PSEUDO-HERMITIAN LAPLACE OPERATORS ON STAR-GRAPHS: REAL SPECTRUM AND SELF-ADJOINTNESS

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ABSTRACT. Pseudo-Hermitian Laplace operators on graphs are constructed using the method of point perturbations. We study whether all such operators with purely real spectrum are self-adjoint or not.

1. INTRODUCTION

Recently there has been a great interest in substituting the mathematical condition of Hermicity for Hamiltonians by a weaker and sometimes more relevant condition of spacetime reflection symmetry ($\mathcal{PT}$ -symmetry), which can be further extended to a more general mathematical context known as *pseudo-Hermicity*. As a result, a wide range of Hamiltonians which a decade ago would have been unfit for the theories of quantum mechanics can now be considered in a new extended pseudo-Hermitian quantum mechanics.[3, 4, 6] It has been proven that Hamiltonians with the pseudo-Hermicity property either have real or conjugated pairs of eigenvalues [13]. Therefore, it is important to investigate the relation between the pseudo-Hermitian Hamiltonians and the reality of their spectra [3, 4, 5, 6, 12, 13, 14]. The present work is a generalization of the studies carried out in [3] on $\mathcal{PT}$-self-adjoint point interactions with the support at the origin and at points $\pm l$, $l > 0$ for the second derivative operator in one dimension. In this instance, a particular pseudo-Hermitian point interaction at the vertex of the star graph for the *Laplace operator* will be studied.

The growing interest in non-Hermitian Hamiltonians was motivated when many researchers found by considering various examples that such Hamiltonians could still have only real spectra. It was then in 1998 that Bender and Boettcher based on the results of various numerical studies, established the field of $\mathcal{PT}$-symmetric quantum mechanics [5, 12]. They showed the existence of a huge class of non-Hermitian Schröndiger eigenvalues problems whose entirely real and positive spectra was due to an unbroken space-time reflection symmetry [5, 6]. It was later that Znojil, Japaridize, Kretcher, and Szymanowski among others carried out analytical studies to investigate the mathematical structure and the interpretation of the $\mathcal{PT}$-symmetric quantum mechanics [10, 11, 12, 17].

However, $\mathcal{PT}$-symmetry is neither a necessary nor a sufficient condition for a Hamiltonian to have a real spectrum. Actually, classes of non $\mathcal{PT}$-symmetric Hamiltonians with real spectra have been constructed [7, 12].

It was Mostafazadeh who first pointed out that because the parity operator $\mathcal{P}$ is Hermitian, it may be used as an intertwining operator [6]. He then showed that the remarkable spectral properties of the $\mathcal{PT}$-symmetric Hamiltonians follow from their

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pseudo-Hermicity [14]. He also considered the class of pseudo-Hermitian hamil-
tonians that have a complete biorthonormal eigenbasis and showed that pseudo-
Hermicity is a necessary condition for having a real spectrum [12].

Our aim is to study the generalization of pseudo-Hermicity for the case of the
Laplace operator on the star graph. Natural generalization for the space reflection
$\mathcal{P}$ is the rotation operator $\mathcal{R}$ which performs cyclic permutation of the edges of the
graph. We are interested in the discrete spectrum of pseudo-Hermitian operators
using $\mathcal{R}$ as the intertwining operator. It is proven that the discrete spectrum is
formed by pairs of eigenvalues conjugated to each other.

Conditions for which the spectrum is real are investigated. In the first step we con-
struct a certain family of pseudo-Hermitian operators (described by the boundary
conditions of type (7)). It appears that if the number of edges of the star-graph is
odd, then all operators from the constructed family have non-trivial discrete spec-
trum which is real if and only if they are self-adjoint. If the number of edges is
even, then there exists a rich set of non-self adjoint pseudo-Hermitian operators
having real spectrum.

2. DEFINITIONS

Pseudo-Hermitian. According to the literature the definition of a pseudo-Hermitian
operator is as below:

A linear densely defined operator A acting in a Hilbert space $\mathcal{H}$ is pseudo-Hermitian
if there exists a Hermitian invertible linear operator $\eta\colon \mathcal{H} \to \mathcal{H}$ such that

$$(1) \qquad\qquad A^* = \eta A \eta^{-1},$$

where A^* is the adjoint operator, and η is called the intertwining operator. The
condition of pseudo-Hermicity coincides with the ordinary Hermicity when η is the
identity and reduces to $\mathcal{PT}$ symmetry when $\eta = \mathcal{P}$ [6].

It was suggested in [3] to call such operators pseudo-self-adjoint. In the present
article we reserve the name of pseudo-Hermitian to a slightly different family of
operators.

Star graph. The star graph $\Gamma_N = \Gamma_{star} = (E, V)$ is a metric graph formed by
a finite set E of semi-infinite closed intervals $\Delta_j = [0, \infty)$, $j = 1, 2, \cdots, N$ called

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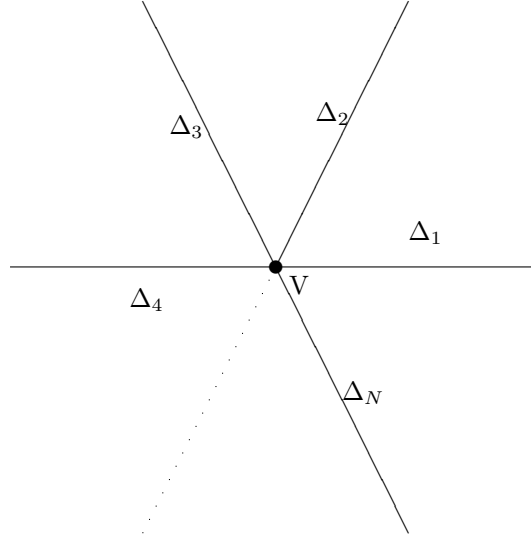
edges, joined together at the unique vertex $V := \{0\}$.

