Varför bör fysiker fråga matematiker?

Pavel Kurasov

26 september 2006, LMS

Sherlock Holmes and Dr. Watson in Kåseberga

Physicists deal with mathematics as criminals do with criminal codes (I. Gelfand)

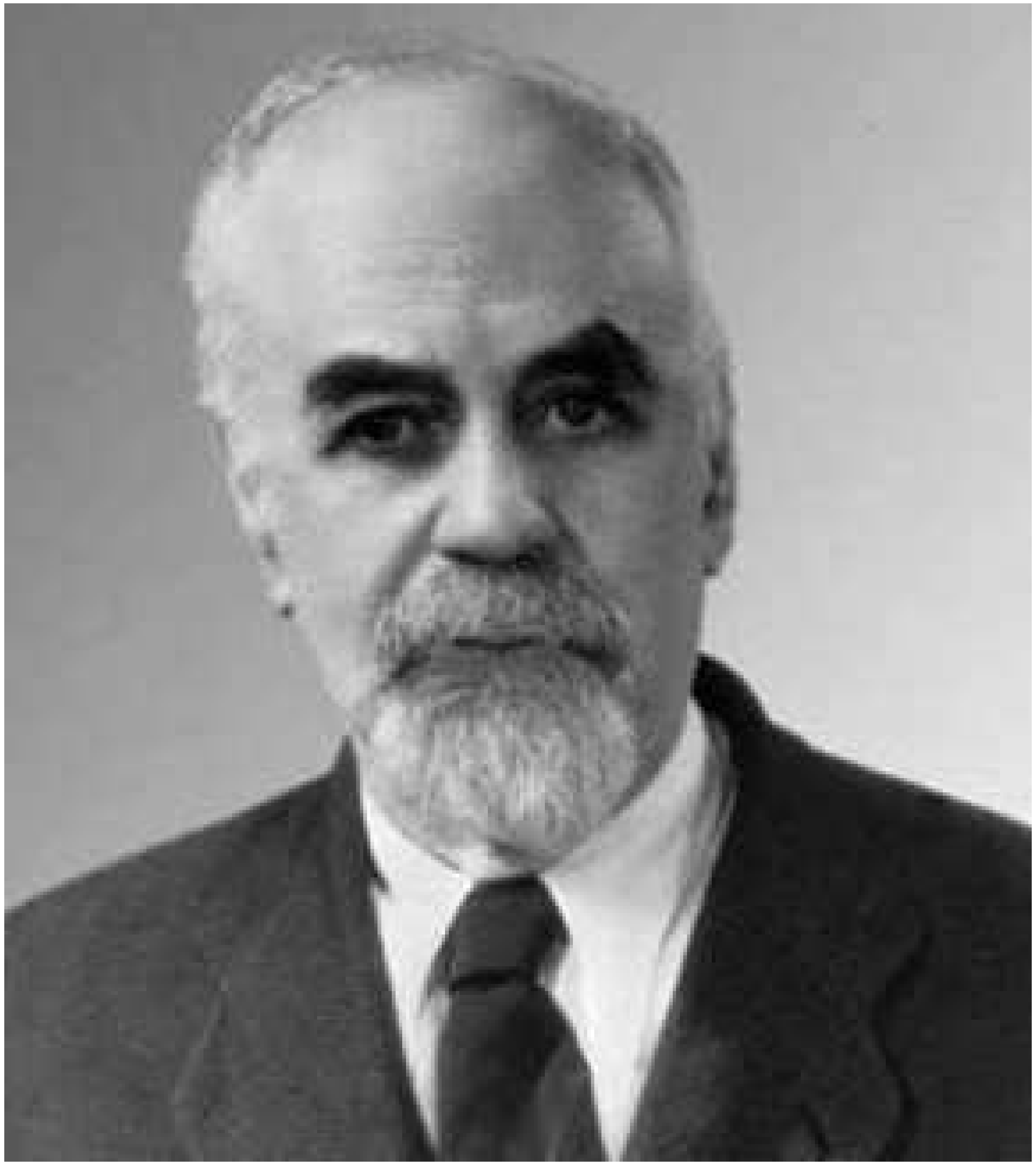
Eugene Wigner and his wife

E.Wigner, The unreasonable effectiveness of mathematics in the natural sciences, *Comm. Pure Appl. Math.*, **XIII**, 1–14 (1960).

”The miracle of the approriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in the future research...”

Examples: Complex numbers $\mathbb{C}$, formulation of quantum mechanics as linear operators in Hilbert space.

”... the mathematician could formulate only a handful of interesting theorems without defining concepts beyond those contained in the axioms and that the concepts outside those contained in the axioms are defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense both as operations and also in their results of great generality and simplicity.”



