

Some special numbers and their hidden combinatorics

Based on joint works with D. Müller, M. Pallich, J. Jorgenson, and L. Smajlović.

The Swedish Mathematical Colloquium, 2026

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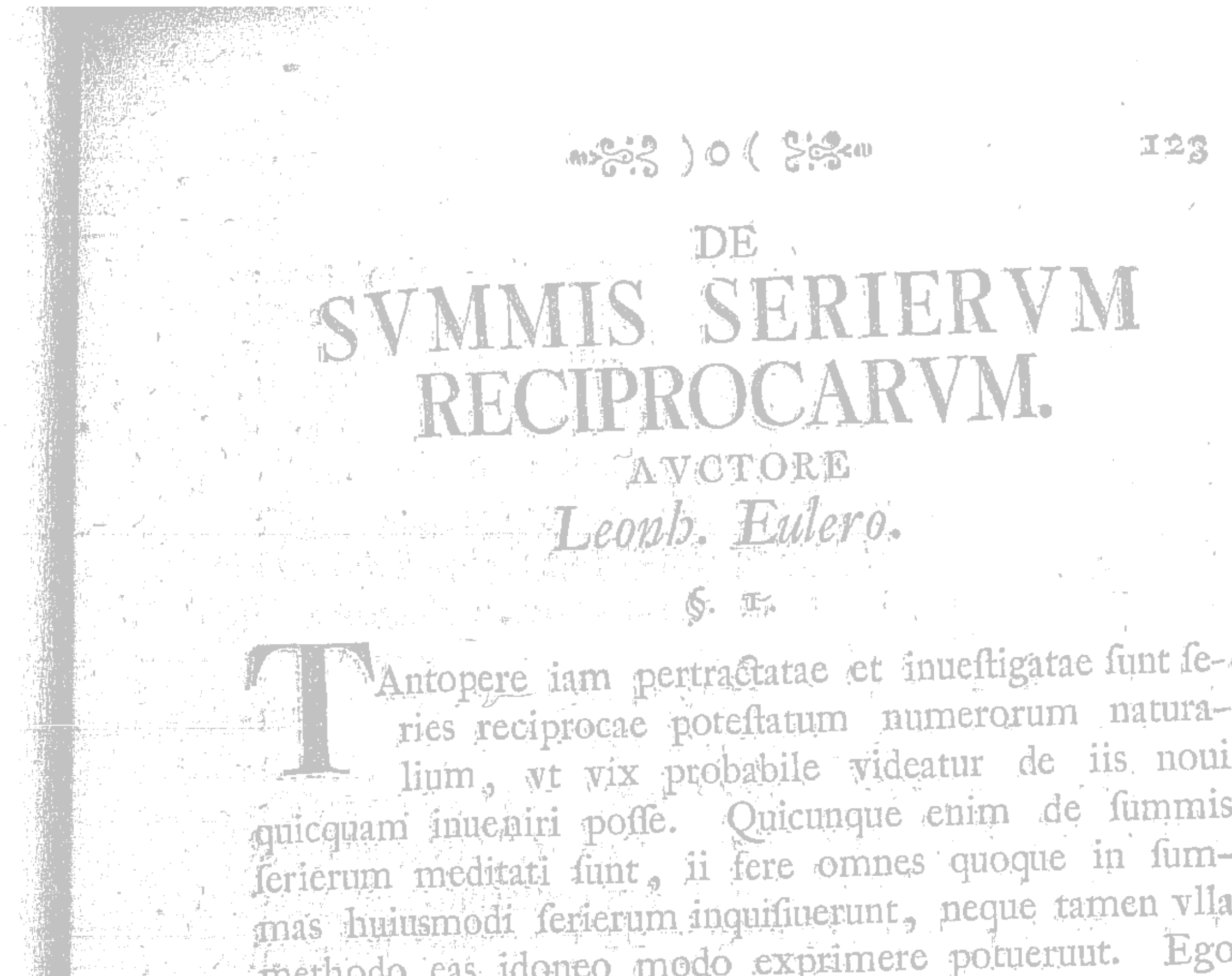
The Basel problem — Euler

The Basel problem

(Mangoli, 1650) What is the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$?

Euler, 1735: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

“So much work has been done on this series, I too, despite repeated efforts could only find approximate values.” “I have recently found, quite unexpectedly, an elegant expression for this sum in terms of quadrature of the circle”



The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1} \quad \operatorname{Re}(s) > 1$$

Special values: $\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \dots$ $\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n!)}$ Euler, 1730s

No formula for $\zeta(\text{odd})$.