Fig.1 Star-like graph.

With Γ_N we associate the following Hilbert space

$$
(2) \qquad L_2(\Gamma_N) = \oplus \sum_{j=1}^{N} L_2(\Delta_j) = \oplus \sum_{j=1}^{N} L_2[0, \infty).
$$

Every function $\mathbf{F} \in L_2(\Gamma_N)$ can be seen as a vector function with components $f_j \in L_2(\Delta_j)$ on the edges $j = 1, \cdots, N$:

$$
\mathbf{F} = \{f_j\}_{j=1}^{N}.
$$

Laplace operator. By the *Laplace operator* in $L_2(\Gamma)$ we mean the (minus) second derivative operator:

$$
(3) \qquad L = \oplus \sum_{j=1}^{N} -\frac{d^2}{dx^2},
$$

defined on the domain:
(4)

$$
\mathrm{Dom}\,(L) = \{f \in \oplus \sum_{j=1}^{N} W(\Delta_j) : \ f \text{ is continuous at the vertex } V \text{ and } \sum_{j=1}^{N} f_j'(0) = 0\}.
$$

Circulant matrix. A *circulant matrix* is a special case of a Toeplitz matrix. The value of the entry a_{ij} of the $N \times N$ *circulant matrix* depends only on the difference $(j - i) \bmod N$, i.e,

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{pmatrix}.$$

The family of pseudo-Hermitian Hamiltonias in $L_2(\Gamma_N)$ will be constructed as point perturbations of the standard Laplace operator.

3. $\mathcal{R}$- pseudo-hermicity of Point Interactions

The method of point-interaction presented in this section is very similar to the one described in [2], which is based on the extension theory of symmetric operators. It is well known that to determine an operator in the Hilbert space, its domain must be specified. To determine the symmetric extensions of a densely defined operator then reduces to determining its domain of definition, since every symmetric extension is a restriction of the adjoint operator. As in [3], we start with the Laplace operator restricted to a certain densely defined operator, and extend the operator, in this case, to a pseudo-Hermitian operator which has non trivial spectrum structure. The bounded linear operator which will be used is the rotation $\mathcal{R}$ with center at the vertex of a star graph , which performs a cyclic permutation of the edges,i.e,

$$\mathcal{R} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \\ \vdots \\ f_1 \end{pmatrix}.$$

We will then define what appears to us to be a natural generalization for the $\mathcal{PT}$-symmetric operator

The *Laplace operator* with the standard domain described in (4) on the star graph is both self-adjoint and $\mathcal{R}$ pseudo-self-adjoint.We will now study point interactions at the vertex to get pseudo-Hermitian operators. In general, all point interactions of L at the vertex can be obtained by restricting L to the set of functions having support separated from the vertex [2]. Then a linear operator $\tilde{L}$ would be a point perturbation at the vertex of the operator L if and only its restriction to $C_0^\infty(\Gamma_N \backslash V)$ coincides with the restriction of the operator L:

$$\tilde{L}|_{C_0^\infty(\Gamma_N \backslash V)} = L|_{C_0^\infty(\Gamma_N \backslash V)} \equiv L_0.$$

All self-adjoint extensions of the operator L_0 determined by the last equation can be described by the boundary condition:

$$(5) \qquad\qquad B\vec{f}'(0) + C\vec{f}'(0) = 0$$

where B, C are certain $N \times N$ matrices satisfying the following two conditions:
1. rank $(B, C) = N$
2. BC^* is Hermitian.[15]
In the current article we are going to study extensions of L_0 described by the boundary conditions of the same form (5), with the matrices B, C satisfyng condition 1, but the second condition will be substituted by:

$$\mathcal{RT}L = L\mathcal{RT}$$

where $\mathcal{T}$ is the antilinear operator of complex conjugation:

$$(\mathcal{T}f)(x) = \overline{f(x)}$$

More precisely:

Definition 3.1. *An operator L is a pseudo-Hermitian extension of L_0 if and only if it is a restriction of L_0^* to the domain of functions satisfying the boundary condition:*

$$Bf' + Cf = 0$$

with the matrix (B,C) having rank N and satisfying the condition $\mathcal{R}\mathcal{T}L = L\mathcal{R}\mathcal{T}$.

L_0^* is the maximal operator L_{max}, where L_{max} is the second derivative operator in $L_2(\Gamma_N \backslash V)$ with the domain $\mathrm{Dom}\,(L_{max}) = W_2^2(\Gamma_N \backslash V)$. It can then be proven that the following boundary conditions define pseudo-Hermitian point interactions of L on the star-graph:

Definition 3.2. *Let A be a certain $N \times N$ matrix, we denote by L_A the restriction of $L_{max} = -(\frac{d^2}{dx^2})$ defined originally on $W_2^2(\Gamma_N \backslash V)$, to the domain of functions satisfying the following additional boundary conditions at the vertex of the star-graph:*

$$(6) \qquad \begin{pmatrix} f_1'(0) \\ f_2'(0) \\ \vdots \\ f_N'(0) \end{pmatrix} = A \begin{pmatrix} f_1(0) \\ f_2(0) \\ \vdots \\ f_N(0) \end{pmatrix}.$$

L_A is a certain point perturbation of the Laplace operator L.