Vladimir Smirnov

Vladimir Fock

Ludwig Faddeev

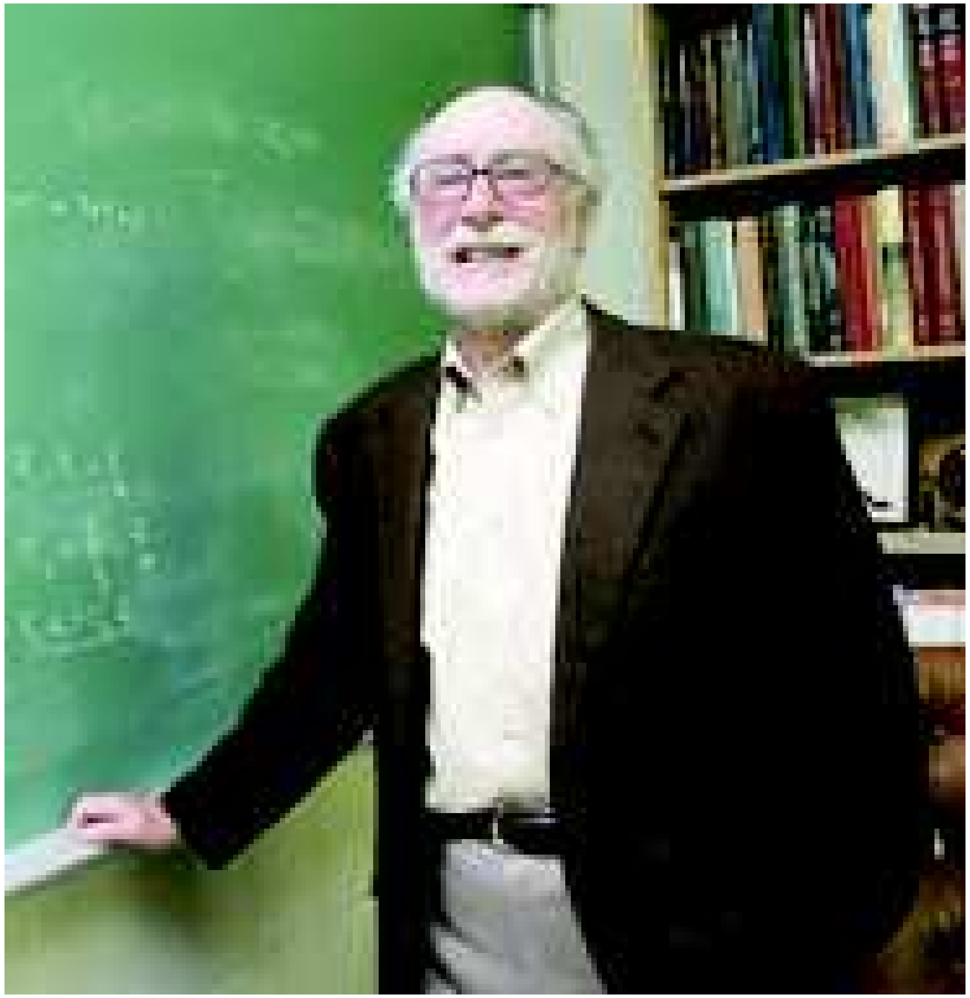
Ludwig Faddeev:

"I consider as the main goal of Mathematical Physics the use of mathematical intuition for the derivation of really new results in fundamental physics."

"... the development of mathematics, which is also to a great extent influenced by applications, has nevertheless its internal logic. Ideas are judged not by their relevance but more by esthetic criteria. The totalitarianism of theoretical physics gives way to a kind of democracy in mathematics and its inherent intuition."

Yuri Demkov

Marcos Moshinsky



Roger Newton

Quantum Graphs - (Γ, H, BC)

- Γ - metric graph
- H - differential operator on the edges
- BC - boundary conditions at the vertices

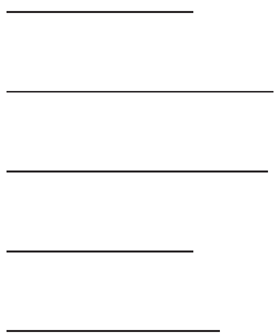
Definitions

1. Metric graph Γ

N edges $(N < \infty)$ $\Delta_j = [x_{2j-1}, x_{2j}] \subset \mathbb{R},\ j = 1, 2, ..., N.$

