

Rearrangements in analysis

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Row and column sums of a matrix

$\mathbf{A} \in \{0,1\}^{n \times n}$... an $(n \times n)$ matrix with entries 0 or 1

$h_{\mathbf{A},i}$ = sum of entries in the i^{th} row

$v_{\mathbf{A},i}$ = sum of entries in the i^{th} column

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} 3 \\ 1 \\ 1 \end{matrix}$$

$\begin{matrix} 2 & 1 & 2 \end{matrix}$

Reconstructing the matrix from its row and column sums

Question:

Let $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{N}_0^n$ and $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{N}_0^n$ satisfy

$$\sum_{i=1}^n p_i = \sum_{i=1}^n q_i.$$

Is there $\mathbf{A} \in \{0, 1\}^{n \times n}$ such that

$$h_{\mathbf{A},i} = p_i \quad \& \quad v_{\mathbf{A},i} = q_i \quad \text{for all } i \in \{1, \dots, n\}?$$

- There might be no solution:

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{matrix} 2 \\ 0 \end{matrix}$$

$2 \quad 0$

Observations

- There might be no solution:

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{matrix} 2 \\ 0 \end{matrix}$$

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- The solution may not be unique:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} 1 \\ 1 \end{matrix}$$

$1 \quad 1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{matrix} 1 \\ 1 \end{matrix}$$

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- Vector $\mathbf{q}$ may be reordered so that $q_1 \geq q_2 \geq \cdots \geq q_m$.

Gale–Ryser theorem (1957)

Let $\mathbf{p}, \mathbf{q} \in \mathbb{N}_0^n$ satisfy $q_1 \geq q_2 \geq \cdots \geq q_n$ and

$$\sum_{i=1}^n p_i = \sum_{i=1}^n q_i.$$

Then there exists $\mathbf{A} \in \{0, 1\}^{n \times n}$ such that

$$h_{\mathbf{A},i} = p_i \quad \& \quad v_{\mathbf{A},i} = q_i \quad \text{for all } i \in \{1, \dots, n\}$$

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if and only if

$$\sum_{i=1}^k q_i \leq \sum_{i=1}^k \hat{p}_i \quad \text{for every } k \in \{1, \dots, n\},$$

where $\hat{p}_i = \text{card} \{j \in \{1, \dots, n\}; p_j \geq i\}$ for $i \in \{1, \dots, n\}$.

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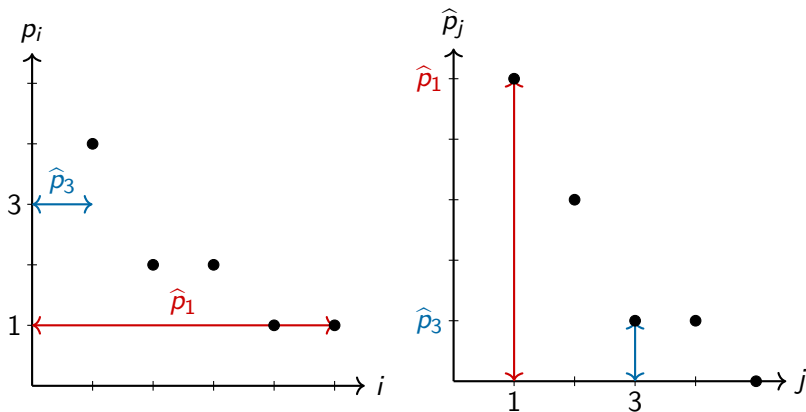
$$\sum_{i=1}^k q_i \leq \sum_{i=1}^k \hat{p}_i \quad \text{for every } k \in \{1, \dots, n\},$$

where $\hat{p}_i = \text{card} \{j \in \{1, \dots, n\}; p_j \geq i\}$ for $i \in \{1, \dots, n\}$.

Note: the entries in $\mathbf{q}$ must be **nonincreasing** for this to be true.

Distribution sequence

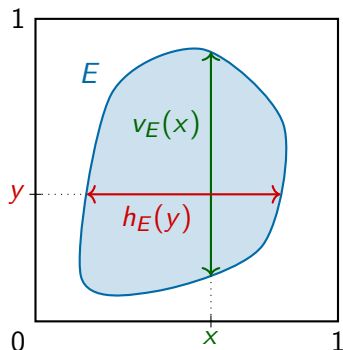
Let $\mathbf{p} = (4, 2, 2, 1, 1)$, $\hat{p}_j = \text{card} \{i \in \{1, \dots, n\}; p_i \geq j\}$.



An analogy: cross-sections of a set

Let $E \subset [0, 1]^2$ be a Lebesgue-measurable set. For $x, y \in [0, 1]$ define

$$v_E(x) = \int_0^1 \chi_E(x, y) dy, \quad h_E(y) = \int_0^1 \chi_E(x, y) dx.$$



Reconstructing a set from its cross-sections

Question:

Let $f, g \in L^1([0, 1])$ be nonnegative functions with $\|f\|_1 = \|g\|_1$.