Theorem 3.1. *Consider the operator L_A defined previously, definition 3.2.*
If $\underline{N\ is\ odd}$ then the operator L_A is pseudo-Hermitian if and only if A is a circulant matrix with real entries:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{pmatrix}, a_j \in \mathbb{R}\, , j = 0, \cdots, N-1.$$

If $\underline{N\ is\ even}$, then the operator L_A is pseudo-Hermitian if and only if A is a complex block circulant matrix of the following form:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ \overline{a_{N-1}} & \overline{a_0} & \overline{a_1} & \overline{a_2} & \cdots & \overline{a_{N-2}} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \cdots & \overline{a_0} \end{pmatrix}, a_j \in \mathbb{C}\, , j = 0, \cdots, N-1.$$

Proof. Let us recall that any pseudo-Hermitian extension of L_0 is a restriction of L_0^*, which in this case is L_{max}. We can then consider the vector space $\mathbb{C}^{2N}$ of boundary values of functions from the domain of L_{max} and the map Λ assigning to any funtion $\mathbf{F} \in W_2^2(\Gamma_N \backslash V)$ its boundary values:

$$
\Lambda \colon \mathbf{F} \to
\begin{pmatrix}
f_1(0) \\
f_1'(0) \\
f_2(0) \\
f_2'(0) \\
\vdots \\
f_N(0) \\
f_N'(0)
\end{pmatrix}
\in \mathbb{C}^{2N}.
$$

Similarly the operator of complex conjugation $\mathcal{T}$ is mapped by Λ into the operator T of complex conjugation in $\mathbb{C}^{2n}$.

The closure of the operator L_0 is defined on the functions having trivial boundary values at the vertex and vice-versa any function from the domain of L_{max} with trivial boundary values belong to $\mathrm{Dom}\,(\overline{L_0})$.

Let us study every edge separately and consider $f_i \in \Delta_i$, $i = 1, \cdots, N$. It can then be seen that letting $g_1, g_2 \in W_2^2(\mathbb{R}^+)$, such that:

$$
\begin{aligned}
g_1(0) &= 1, \quad g_2(0) = 0, \\
g_1'(0) &= 0, \quad g_2'(0) = 1.
\end{aligned}
$$

$$
f - f_i(0)g_1 - f_i'(0)g_2 \in \mathrm{Dom}\,(\overline{L_0}).
$$

Then, defining L_{max} and $\overline{L_0}$ on each edge, the dimension of the quotient space $\mathrm{Dom}\,(L_{max})\backslash\mathrm{Dom}\,(\overline{L_0})$ is 2. As a result, the dimension of $\mathrm{Dom}\,(L_{max})\backslash\mathrm{Dom}\,(\overline{L_0})$ on the star-graph is $2N$.

Extending $\overline{L_0}$ amounts to removing certain boundary conditions, thereby enlarging the $\mathrm{Dom}\,(\overline{L_0})$ and reducing $\mathrm{Dom}\,(L_{max})$. Functions belonging to the pseudo-Hermitian extensions of L_0 have boundary values from an N dimensional subspace $\mathcal{L}$ of $\mathbb{C}^{2N}$ and belong to the domain of the restriction of L_{max}. Every such subspace can be described by N (linearly independent) boundary conditions using a $N \times 2N$ dimensional rank N matrix $Q = \{q_{ij}\}$ as follows:

$$
\text{(7)} \qquad \Lambda\mathbf{F} \in \mathcal{L} \Leftrightarrow
\begin{cases}
q_{11}f_1(0) + q_{12}f_1'(0) + \cdots + q_{1(2N)}f_N'(0) = 0, \\
\qquad\qquad\qquad \vdots \\
q_{N1}f_1(0) + q_{N2}f_1'(0) + \cdots + q_{N(2N)}f_N'(0) = 0
\end{cases}.
$$

The restriction of the maximal operator to a N-dimensional subspace $\mathcal{L}$ posseses the pseudo-Hermitian property if and only if

$$
\mathbf{X} \in \mathcal{L} \Leftrightarrow \mathcal{R}T\mathbf{X} \in \mathcal{L}.
$$

Let us start by observing that since the Q appearing in (9) has rank N, at least one of its $N \times N$ minors is non degenerate. Let us suppose that:

$$
\begin{vmatrix}
q_{11} & q_{13} & \cdots & q_{1(2N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
q_{N1} & q_{N3} & \cdots & q_{N(2N-1)}
\end{vmatrix}
\neq 0.
$$

The boundary condition can then be written using a $N \times N$ matrix

$$
A =
\begin{pmatrix}
a_{00} & a_{01} & \cdots & a_{1(N-1)} \\
a_{10} & a_{11} & \cdots & a_{2(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N-1(0)} & a_{N-1(1)} & \cdots & a_{(N-1)(N-1)}
\end{pmatrix},
$$