$\mathbf{E} = \{x_i\}_{i=1}^{2N}$ - the set of endpoints

$\mathbf{E} \sim \cup_{m=1}^{M} E_m,\ E_m$ - vertex



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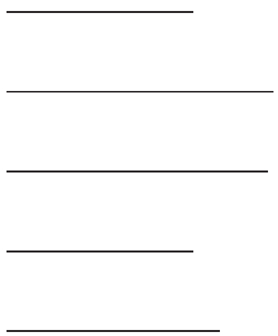
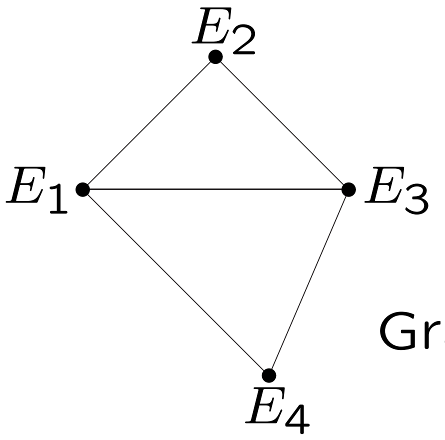




Graph Γ

2. Laplace operator

$$L = -\frac{d^2}{dx^2}$$

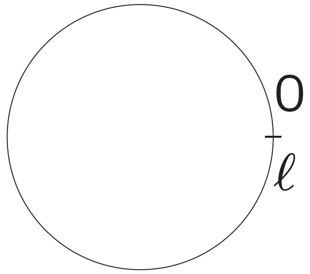
standard boundary conditions

$$\begin{cases} f \text{ continuous} \\ \sum_{x_i \in E_m} \partial_n f(x_i) = 0, \; m = 1, 2, \ldots, M. \end{cases}$$

NB! The metric graph Γ determines the Laplace operator completely.

Spectral theory

Example $L = -\frac{d^2}{dx^2}$, $\mathcal{H} = L_2(0, \ell)$



$$\begin{cases} f'(0) = 0 \\ f'(\ell) = 0 \end{cases} \qquad \begin{cases} f(0) = f(\ell) \\ f'(0) = f'(\ell) \end{cases}$$

$$\boxed{-\frac{d^2}{dx^2}\psi(x,\lambda) = \lambda\psi(x,\lambda),\ k^2 = \lambda}$$

$\psi(x,\lambda) = \alpha \sin kx + \beta \cos kx$

$\psi'(0,\lambda) = 0$

$\Rightarrow \psi(x,\lambda) = \beta \cos kx$

$\psi'(\ell,\lambda) = 0 \Rightarrow \sin k\ell = 0$

$\Rightarrow k = \frac{\pi n}{\ell}$

$\Rightarrow \lambda = 0, \left(\frac{\pi}{\ell}\right)^2, \left(\frac{2\pi}{\ell}\right)^2, \ldots$

$\psi(x,\lambda) = ae^{ikx} + be^{-ikx}$

$$\begin{cases} a + b &= ae^{ik\ell} + be^{-ik\ell} \\ a - b &= ae^{ik\ell} - be^{-ik\ell} \end{cases}$$

$e^{ik\ell} = 1 \wedge e^{-ik\ell} = 1$

$k = \frac{2\pi}{\ell}n$

$\Rightarrow \lambda = 0, \left(\frac{2\pi}{\ell}\right)^2, \left(\frac{2\pi}{\ell}\right)^2, \ldots$

$$\lambda_n = \left(\frac{\pi}{\ell}n\right)^2 + O(n),\ n \to \infty$$

$$u(k) \equiv 2m_s(0)\delta(k) + \sum_{n=1}^{\infty} \delta(k - k_n)$$

$$= \frac{\mathcal{L}}{\pi} + \frac{1}{2\pi} \sum_{p\in\mathcal{P}} l(\mathrm{prim}(p))S(p)\ (e^{ikl(p)} + e^{-ikl(p)}),$$

Theorem (Trace formula)

Let $L(\Gamma)$ be the Laplace operator on a (compact) graph Γ with Euler characteristic χ, then

$$u(k) \equiv 2m_s(0)\delta(k) + \sum_{n=1}^{\infty} (\delta(k-k_n) + \delta(k+k_n))$$

$$= \chi\delta(k) + \frac{\mathcal{L}}{\pi} + \frac{1}{2\pi}\sum_{p\in\mathcal{P}} l(\text{prim}(p))S(p)\left(e^{ikl(p)} + e^{-ikl(p)}\right),$$

