LECTURE 7

Distribution theory and Poisson's summation formula

7.1. Introduction to distribution theory

In what follows we shall use distribution theory in order to express the spectral measure associated with metric graphs. This lecture provides an extremely short introduction into distribution theory. Every L_1-function can be seen as a distribution applied to a test function $\varphi \in \mathscr{D} = C_0^\infty(\mathbb{R})$

$$f[\varphi] = \int_{\mathbb{R}} f(x)\varphi(x)dx. \tag{7.1}$$

The set $\mathscr{D}$ of test functions can be seen as a topological vector space:

DEFINITION 7.1. *A sequence $\{\varphi_n\}$ of test functions is* **converging** *in $\mathscr{D}$ to the test function φ*

$$\varphi_n \stackrel{\mathscr{D}}{\to} \varphi$$

iff

1. *there is a bounded interval $I = [a, b]$ such that all functions φ_n have their support inside I;*
2. *$\varphi_n(x) \to \varphi(x)$ as $n \to \infty$ uniformly in $x \in I$;*
3. *$\left(\frac{d}{dx}\right)^k \varphi_n(x) \to \left(\frac{d}{dx}\right)^k \varphi(x)$ as $n \to \infty$ for all $k = 1, 2, 3, ...$, and uniformly in $x \in I$.*

Note that this convergence in $\mathscr{D}$ cannot be defined using any norm.

Then generalising (7.1) the set of distributions is defined as a set of continuous functionals on $\mathscr{D}$.

DEFINITION 7.2. *A functional f is called* **continuous** *iff*

$$\varphi_n \stackrel{\mathscr{D}}{\to} \varphi \Rightarrow f[\varphi_n] \to f[\varphi].$$

DEFINITION 7.3. *The set $\mathscr{D}'$ of* **distributions** *is the set of continuous linear functionals on $\mathscr{D}$.*

$f \in \mathscr{D}' \Leftrightarrow$

1. *f is a functional on $\mathscr{D}$ i.e. $f[\varphi] \in \mathbb{C}$ and is defined for every $\varphi \in \mathscr{D}$;*
2. *f is continuous;*
3. *f is linear, i.e. $f[\alpha_1\varphi_1 + \alpha_2\varphi_2] = \alpha_1 f[\varphi_1] + \alpha_2 f[\varphi_2]$, $\alpha_j \in \mathbb{C}, \varphi_j \in \mathscr{D}$.*

PROBLEM 28. *Check that formula* (7.1) *determines a distribution on $\mathscr{D}$ in accordance with the definition above.*

EXAMPLE 7.4. *Dirac's delta distribution (delta-function) determined by*

$$\delta[\varphi] = \varphi(0) \tag{7.2}$$

is an example of a distribution that cannot be defined by formula (7.1)*. Assume on the contrary that there exists a function* $f \in L_1(\mathbb{R})$ *such that*

$$\int_{\mathbb{R}} f(x)\varphi(x)dx = \varphi(0)$$

holds. Then f *must be equal to zero almost everywhere on* $x \neq 0$*. Every such function gives zero distribution* $\int_{\mathbb{R}} f(x)\varphi(x)dx = 0$ *instead of* (7.1)*.*

Many operations on distributions are defined copying how these operations act on L_1-functions. For example let T_{x_0} denote the shift transformation given by

$$\big(T_{x_0}f\big)(x) = f(x - x_0)$$

on usual functions. Then to determine $\delta_{x_0} = T_{x_0}\delta$ we start with the following calculation

$$\begin{aligned}\big(T_{x_0}f\big)[\varphi] &= \int_{\mathbb{R}} \big(T_{x_0}f\big)(x)\varphi(x)dx = \int_{\mathbb{R}} f(x-x_0)\varphi(x)dx = \int_{\mathbb{R}} f(x)\varphi(x+x_0)dx \\ &= f[T_{-x_0}\varphi],\end{aligned}$$

valid for any $f \in L_{1,loc}(\mathbb{R})$ and $\varphi \in \mathscr{D}$. Then we use formula

$$\big(T_{x_0}f\big)[\varphi] = f[T_{-x_0}\varphi] \tag{7.3}$$

to define the shift T_{x_0} of any distribution since the shift of test functions $T_{-x_0}\varphi$ is well defined.

