

Constraint Maps and Free Boundaries

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Geodesics with obstacles

Set

- ▶ $m \geq 2$: dimension of the target space.
- ▶ $M \subset \mathbb{R}^m$: a smooth open-connected set.
- ▶ M^c : **obstacle** (forbidden region!).

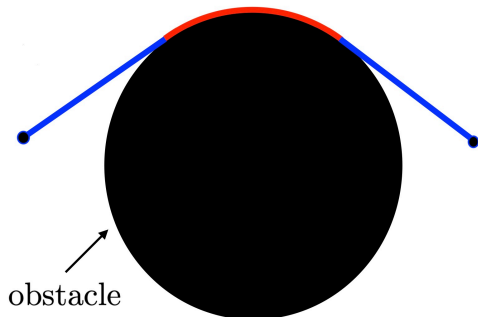
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Problem

Given any two points $y_0, y_1 \in \overline{M}$, find the **geodesic(s)** (i.e., length-minimizing curves) contained in $\overline{M}$ joining y_0 and y_1 .



Geodesics with obstacles

Analytical solutions can be found via **Calculus of Variations**:

- ▶ $I := (0, 1) \subset \mathbb{R}$.
- ▶ $W^{1,2}(I; \mathbb{R}^m) := \{v \in L^2(I; \mathbb{R}^m) : |v'| \in L^2(I)\}$.
- ▶ $W^{1,2}(I; \overline{M}) := \{v \in W^{1,2}(I; \mathbb{R}^m) : v(x) \in \overline{M}, \forall x \in I\}$.

Geodesics with obstacles

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Minimize

$$E[v] := \int_0^1 |v'|^2 dx,$$

among all $v \in W^{1,2}(I; \overline{M})$ with $v(0) = y_0$, $v(1) = y_1$.

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- ▶ By the [Cauchy-Schwarz inequality](#),

$$L[v] := \int_0^1 |v'| dx \leq \sqrt{E[v]},$$

for any $v \in W^{1,2}(I; \mathbb{R}^m)$ with “=” when $|v'| = \text{const.}$ a.e. in I .

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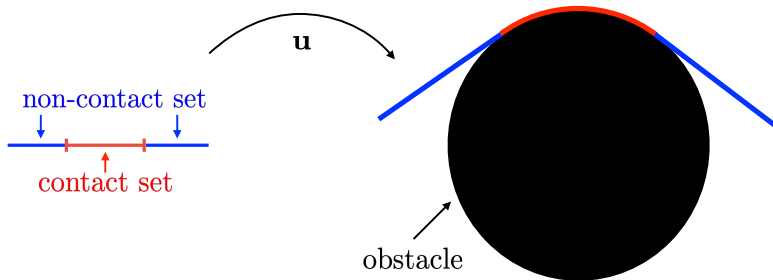
- ▶ Since any rectifiable curve in $\overline{M}$ joining y_0 and y_1 admits a constant-speed parametrization $v \in W^{1,2}(I; \overline{M})$,

$$L[u] = \sqrt{E[u]} \leq \sqrt{E[v]} = L[v],$$

which shows that $u(I)$ is indeed a geodesic.

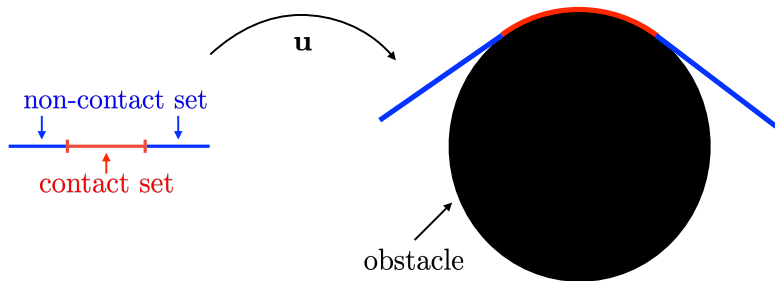
Geodesics with obstacles

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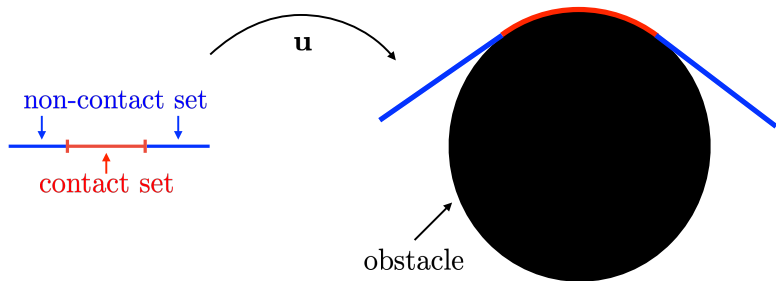
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- ▶ Non-contact set $\mathcal{N}$: admits free variations, hence $u'' = 0$ in $\mathcal{N}$.
- ▶ Contact set $\mathcal{C}$: geodesic in ∂M .
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- ▶ Free boundary: $I \cap \partial \mathcal{N}$.
- ▶ $u \in C^1(I)$.