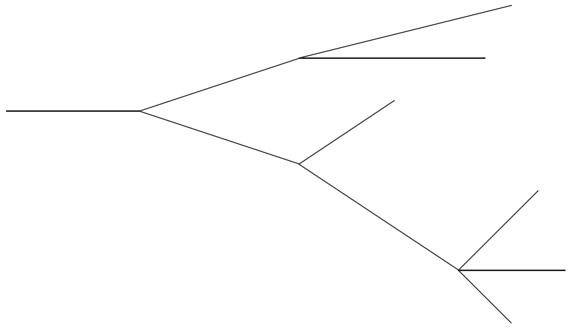
where:
$\mathcal{L}$ - the total length of Γ;
$m_s(0)$ - the mutiplicity of the eigenvalue $\lambda = 0$;
p - closed paths on Γ;
$\mathcal{P}$ - the set of all closed paths;
$l(p)$ - the length of p;
$\lambda_n = k_n^2$ - eigenvalues of $L(\Gamma)$;
$(S_v^m)_{ij} = \begin{cases} \frac{2}{v_m}, & i \neq j \\ \frac{2-v_m}{v_m}, & i = j, \end{cases}$, $v_m -$ the multiplicity of E_m;
$S(p)$ - product of all scattering coefficients along the path p.

Euler characteristic

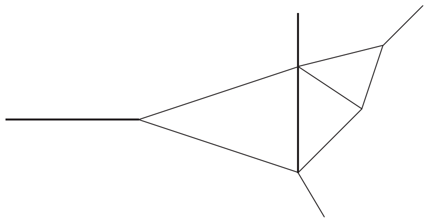
$$\chi = M - N$$

M - the number of vertices;
N - the number of edges.

Γ - a tree $\Rightarrow M = N + 1 \Leftrightarrow N = M - 1 \Rightarrow \chi(\Gamma) = 1$



Γ - a graph with n cycles $\Rightarrow N = M - 1 + n \Rightarrow \chi = 1 - n$
n - the number of generators for the fundamental group of Γ



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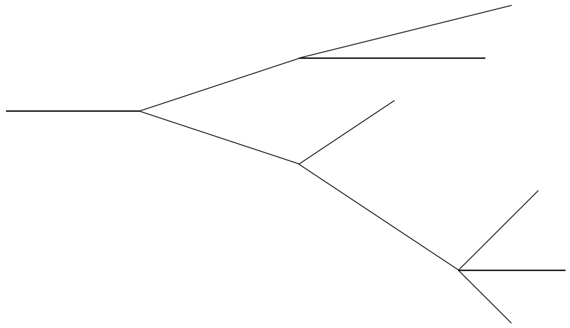

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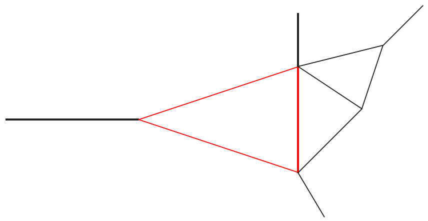

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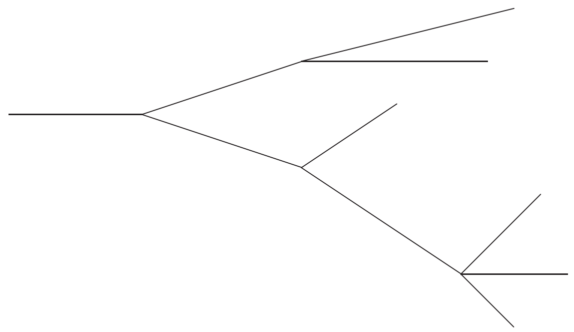

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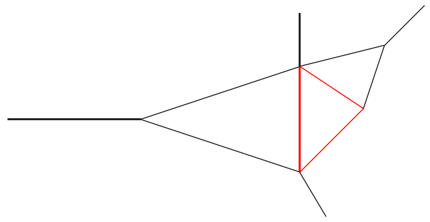

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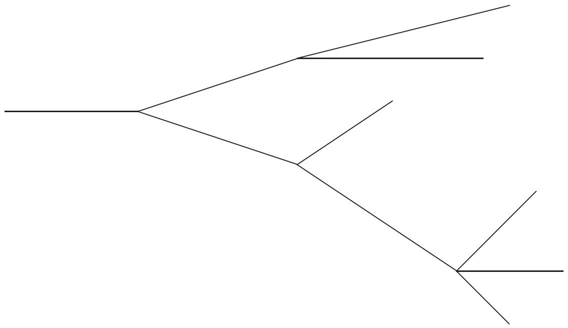

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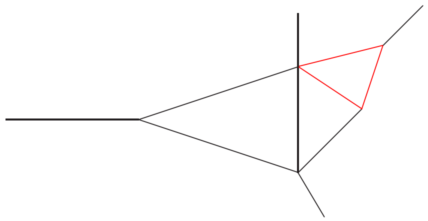

$$\boxed{n = 1 - \chi}$$

Uniqueness theorem

Isospectral graphs have
the same Euler characteristics,
the same number of connected components and
the same total length.