Does there exist a measurable set $E \subset [0, 1]^2$ such that

$$v_E(x) = f(x),$$

$$h_E(y) = g(y)$$

for almost every $x \in [0, 1]$ and $y \in [0, 1]$?

Rearranging a function

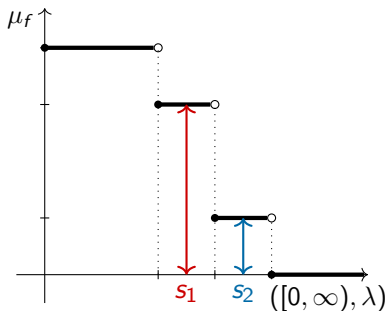
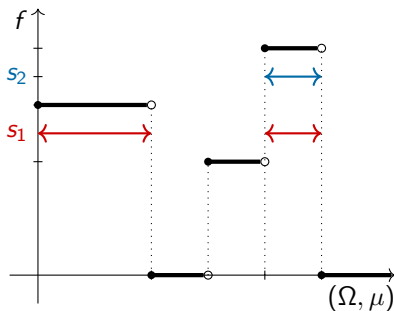
In the original setting, we first reordered a vector.

How to “reorder” a function?

Distribution function

Let (Ω, μ) be a σ -finite measure space and f be a μ -measurable function. Its **distribution function** is defined by

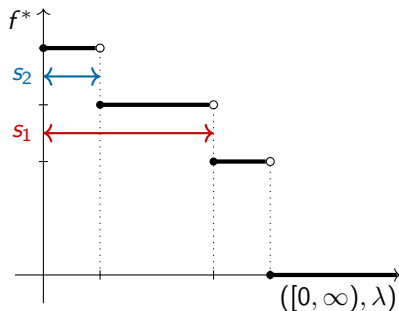
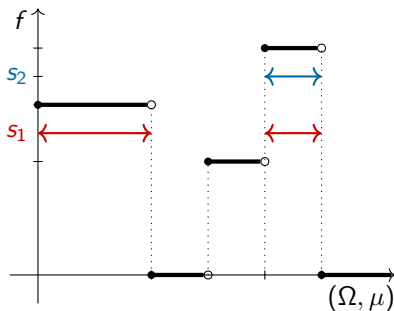
$$\mu_f(s) = \mu(\{x \in \Omega; |f(x)| > s\}), \quad s > 0.$$



Nonincreasing rearrangement

The **nonincreasing rearrangement** of f is given by

$$f^*(t) = \inf\{s > 0; \mu_f(s) \leq t\}, \quad t > 0.$$



Solution to the G–R problem with functions

Let $f, g \in L^1([0, 1])$ be nonnegative functions with $\|f\|_1 = \|g\|_1$.

The following assertions are equivalent:

- (i) There exists a measurable set $E \subset [0, 1]^2$ which satisfies $\nu_E(x) = f(x)$ and $h_E(y) = g(y)$ for almost every $x \in [0, 1]$ and $y \in [0, 1]$.
- (ii) For every $t > 0$ it holds that

$$\int_0^t f^*(s) \, ds \leq \int_0^t \lambda_g(s) \, ds.$$

- (iii) For every $t > 0$ it holds that

$$\int_0^t g^*(s) \, ds \leq \int_0^t \lambda_f(s) \, ds.$$

(Lorentz 1949; Boza, Křepela, Soria 2022)

Hardy–Littlewood inequality

Let f, g be μ -measurable functions defined on Ω . Then

$$\int_{\Omega} |fg| \, d\mu \leq \int_0^{\infty} f^*(t)g^*(t) \, dt.$$

(f_* nondecreasing rearrangement)

Hardy–Littlewood inequality

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$$\int_0^\infty f_*(t)g^*(t) \, dt \leq \int_\Omega |fg| \, d\mu \leq \int_0^\infty f^*(t)g^*(t) \, dt.$$

(f_* nondecreasing rearrangement)

Let $p \in [1, \infty)$, $f \in W^{1,p}(\mathbb{R})$. Then

$$\int_0^\infty |(f^*)'(t)|^p dt \leq \int_0^\infty |f'(t)|^p dt.$$

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If $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |\nabla(f^\circ)(x)|^p dx \leq \int_{\mathbb{R}^n} |\nabla f(x)|^p dx.$$

(f° radially decreasing rearrangement)

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ define (with supremum over cubes):

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

Then

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) \, ds \quad \text{for all } t > 0.$$

A Banach function space X with a norm $\|\cdot\|_X$ is called **rearrangement-invariant (r.i.)** if, for all $f, g \in X$, it satisfies:

$$(f^* = g^*) \quad \Rightarrow \quad (\|f\|_X = \|g\|_X).$$

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Examples of r.i. spaces:

Lebesgue, Orlicz, Lorentz

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Examples of r.i. spaces:

Lebesgue, Orlicz, Lorentz

Examples of non-r.i. spaces:

Sobolev, Besov, C^k

Thank you!