For example applying formula (7.3) to the delta distribution we get:

$$\delta_c[\varphi] = T_c\delta[\varphi] = \delta[T_{-c}\varphi] = T_{-c}\varphi(0) = \varphi(0 + c) = \varphi(c).$$

This distribution is sometimes denoted with abuse of notations by

$$\delta(x - c)[\varphi] = \big(T_c\delta\big)[\varphi] = \varphi(c). \tag{7.4}$$

To differentiate distributions we use that for $f \in C^1(\mathbb{R})$ the following formula holds

$$\frac{d}{dx}f[\varphi] = \int_{\mathbb{R}} \frac{df}{dx}(x)\varphi(x)dx = -\int_{\mathbb{R}} f(x)\frac{d\varphi}{dx}dx = -f\big[\frac{d}{dx}\varphi\big],$$

where we used that φ has compact support. All distributions can be differentiated using

$$\frac{d}{dx}f[\varphi] = -f\big[\frac{d}{dx}\varphi\big]. \tag{7.5}$$

Problem 29. *Check that formula* (7.5) *determines a distribution in* $\mathscr{D}'$*, provided that* f *is a distribution from* $\mathscr{D}'$*.*

Note the minus sign appearing in the formula above.

In particular we have

$$\delta = \frac{d}{dx}\Theta \quad \text{and} \quad \frac{d}{dx}\delta'[\varphi] = -\frac{d}{dx}\varphi(0),$$

where $\Theta(x)$ is the Heaviside function: $\Theta(x) = \begin{cases} 1, & x > 0; \\ 0, & x < 0. \end{cases}$

In this way we are able to differentiate any L_1-function. The corresponding derivatives are called **generalised derivatives**. The derivatives used in the rigorous definition of the Schrödinger operator in Section 1.1 should in fact be understood as generalised derivatives. For example the domain for the Laplacian and the Schrödinger operator with L_∞ potential consists of L_2-functions whose second generalised derivative is again an L_2-function.

DEFINITION 7.5. *Let E be an open interval $E \subset \mathbb{R}$. Let $\mathscr{D}(E)$ be the set of infinitely many times differentiable functions with compact support inside E and convergence defined by Definition 7.1. Then the* **Sobolev space** $W_2^2(E)$ *is the space of all L_2-functions on E whose second generalised derivative also belongs to $L_2(E)$. The norm in the space is given by*

$$\|u\|^2_{W_2^2(E)} = \int_E \Big(|u(x)|^2 + |u''(x)|^2 \Big) dx.$$

We introduce the Sobolev space $W_2^2(\Gamma \setminus \mathbf{V})$ as the orthogonal sum of Sobolev spaces on the edges (with the end points removed):

$$W_2^2(\Gamma) = \bigoplus_{n=1}^{N} W_2^2(x_{2n-1}, x_{2n}). \tag{7.6}$$

Sobolev embedding theorems imply that the functions from $W_2^2(x_{2n-1}, x_{2n})$ together with their first derivatives are continuous, hence it is possible to introduce the boundary values of the functions and the normal derivatives as it was done in (1.12) and (1.13).

Hence the Laplacian and the Schrödinger operator with L_∞ potential should be defined on the functions from the Sobolev space $W_2^2(\Gamma)$ in addition satisfying vertex conditions (3.24) as it was done in Definition 3.3.

The **Fourier transform** of distributions can be defined only if the set of test functions $\mathscr{D}$ is extended by allowing rapidly decaying at infinity functions (instead of functions with compact support).

DEFINITION 7.6. *Let $\mathscr{S}$ be the set of all $\phi \in C^\infty(\mathbb{R})$ such that*

$$\sup_x \left| x^\beta \left(\frac{d}{dx}\right)^\alpha \phi(x) \right| < \infty \tag{7.7}$$

for all $\alpha, \beta \in \mathbb{N}$. The topology (convergence) in $\mathscr{S}$ is defined by the seminorms in the left-hand side of (7.7)*.*

The corresponding distributions are denoted by $\mathscr{S}'$ and are called **tempered**. The reason to extend $\mathscr{D}$ is that the most natural way to define Fourier transform of distributions is via the formula

$$\hat{f}[\varphi] = f[\hat{\varphi}]. \tag{7.8}$$

In order to use this formula one needs that the Fourier transform of the test function $\hat{\varphi}$ is again a test function. The Fourier transform of any function with compact support never has compact support, in other words the space of test functions $\mathscr{D}$ is not invariant under the Fourier transform.

