a connected quantum tree equipped with a Schrödinger-type operator H and δ-type vertex conditions on all vertices, such that $\alpha_v < \infty$ for every vertex v with $\deg(v) > 1$.

Suppose $\lambda \in \sigma_{\mathfrak{p}}(H)$ and that there exists an eigenfunction $h \in \ker(\lambda - H)$ such that $h(v) \neq 0$ for all $v \in V$ with $\alpha_v < \infty$. Then the following statements hold:

- *The set $\{f|_V \mid f \in \ker(\lambda - H)\}$ is a one-dimensional subspace of $\ell^2(V)$.*
- *If T is finite, then $\dim(\ker(\lambda - H)) = 1$.*

A Central Lemma

- In a sense, this is the central lemma on which everything is built. Its discrete version goes back to Aomoto and is used centrally also by Banks-Garza-Vargas-Mukherjee. The proof in the discrete case is simple. A compact proof in the metric case requires a somewhat elaborate construction.

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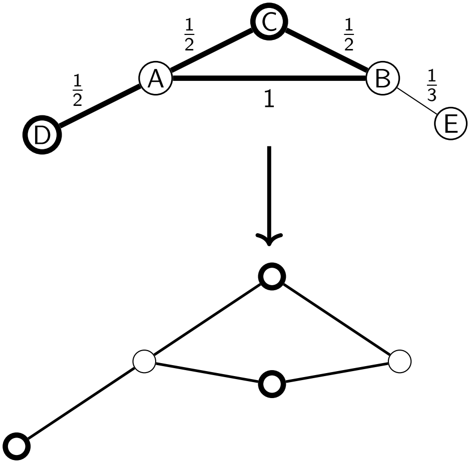
- Once this lemma is in place, the acyclicity of the components of $X_\lambda(\Gamma)$ follows.
- Now a combinatorial graph is constructed by shrinking each component (including isolated edges) to a single vertex and connecting them to the boundary vertices. The edge weights are defined through normalized restricted eigenfunctions.

Reduction to a Discrete Graph



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- Generic absence of eigenvalues for quantum trees with a non-zero potential.
- Spectral band structure (Structure? Bethe-Sommerfeld conjecture?)
- Bloch-Floquet theory.

Thank you for your attention!

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Reference:

The Point Spectrum of Periodic Quantum Trees — arXiv:2603.08362 (2026)