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$$
\begin{pmatrix} f_1'(0) \\ f_2'(0) \\ \vdots \\ f_N'(0) \end{pmatrix} = A \begin{pmatrix} f_1(0) \\ f_2(0) \\ \vdots \\ f_N(0) \end{pmatrix}.
$$

Suppose the function $\mathbf{F}$ satisfies the boundary conditions. The boundary conditions for the function $\mathcal{R}T\mathbf{F}$ are given by:

$$
\begin{pmatrix} f_1'(0) \\ f_2'(0) \\ \vdots \\ f_N'(0) \end{pmatrix} = \begin{pmatrix} \overline{a}_{(N-1)(N-1)} & \overline{a}_{(N-1)(0)} & \cdots & \overline{a}_{(N-1)(N-2)} \\ \overline{a}_{0(N-1)} & \overline{a}_{00} & \cdots & \overline{a}_{0(N-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{(n-2)(n-1)} & \overline{a}_{(N-2)(0)} & \cdots & \overline{a}_{(N-2)(N-2)} \end{pmatrix} \begin{pmatrix} f_1(0) \\ f_2(0) \\ \vdots \\ f_N(0) \end{pmatrix}.
$$

Then we have the following equality:

(8)
$$
\begin{pmatrix} \overline{a}_{(N-1)(N-1)} & \overline{a}_{(N-1)(0)} & \cdots & \overline{a}_{(N-1)(N-2)} \\ \overline{a}_{0(N-1)} & \overline{a}_{00} & \cdots & \overline{a}_{0(N-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{(N-2)(N-1)} & \overline{a}_{(N-2)(0)} & \cdots & \overline{a}_{(N-2)(N-2)} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{1(N-1)} \\ a_{10} & a_{11} & \cdots & a_{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1(0)} & a_{N-1(1)} & \cdots & a_{(N-1)(N-1)} \end{pmatrix}.
$$

Consider the entries on the main diagonal to obtain the following set of N equalities

$$
\overline{a}_{(N-1)(N-1)} = a_{00},
$$
$$
\overline{a}_{00} = a_{11},
$$
(9)
$$
\vdots
$$
$$
\overline{a}_{(N-3)(N-3)} = a_{(N-2)(N-2)},
$$
$$
\overline{a}_{(N-2)(N-2)} = a_{(N-1)(N-1)}.
$$

When $\underline{N \text{ is odd}}$, the previous set of equalities (10) can be rewritten as

$$
a_{(N-1)(N-1)} = \overline{a}_{00} = a_{11} = \cdots = \overline{a}_{(N-3)(N-3)} = a_{(N-2)(N-2)} = \overline{a}_{(N-1)(N-1)}.
$$

Then, all entries on the diagonal are equal and real and will be denoted by a_0, i.e,

$$
a_0 = a_{00} = a_{11} = \cdots = a_{(N-1)(N-1)} \in \mathbb{R}.
$$

Similarly,

$$
a_1 = a_{(N-1)(0)} = a_{01} = a_{12} = \cdots = a_{(N-2)(N-1)} \in \mathbb{R},
$$
$$
\vdots
$$
$$
a_{N-1} = a_{0(N-1)} = a_{10} = a_{21} = \cdots = a_{(N-1)(N-2)} \in \mathbb{R}
$$

This shows that when N is odd the entries a_{ij} of A depends only on the difference $(j - i) \bmod N$, therefore it is a circulant matrix and $a_i \in \mathbb{R}$ for $i = 0, \cdots, N - 1$.

If $\underline{N \text{ is even}}$, let us consider again the set of N equalities in (10) to obtain the following equalities :

$$
a_{00} = \overline{a}_{11} = a_{22} = \cdots = \overline{a}_{(N-2)(N-2)} = a_{(N-1)(N-1)}.
$$

The entries in the odd rows of the main diagonal are all equal. Let us denote them by a_0. Then, $\overline{a_0}$ are the entries of the main diagonal in the even rows, i.e,

$$a_0 = a_{00} = a_{22} = \cdots = a_{(N-2)(N-2)},$$
$$\overline{a_0} = a_{11} = \cdots = a_{33} = a_{(N-1)(N-1)}.$$

Similarly,

$$a_1 = a_{01} = a_{23} = \cdots = a_{(N-2)(N-1)},$$
$$\overline{a_1} = a_{12} = a_{34} = \cdots = a_{(N-1)0},$$
$$\vdots$$
$$a_{N-1} = a_{21} = a_{43} = \cdots = a_{0(N-1)},$$
$$\overline{a}_{N-1} = a_{21} = a_{32} = \cdots = a_{(N-1)(N-2)}.$$

Then each row of the matrix A is the conjugated of the previous row shifted to the right, i.e,

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ \overline{a}_{N-1} & \overline{a_0} & \overline{a_1} & \overline{a_2} & \cdots & \overline{a}_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \cdots & \overline{a_0} \end{pmatrix},$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The family L_A describes almost all pseudo-Hermitian extensions of the operator L_0. However, there exist other extensions which are not considered in the present article.