Formula (7.8) holds for rapidly decaying at infinity functions and therefore is used to defined the Fourier transform of the distributions. In particular we have

$$\hat{\delta}[\varphi] = \delta[\hat{\varphi}] = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) dx = 1[\varphi],$$

implying that

$$\hat{\delta} = 1.$$

Here 1 is the function on $\mathbb{R}$ assigning number 1 to all x. In a similar way we prove that

$$\hat{\delta}_c = e^{-icx}. \tag{7.9}$$

PROBLEM 30. *Prove formula* (7.9) *using definition of the Fourier transform.*

It is very important that all readers use any standard book on distribution theory to learn more about this important branch of modern mathematical physics. The book by Robert Strichartz *A Guide to Distribution Theory and Fourier Transforms* and the first volume of Lars Hörmander *The Analysis of Linear Partial Differential Operators* are recommended.

7.2. Poisson's summation formula

Proving trace formula we shall use Sokhotski-Plemelj's formula (see *e.g.* formula (3.2.11) in [?Hor])

$$\delta = \frac{1}{2\pi i}\left(\frac{1}{x-i0} - \frac{1}{x+i0}\right).$$

This formula can be obtained by integrating along the real axis and taking the limit

$$\lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\left(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\right).$$

A generalisation of this formula can be obtained integrating the jump of the logarithmic derivative of an analytic function on the real axis. Let p be an analytic function with real zeroes $z_j \in \mathbb{R}$, then it holds

$$\frac{1}{2\pi i}\ \frac{d}{dx}\Big(\log p(x-i0) - \log p(x+i0)\Big) = \sum_{z_j}\delta_{z_j}.$$

One may use the following reasoning to justify the formula. Let φ be a $C_0^\infty(\mathbb{R})$ function with the support in $[a,b]$ containing just one of the zeroes of the function p, say a simple zero z_j. In this case we have:

$$\begin{aligned}
& \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{p'(k-i\epsilon)}{p(k-i\epsilon)} - \frac{p'(k+i\epsilon)}{p(k+i\epsilon)}\right)\varphi(k)dk \\
= \ & \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\left(\underbrace{\int_a^{z_j-\chi}}_{\to 0} + \int_{z_j-\chi}^{z_j+\chi} + \underbrace{\int_{z_j+\chi}^{b}}_{\to 0}\right)\left(\frac{p'(k-i\epsilon)}{p(k-i\epsilon)} - \frac{p'(k+i\epsilon)}{p(k+i\epsilon)}\right)\varphi(k)dk \\
= \ & \lim_{\epsilon\searrow 0}\left(\frac{1}{2\pi i}\int_{z_j-\chi}^{z_j+\chi}\left(\frac{p'(k-i\epsilon)}{p(k-i\epsilon)} - \frac{p'(k+i\epsilon)}{p(k+i\epsilon)}\right)\varphi(z_j)dk\right. \\
& \left. + \frac{1}{2\pi i}\int_{z_j-\chi}^{z_j+\chi}\left(\frac{p'(k-i\epsilon)}{p(k-i\epsilon} - \frac{p'(k+i\epsilon)}{p(k+i\epsilon}\right)(\varphi(k)-\varphi(z_j))dk\right),
\end{aligned}$$

where we used that $\frac{p'(k-i0)}{p(k-i0} = \frac{p'(k+i0)}{p(k+i0}$ for $k \neq z_j$. The first integral can be transformed to an integral along a small contour $\gamma(z_j)$ around z_j and then calculated using residue calculus

$$\frac{1}{2\pi i}\int_{z_j-\chi}^{z_j+\chi}\left(\frac{p'(k-i\epsilon)}{p(k-i\epsilon} - \frac{p'(k+i\epsilon)}{p(k+i\epsilon}\right)\varphi(z_j)dk = \frac{\varphi(z_j)}{2\pi i}\int_{\gamma(z_j)}\frac{p'(k)}{p(k)}dk = \varphi(z_j).$$