Minimal surfaces with obstacles

Problem

Given a smooth Jordan curve $\Gamma \subset \overline{M} \subset \mathbb{R}^m$, find the **minimal surface(s)** (i.e., area-minimizing surfaces) contained in $\overline{M}$ spanning Γ .

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- ▶ The modern approach (by Courant in 1950): set $D := \{(x, y) : x^2 + y^2 < 1\}$, and minimize

$$E[v] := \iint_D |\nabla v|^2 dA, \quad \nabla v = (v_x, v_y),$$

among all $v \in W^{1,2}(D; \mathbb{R}^m) \cap C(\overline{D}; \mathbb{R}^m)$ for which $v|_{\partial D}$ maps monotonically onto Γ under the three-point condition.

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- ▶ If $M^c \neq \emptyset$, this is the **Plateau problem with obstacles**, solved by Hildebrandt and Tomi (also independently) in the early 1970s.

Minimal surfaces with obstacles

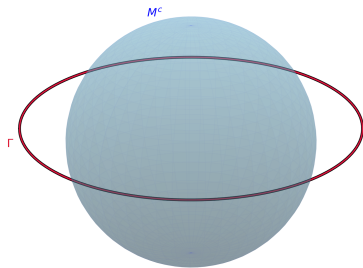
Example

Set

- ▶ $M := \{y \in \mathbb{R}^3 : |y| > 1\}$, (**obstacle** = $\{|y| \leq 1\}$).
- ▶ $\Gamma =$ any circle in M .

Then $\exists!$ *axially symmetric* minimal surface with obstacle M^c :

$$u(r, \theta) = (R(r) \cos \theta, R(r) \sin \theta, Z(r)).$$



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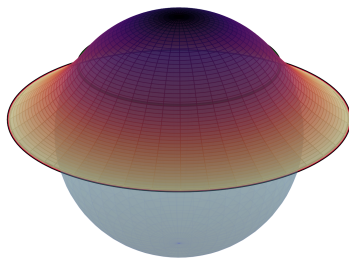
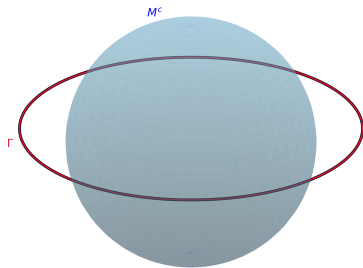
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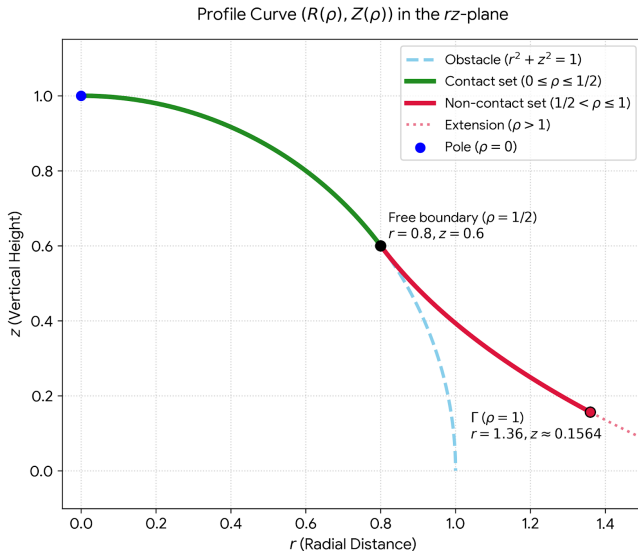
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Minimal surfaces with obstacles



Constraint maps: general framework

In general, we can consider:

- ▶ $\Omega \subset \mathbb{R}^n$, a bounded domain, $n \geq 1$;
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We call $u : \Omega \rightarrow \overline{M}$ a **Constraint Map**, if it minimizes

$$E[v] := \int_{\Omega} |\nabla v|^2 dx,$$

among all maps $v : \Omega \rightarrow \overline{M}$ with finite energy ($E[v] < \infty$) such that $v = g$ on $\partial\Omega$. Here $\nabla v = (v_{x_1}, \dots, v_{x_n})$.

Discontinuities

Constraint maps can be **discontinuous** for $n \geq 3$, **due to topological reasons**:

Example

*There are no continuous maps from $u : \overline{B}_1 \rightarrow \overline{B_1 \setminus B_{1/2}}$, where $u|_{\partial B_1} = \text{id}$, by **degree theory**. However,*

$$E\left[\frac{x}{|x|}\right] < \infty, \quad \text{over } B_1 \text{ if } n \geq 3!$$

Discontinuities

How large can the set of discontinuities be?

Discontinuities

Partial regularity (Duzaar-Fuchs, 86)

Let $u : \Omega \rightarrow \overline{M}$ be a constraint map. Then $\exists \Sigma \subset \Omega$ such that

- ▶ $u \in C^1(\Omega \setminus \Sigma)$;
- ▶ $\dim \Sigma \leq n - 3$.

A more fundamental issue

Do discontinuous points appear on the free boundary?

Free boundary vs. discontinuity

Theorem (Figalli-Guerra-K.-Shahgholian, '24)

No, if the obstacle is **uniformly convex**, and this is optimal!

Thank you for your attention!