To the Birman-Krein-Visic Theory

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Let A be a closed non-negative symmetric densely defined operator in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H}_1 := \operatorname{ran}(I + A)$. By the Stone – Friedrichs theorem the set $\operatorname{Ext}_A(0,\infty)$ of all nonnegative selfadjoint extensions $\widetilde{A} = \widetilde{A}^*$ of A is nonempty. M. Krein established that the set $\operatorname{Ext}_A(0,\infty)$ forms an operator segment with two endpoints: the maximal (the Friedrichs) and the minimal (the Krein) extensions $\widehat{A}_F$ and $\widehat{A}_K$. They are uniquely characterized by means of the inequalities: $\widehat{A}_K \le \widetilde{A} \le \widehat{A}_F$, $\widetilde{A} \in \operatorname{Ext}_A(0,\infty)$, which are understood in the form sense.

Krein's theory has substantially been completed by M. Vicik and M. Birman.

If A is positive definite, then $\widehat{A}_K$ admits a representation $\widehat{A}_K = \widehat{A}'_K \oplus (\mathbb{O} \upharpoonright \mathfrak{N}_0)$ where $\mathfrak{N}_0 := \ker A^*$. The operator $\widehat{A}'_K$ is called the reduced Krein extension.

M.Krein discovered the following implication

$$(I_{\mathfrak{H}} + \widehat{A}_F)^{-1} \in \mathfrak{S}_\infty \implies (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}_\infty. \tag{0.1}$$

First we will discuss the following its improvement (see [1]):

$$P_1(I_{\mathfrak{H}} + A)^{-1} \in \mathfrak{S}(\mathfrak{H}_1) \iff (I_{\mathfrak{M}_0} + \widehat{A}'_K)^{-1} \in \mathfrak{S}(\mathfrak{M}_0). \tag{0.2}$$

Here P_1 is the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{H}_1 = \operatorname{ran}(I + A)$, $\mathfrak{M}_0 = \mathfrak{N}_0^\perp$, and $\mathfrak{S}$ is any symmetrically normed ideal (including ideals $\mathfrak{S}_p$, Σ_p, Σ_p^0, etc.).

We will discuss the following problems:

(i) Additional conditions on A that ensure the following equivalence

$$\lambda_n\big(\widehat{A}_F)\big) = a^{-1}n^{1/p}\,(1 + o(1)) \iff \lambda_n\big(\widehat{A}'_K\big) = a^{-1}n^{1/p}\,(1 + o(1))\,.$$

which completes (**??**) in the case of $\mathfrak{S} = \Sigma_p$

(ii) The Birman problem and the validity of the inverse implication to (**??**).

(iii) Solution to the abstract Alonso-Simon problem.

(iv) Improvement of certain results by Birman and Grubb regarding the LSB-property of A and its application to elliptic BV problems in unbounded domains.

The talk is based on results published in [1]-[3] and their recent developments.

References

1. M. M.Malamud, To Birman–Krein–Vishik Theory, Doklady Mathematics, V. 107, No. 1 (2023), pp. 44–48.

2. M.M. Malamud, On the Birman problem on positive symmetric operators with compact inverse, Func. Anal. Appl., V.57, No 2 (2023), p. 111–116.

3. M. Malamud, Explicit Solution to the Birman Problem for the $2D$-Laplace operator, Russian Jour. of Math. Phys., 2024, V. 31, No.3, p. 495-504.