4. The spectrum of the pseudo-hermitian operator

We can now look at the discrete spectra of the constructed pseudo-Hermitian operators on the star graph. A function $\mathbf{F}$ is an eigenfunction of L_A if and only if it satisfies the differential equation:

$$(10) \qquad\qquad\qquad\qquad -\mathbf{F}'' = E\mathbf{F},$$

where $E = -r^2$, and the boundary conditions described in (6).
The differential equation can be easily calculated and it solution belonging to the domain $L_2(\Gamma_N \backslash V)$ on each edge is as follows:

$$(11) \qquad\qquad\qquad\qquad f_i = c_i e^{-rx} \ , \ \Re r > 0.$$

$$i = 1, \cdots, N$$

Let us then find the eigenvalues of the pseudo-Hermitian operator.
Again, we look first at the odd case:
We recall that we have the following boundary conditions:

$$\begin{pmatrix} f_1'(0) \\ f_2'(0) \\ \vdots \\ f_N'(0) \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} f_1(0) \\ f_2(0) \\ \vdots \\ f_N(0) \end{pmatrix}.$$

Substituting the corresponding values of the functions at the edges, we obtain:

$$
\begin{pmatrix} -rc_1 \\ -rc_2 \\ \vdots \\ -rc_N \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix},
$$

$$
(12) \quad \left[\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{pmatrix} + \begin{pmatrix} r & & & \\ & r & & \\ & & \vdots & \\ & & & r \end{pmatrix} \right] \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = 0.
$$

Letting $\lambda = -r$, then the last equation has non trivial solutions if and only if

$$
\det (A - \lambda I) = 0.
$$

Remembering that A is a circulant matrix, it is known that for these type of matrices the determinant and the eigenvalues can be nicely calculated as follows [8]:

$$
(13) \qquad \det A = \prod_{j=0}^{n-1} (a_0 + a_1 z^j + a_2 z^{2j} + \cdots + a_{N-1} z^{(N-1)j}),
$$

and the eigenvalues λ_j are

$$
(14) \qquad \lambda_j = a_0 + a_1 z^j + a_2 z^{2j} + \cdots + a_{N-1} z^{(N-1)j},
$$

$$
j = 0, \cdots, N-1,
$$

where z is the $N-th$ root of the unity:

$$
z = e^{\frac{2\pi i}{N}}.
$$

It is easily seen that $(a_0 + a_1 + a_2 + a_3 + \cdots + a_{N-1})$ is an eigenvalue of the circulant matrix, by simple column operations

$$
\det (A - \lambda I) = \begin{vmatrix} a_0 - \lambda & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 - \lambda & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 - \lambda & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 - \lambda \end{vmatrix}
$$

$$
= \begin{vmatrix} a_0 - \lambda + a_1 + a_2 + \cdots + a_{N-1} & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} + a_0 - \lambda + a_1 + \cdots + a_{N-2} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} + a_{N-1} + a_0 - \lambda + \cdots + a_{N-3} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 + a_2 + a_3 + \cdots + a_0 - \lambda & a_2 & a_3 & a_4 & \cdots & a_0 \end{vmatrix}
$$

$$= (a_0 - \lambda + a_1 + \cdots + a_{N-1}) \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ 1 & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ 1 & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{vmatrix}.$$

Now, following [1] let $\Omega = \operatorname{diag}(1, z, z^2, \cdots, z^{N-1})$ where

$$z = e^{\frac{2\pi i}{N}}.$$

Then

$$\Omega^k = \operatorname{diag}(1, z^k, z^{2k}, \cdots, z^{(N-1)k}).$$

The matrix $\Omega^{-k} A \Omega^k$ is also a circulant matrix determined by the entries $a_j z^{j-1}$ instead of a_j, $j = 1, \cdots, N$. Now using that $\det(A - \lambda I) = \det(\Omega^{-k}(A - \lambda I)\Omega^k)$, then the value of $\det(A - \lambda I)$ can be calculated as follows:

$$\det(A - \lambda I) = \det(\Omega^{-k} A \Omega^k - \lambda I)$$

$$= \begin{vmatrix} a_0 - \lambda & a_1 z^k & a_2 z^{2k} & \cdots & a_{N-1} z^{(N-1)k} \\ a_{N-1} z^{-k} & a_0 - \lambda & a_1 z^k & \cdots & a_{N-2} z^{(N-2)k} \\ a_{N-2} z^{-2k} & a_{N-1} z^{-k} & a_0 - \lambda & \cdots & a_{N-3} z^{(N-3)k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 z^{(1-N)k} & a_2 z^{(2-N)k} & a_3 z^{(3-N)k} & \cdots & a_0 - \lambda \end{vmatrix}$$

$$= \begin{vmatrix} a_0 - \lambda & a_1 z^k & a_2 z^{2k} & \cdots & a_{N-1} z^{(N-1)k} \\ a_{N-1} z^{(N-1)k} & a_0 - \lambda & a_1 z^k & \cdots & a_{N-2} z^{(N-2)k} \\ a_{N-2} z^{(N-2)k} & a_{N-1} z^{(N-1)k} & a_0 - \lambda & \cdots & a_{N-3} z^{(N-3)k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 z^k & a_2 z^{2k} & a_3 z^{3k} & \cdots & a_0 - \lambda \end{vmatrix}.$$

Again by simple column operations it can be seen that $(a_0 - \lambda + a_1 z + a_2 z^{2k} + \cdots + a_{N-1} z^{(N-1)k})$ is a factor of $\det(A - \lambda I)$ for $k = 1 \cdots N - 1$, and together with the factor $a_0 - \lambda + a_1 + \cdots + a_{N-1}$, these are the N factors in $\det(A - \lambda I)$.

