Characterization of spectral measures and hyperrigidity

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One of the features of a Borel spectral measure on $\mathbb{R}$ is the multiplicativity of the corresponding Stone-von Neumann functional calculus. In particular, if E is a Borel spectral measure on $\mathbb{R}$ with compact support, then the following identities hold

$$\Big(\int_{\mathbb{R}} xE(\mathrm{d}x)\Big)^n = \int_{\mathbb{R}} x^n E(\mathrm{d}x), \quad n = 1, 2, \ldots. \tag{0.1}$$

Hence, all operator moments of E are determined by the first one, and according to the spectral theorem there is a one-to-one correspondence between Borel spectral measures on $\mathbb{R}$ and their first operator moments. This is no longer true for general Borel semispectral measures on $\mathbb{R}$. It turns out, however, that the single equality in (**??**) with $n = 2$ guarantees spectrality.

It turns out that, from a mathematical and physical point of view, it is important to investigate the relationship between semispectral and spectral measures. In the classical von Neumann description of quantum mechanics selfadjoint operators or, equivalently, Borel spectral measures on the real line represent observables. This approach is insufficient in describing many natural properties of measurements, such as measurement inaccuracy. Therefore, in standard modern quantum theory, the generalization to semispectral measures is widely used.

In 2006 Kiukas, Lahti and Ylinen asked the following general question. *When is a positive operator measure projection valued?* A version of this question formulated in terms of operator moments was posed in [**?**]. *Let T be a selfadjoint operator and F be a Borel semispectral measure on the real line with compact support. For which positive integers $p < q$ do the equalities $T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$, $k = p, q$, imply that F is a spectral measure?* In this talk, we show that the answer is affirmative if p is odd and q is even, and negative otherwise.

Motivated both by the fundamental role of the classical Choquet boundary in classical approximation theory, and by the importance of approximation in the contemporary theory of operator algebras, Arveson introduced hyperrigidity as a form of approximation that captures many important operator-algebraic phenomena. We discuss the relationship between hyperrigidity and the aforementioned spectral measure characterisation. This talk is based on joint work with Jan Stochel.

[1] P. Pietrzycki, J. Stochel, Subnormal nth roots of quasinormal operators are quasinormal, *J. Funct. Anal.* **280** (2021), 109001.

[2] P. Pietrzycki, J. Stochel, Two-moment characterization of spectral measures on the real line, *Canad. J. Math.* vol. 75 (4) (2023), 1369-1392

[3] P. Pietrzycki, J. Stochel, Hyperrigidity I: singly generated commutative C^*-