To calculate the second integral we note that $\frac{p'(k)}{p(k)}(\varphi(k) - \varphi(z_j))$ is uniformly bounded, since $\frac{p'(k)}{p(k)}$ has a first order pole at z_j and $\varphi(k) - \varphi(z_j)$ – a first order zero, and therefore the integral is zero in the limit. Thus we have

$$\lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \left(\frac{p'(k - i\epsilon)}{p(k - i\epsilon)} - \frac{p'(k + i\epsilon)}{p(k + i\epsilon)} \right) \varphi(k) dk = \varphi(z_j) \equiv \delta_{z_j}[\varphi].$$

Generalisation for the case of several and multiple zeroes is straightforward.

Consider the analytic function

$$p(k) = 1 - e^{2i\pi k}.$$

Its zeroes are situated at the integer points $k = n \in \mathbb{Z}$. We use Taylor's expansion of the logarithm

$$\ln(1 - x) = -\left(x + \frac{x^2}{2} + \cdots + \frac{x^m}{m} + \dots \right) \tag{7.10}$$

to get expansions for $p(k \pm i\epsilon)$

$$\begin{aligned}
p(k + i\epsilon) &= \log\left(1 - e^{2\pi i k} e^{-2\pi\epsilon}\right) = -\sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i k m} e^{-2\pi\epsilon m}, \\
p(k - i\epsilon) &= \log\left(1 - e^{2\pi i k} e^{2\pi\epsilon}\right) = \log\left(e^{2\pi i k} e^{2\pi\epsilon}\right) + \log(-1) + \log\left(1 - e^{-2\pi i k} e^{-2\pi\epsilon}\right) \\
&= 2\pi i k + 2\pi\epsilon + \pi i - \sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i k m} e^{-2\pi\epsilon m}.
\end{aligned}$$

Differentiating with respect to k and taking the jump we arrive at

$$\begin{aligned}
\frac{d}{dk}\Big(\log p(k - i\epsilon) - \log p(k + i\epsilon) \Big) &= 2\pi i + 2\pi i \sum_{m=1}^{\infty} e^{-2\pi i k m} e^{-2\pi\epsilon m} + 2\pi i \sum_{m=1}^{\infty} e^{2\pi i k m} e^{-2\pi\epsilon m} \\
&= 2\pi i \sum_{m=-\infty}^{\infty} e^{-2\pi i k m} e^{-2\pi\epsilon|m|}.
\end{aligned}$$

It remains to take the limit as $\epsilon \searrow 0$. This limit should be again taken in the distributional sense (applied to an arbitrary test function $\varphi \in \mathscr{S}$) since the series $\sum_{m=-\infty}^{\infty} e^{-2\pi i k m}$ is diverging in the conventional sense (all terms have absolute value 1 and do not go to 0). On the other hand, this series is converging in the distributional sense since the estimate

$$|e^{-2\pi i k m}[\varphi]| = |\int_{\mathbb{R}} e^{-2\pi i k m} \varphi(k) dk| = |\hat{\varphi}(2\pi m)| \leq \frac{C}{1 + m^2}$$

holds for any function $\varphi \in \mathscr{S}$. The same estimate holds for $\left(e^{-2\pi i k m} e^{-2\pi\epsilon m}\right)[\varphi]$ implying that

$$\sum_{m=-\infty}^{\infty} e^{-2\pi i k m} e^{-2\pi\epsilon m}[\varphi] \to_{\epsilon \to 0} \sum_{m=-\infty}^{\infty} e^{-2\pi i k m}[\varphi].$$

Removing the test function φ we get **Poisson's summation formula**

$$\sum_{n \in \mathbb{Z}} \delta_n = \sum_{m \in \mathbb{Z}} e^{-2\pi i m k}. \tag{7.11}$$

This formula should be understood in the distributional sense applied to any test function φ

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{m\in\mathbb{Z}} \hat{\varphi}(2\pi m). \tag{7.12}$$

Poisson's summation formula shows that the values of the function and its Fourier transform on the integer lattice and its dual are connected. This formula will play a very important role in our studies later on since it provides a trivial example of so-called crystalline measures.