$$\det(A - \lambda I) = \prod_{j=0}^{N-1} (a_0 - \lambda + a_1 z^j + a_2 z^{2j} + \cdots + a_{N-1} z^{(N-1)j})$$

$$= (a_0 - \lambda + a_1 + a_2 + \cdots + a_{N-1}) \times$$

$$\left(\prod_{j=1}^{\frac{N-1}{2}} (a_0 + \lambda + a_1 z^j + \cdots + a_{N-1} z^{(N-1)j}) \times (a_0 + \lambda + a_1 z^{-j} + \cdots + a_{N-1} z^{-(N-1)j}) \right).$$

The following theorem concerning the spectrum of the operator can then be proven:

Theorem 4.1. *When N is odd, consider the pseudo-hermitian operator L_A on the star-graph Γ_N defined in 3.2. If the operator has N eigenvalues and these eigenvalues are real then the operator is self-adjoint.*

Proof. If N is odd, A is a circulant matrix with the characteristic polynomial

$$det(A + \lambda I) = \prod_{j=0}^{N-1} (a_0 + \lambda + a_1 z^j + a_2 z^{2j} + \cdots + a_{N-1} z^{(N-1)j})$$

$$= (a_0 + \lambda + \cdots + a_{N-1}) \times$$

$$\left(\prod_{j=1}^{\frac{n-1}{2}} (a_0 + \lambda + \cdots + a_{n-1} z^{(n-1)j}) \times (a_0 + \lambda + \cdots + a_{n-1} z^{-(n-1)j}) \right).$$

Let

$$\mu_j = a_0 + a_1 z^j + a_2 z^{2j} + \cdots + a_{\frac{N-1}{2}} z^{\frac{N-1}{2}j} + a_{\frac{N+1}{2}} z^{\frac{N+1}{2}j} + \cdots + a_{N-2} 1 z^{(N-2)j} + a_{N-1} z^{(N-1)j}$$

$$= a_0 + a_1 z^j + a_2 z^{2j} + \cdots + a_{\frac{N-1}{2}} z^{\frac{-j}{2}} + a_{\frac{N+1}{2}} z^{\frac{j}{2}} + \cdots + a_{N-2} 1 z^{-2j} + a_{N-1} z^{-j},$$

$$\mu_{N-j} = a_0 + a_1 z^{N-j} + a_2 z^{2(N-j)} + \cdots + a_{\frac{N-1}{2}} z^{\frac{N-1}{2}(N-j)} + a_{\frac{N+1}{2}} z^{\frac{N+1}{2}(N-j)} + \cdots + a_{N-2} z^{(N-2)(N-j)}$$

$$+ a_{N-1} z^{(N-1)(N-j)}$$

$$= a_0 + a_1 z^{-j} + a_2 z^{-2j} + \cdots + a_{\frac{N-1}{2}} z^{-\frac{N-1}{2}(j)} + a_{\frac{N+1}{2}} z^{-\frac{N+1}{2}(j)} + \cdots + a_{N-2} z^{2j} + a_{N-1} z^j$$

$$= a_0 + a_1 z^{-j} + a_2 z^{-2j} + \cdots + a_{\frac{N-1}{2}} z^{\frac{j}{2}} + a_{\frac{N+1}{2}} z^{-\frac{j}{2}} + \cdots + a_{N-2} z^{2j} + a_{N-1} z^j,$$

$$j = 1, \cdots, N-1.$$

Therefore, $\mu_j = \overline{\mu_{N-j}}$.
Let us then consider the following linear combination of μ_j's

$$P_1(z) = \mu_0 + z^{N-1} \mu_1 + z^{N-2} \mu_2 + \cdots + z^2 \mu_{N-2} + z \mu_{N-1},$$

and taking into account that $\sum_{k=0}^{N-1} z^k = 0$, we obtain that $P_1(z) = N a_1$.
Similarly,

$$P_{N-1}(z) = \mu_0 + z \mu_1 + z^2 \mu_2 + \cdots + z^{N-2} \mu_{N-2} + z^{N-1} \mu_{N-1} = N a_{N-1}.$$

If $\lambda \in \mathbb{R}$, then $\mu_j = \mu_{N-j}$ and

$$P_1(z) = P_{N-1}(z),$$

which means that $a_1 = a_{N-1}$.
In the same way linear combinations of μ_j's can be found such that $P_j(z) = N a_j$
and then use the reality of the eigenvalues to show that

$$a_2 = a_{N-2},$$
$$a_3 = a_{N-3},$$
$$\vdots$$
$$a_{\frac{N-1}{2}} = a_{\frac{N+1}{2}}.$$

$a_j = a_{N-j}, j = 1, \cdots, N-1.$ The matrix A must be symmetric. $\qquad \square$

Let us now look at the even case:
We recall that we have the following boundary conditions

$$(15) \quad \begin{pmatrix} f'_1(0) \\ f'_2(0) \\ \vdots \\ f'_N(0) \end{pmatrix} = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots & b_{k-1} \\ b_{k-1} & b_0 & b_1 & b_2 & \cdots & b_{k-2} \\ b_{k-2} & b_{k-1} & b_0 & b_1 & \cdots & b_{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_0 \end{pmatrix} \begin{pmatrix} f_1(0) \\ f_2(0) \\ \vdots \\ f_N(0) \end{pmatrix},$$

where $k = \frac{N}{2}$, and

$$b_i = \begin{pmatrix} a_{2i} & a_{2i+1} \bmod N \\ \overline{a}_{2i-1} \bmod N & \overline{a}_{2i} \end{pmatrix}.$$

Now letting

$$w = \begin{bmatrix} v \\ zv \\ z^2 v \\ \vdots \\ z^{k-1} v \end{bmatrix},$$

where v is a non-zero 2-vector and z is the $k-th$ root of the unity :

$$z = e^{\frac{2\pi}{k} j}, \quad j = 0, \cdots, k-1.$$

The vector w is an eigenvector of A with eigenvalue λ if and only if

$$(16) \quad Aw = \lambda w.$$

Extending (16) the following set of k equations are obtained:

$$(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots + b_{k-1} z^{k-1})v = \lambda v,$$
$$(b_{k-1} + b_0 z + b_1 z^2 + b_2 z^3 + \cdots + b_{k-2} z^{k-1})v = z\lambda v,$$
$$\vdots$$
$$(b_1 + b_2 z + b_3 z^2 + b_4 z^3 + \cdots + b_0 z^{k-1})v = z^{k-1}\lambda v.$$

Diving the kth equation by z^k, $k = 2, \ldots, \frac{N}{2}$, then every equation reduces to the first one. Let us now rewrite it as an eigenvector equation:

$$(17) \quad Hv = \lambda v,$$

where the square matrix H is

$$(18) \quad H = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots + b_{k-1} z^{k-1}$$

$$= \begin{pmatrix} a_0 & a_1 \\ \overline{a}_{N-1} & \overline{a}_0 \end{pmatrix} + \begin{pmatrix} a_2 & a_3 \\ \overline{a}_1 & \overline{a}_2 \end{pmatrix} z + \cdots + \begin{pmatrix} a_{N-2} & a_{N-1} \\ \overline{a}_{N-3} & \overline{a}_{N-2} \end{pmatrix} z^{k-1}$$

$$= \begin{pmatrix} a_0 + a_2 z + \cdots + a_{N-2} z^{k-1} & a_1 + a_3 z + \cdots + a_{N-1} z^{k-1} \\ \overline{a}_{N-1} + \overline{a}_1 z + \cdots + \overline{a}_{N-3} z^{k-1} & \overline{a}_0 + \overline{a}_2 z + \cdots + \overline{a}_{N-2} z^{k-1} \end{pmatrix}.$$

Now for each value of z corresponding to different j's the eigenvectors v of H, give eigenvectors w of the block circulant matrix A, with that eigenvalue λ [16].
The following theorem concerning the spectrum of the pseudo-hermitian operators can then be proven.

Theorem 4.2. *Let the number of edges of the star-graph Γ_N be even. Then among the pseudo-Hermitian operators L_A there are non self-adjoint operators with N real eigenvalues.*

Proof. Since each eigenvector of H, gives an eigenvector w of A with the same λ, we shall now calculate the eigenvalues of H and as a result the eigenvalues of A:

$$
(19) \qquad
\begin{aligned}
\det(H - \lambda I) = \; & (a_0 + \cdots + a_{N-2}z^{k-1} - \lambda)(\overline{a_0} + \cdots + \overline{a}_{N-2}z^{k-1} - \lambda) \\
& -(a_1 + \cdots + a_{N-1}z^{k-1})(\overline{a}_{N-1} + \cdots + \overline{a}_{N-3}z^{k-1}).
\end{aligned}
$$

Let us use the following notations:

$$
G_e(z) = a_0 + a_2 z + \cdots + a_{N-2}z^{k-1},
$$

$$
G_0(z) = a_1 + a_3 z + \cdots + a_{N-1}z^{k-1}.
$$

Then we have:

$$
\overline{G_e(1/z)} = \overline{a_0} + \overline{a_2}z + \cdots + \overline{a}_{N-2}z^{k-1},
$$

and

$$
z\overline{G_0(1/z)} = \overline{a}_{N-1} + \overline{a_1}z + \cdots + \overline{a}_{N-3}z^{k-1}.
$$

The determinant $\det(H - \lambda I)$ can then be rewritten as:

$$
\det(H - \lambda I) = \lambda^2 - \lambda(G_e(z) + \overline{G_e(1/z)}) + (G_e(z)\overline{G_e(1/z)} - zG_0(z)\overline{G_0(1/z)}).
$$

Then the eigenvalues corresponding to each z are as follows:

$$
\lambda_\pm(z) = \left(G_e(z) + \overline{G_e(1/z)} \pm \sqrt{(G_e(z) - \overline{G_e(1/z)})^2 + 4zG_0(z)\overline{G_0(1/z)}} \right)/2.
$$

Let us consider $z \notin \mathbb{R}^+$ and use that $\sqrt{\bar{z}} = -\overline{\sqrt{z}}$. Let us also introduce the notation $u = 1/z$. We obtain that:

$$
\overline{\lambda_\pm}(z) = \left(G_e(u) + \overline{G_e(1/u)} \mp \sqrt{(G_e(u) - \overline{G_e(1/u)})^2 + 4uG_0(u)\overline{G_0(1/u)}} \right)/2.
$$

If $\lambda_\pm(z)$ is an eigenvalue of A then $\overline{\lambda_\pm} = \lambda_\mp(1/z)$ is also an eigenvalue of A. Therefore the complex eigenvalues of A come in conjugated pairs. Now let us study when the eigenvalues are real. Let us use the following notations:

$$
\begin{aligned}
U(z) &= G_e(z) + \overline{G_e(1/z)}, \\
V(z) &= G_e(z) - \overline{G_e(1/z)}, \\
W(z) &= zG_0(z)\overline{G_0(1/z)}.
\end{aligned}
$$

Then we have:

$$
\lambda_\pm(z) = \left(U(z) \pm \sqrt{V^2(z) + 4W(z)} \right)/2.
$$

To prove the theorem it is enough to present a family of non self-adjoint pseudo-Hermitian operators with N real eigenvalues. Therefore we must show that it is posssible to find entries a_i of A such that the following conditions are satisfied:
1. $U(z) < 0$,
2. $V^2(z) + 4W(z) \geq 0$,
3. $V^2(z) + 4W(z) \leq U(z)^2$.
Let us then take the following values for the a_i's:

$$
(20) \qquad\qquad a_2 = a_4 = \cdots = a_{N-2},
$$

$$
(21) \qquad\qquad a_1 = a_3 = \cdots = a_{N-1},
$$

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$$(22) \qquad\qquad \Im a_0 = \Im a_2.$$

In particular:

$$(23) \qquad\qquad a_0 = -8k + 2 + i, \quad k > 0,$$

$$(24) \qquad\qquad a_1 = 1 + i,$$

$$(25) \qquad\qquad a_2 = 2 + i,$$

let us then see that condition 1 is fullfilled, i.e,

$$(26) \qquad U(z) = (a_0 + \overline{a_0}) + (a_2 + \overline{a_2})z^j + \cdots + (a_{N-2} + \overline{a}_{N-2})z^{(k-1)j} \in \mathbb{R}^-.$$

If (20) holds then:

$$\Re a_2 = \cdots = \Re a_{N-2}.$$

Then:

$$U(z) = 2\Re a_0 + 2\Re a_2 \left(\sum_{m=1}^{k-1} z^m \right).$$

Let us now substitute values of a_0 and a_2 described in (24) and (25) respectively.
If $z \neq 1$ then:

$$U(z) = 2(\Re a_0 - \Re a_2) = -16k.$$

Otherwise $z = 1$ and:

$$U(z) = 2(\Re a_0 + (k-1)\Re a_2) = -12k.$$

Then in both cases $U(z) < 0$.
We must also show that $V^2(z) + 4W(z) \geq 0$. Let us recall that

$$V^2(z) + 4W(z)$$

$$= ((a_0 - \overline{a_0}) + (a_2 - \overline{a_2})z + \cdots + (a_{N-2} - \overline{a}_{N-2})z^{k-1})^2 + 4(a_1 + \cdots + a_{N-1}z^{k-1})(\overline{a}_{N-1} + \overline{a_1}z + \cdots + \overline{a}_{N-3}z^{k-1})$$

$$= 4[(a_1 + \cdots + a_{N-1}z^{k-1})(\overline{a}_{N-1} + \overline{a_1}z + \cdots + \overline{a}_{N-3}z^{k-1}) - (\Im a_0 + \Im a_2 z + \cdots + \Im a_{N-2}z^{k-1})^2].$$

Substituting (20),(21), and (22) gives:

$$V^2(z) + 4W(z) = 4\left[\left(a_1 \left(\sum_{m=0}^{k-1} z^m \right) \right) \left(\overline{a_1} \left(\sum_{m=0}^{k-1} z^m \right) \right) - \left(\Im a_0 \left(\sum_{m=0}^{k-1} z^m \right) \right)^2 \right]$$

Substituting values of a_1 and a_0 described in (23) and (24) respectively we obtain
that: If $z \neq 1$:

$$V^2(z) + 4W(z) = 0$$

otherwise $z = 1$ and:

$$V^2(z) + 4W(z) = 4[(ka_1)(k\overline{a_1}) - k^2(\Im a_0)^2] = 4k^2(|a_1|^2 - (\Im a_0)^2) = 4k^2$$

Then conditions 2 is satisfied.
We have obtained that if $z = 1$:

$$U(z)^2 = (12k)^2$$

and

$$V(z)^2 + 4W(z) = 4k^2$$

This means that $V^2 + 4W(z) \leq U(z)^2$, which is condition 3.
If $z = 1$ then

$$U(z)^2 = (16k)^2$$

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and
$$V(z)^2 + 4W(z) = 0$$
Then it is clear that $V^2 + 4W(z) \leq U(z)^2$, and condition 3 is also satisfied in this case.

Therefore, let the entries $a_i \in \mathbb{C}$, $i = 0, \cdots, N-1$, be such that they satisfy (20)-(25), then A is of the following form:

$$A = \begin{pmatrix} (-8k+2)+i & 1+i & 2+i & \cdots & 2+i & 1+i \\ 1-i & (-8k+2)-i & 1-i & \cdots & 1-i & 2-i \\ 2+i & 1+i & (-8k+2)+i & \cdots & 2+i & 1+i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2+i & 1+i & 2+i & \cdots & (-8k+2)+i & 1+i \\ 1-i & 2-i & 1-i & \cdots & 1-i & -8k+2)-i \end{pmatrix},$$

and L_A is pseudo-Hermitian with N real eigenvalues but it is not self-adjoint. $\quad\square$

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