We will see:

- How π enters
- Calculate these values from counting spanning forests
- $\zeta(2n + 1)$, no combinatorics

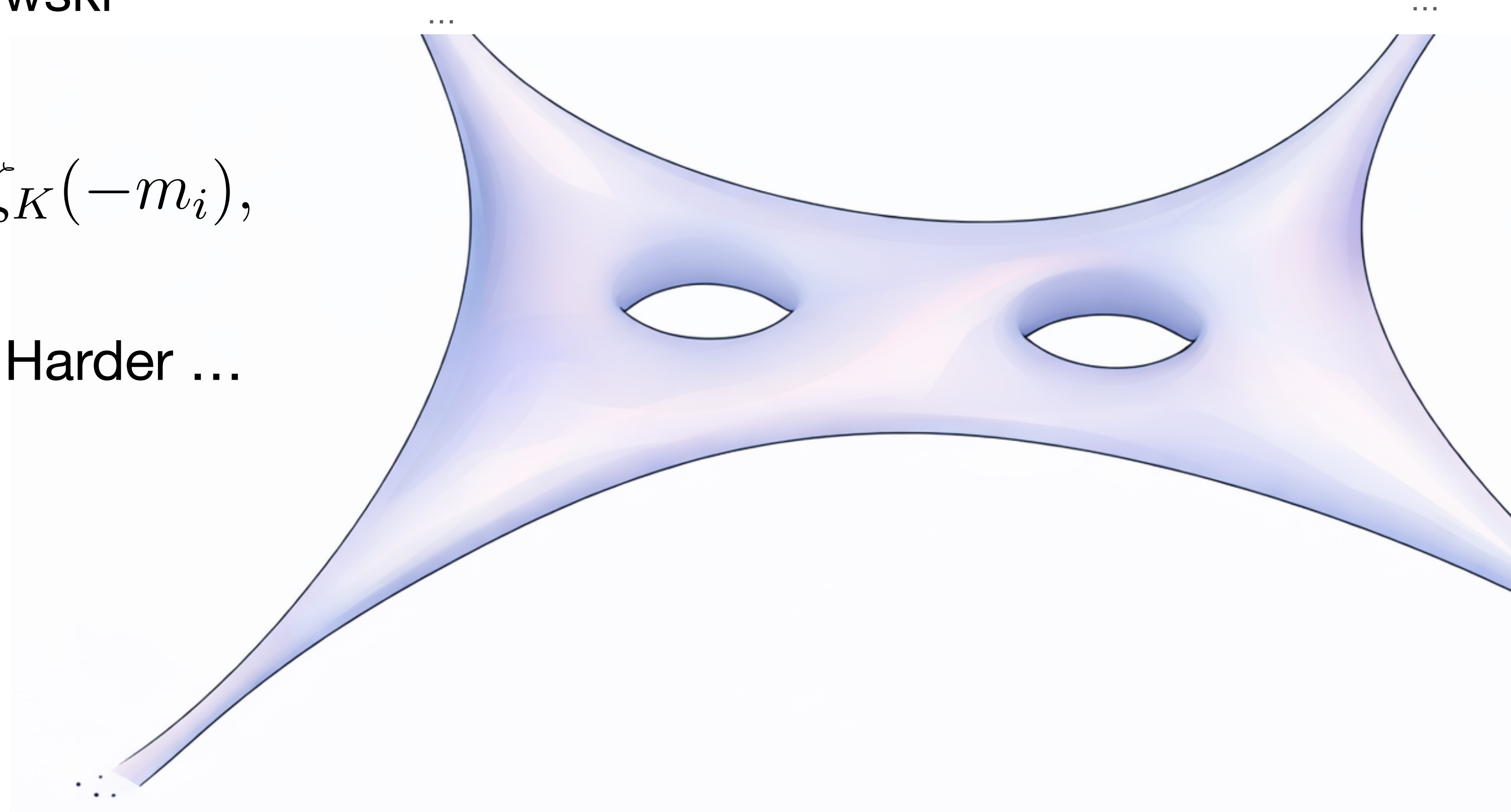
Special values in volume formulas

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \zeta(2)\zeta(3)\dots\zeta(n)$$

Minkowski

$$\mathrm{vol}(G/\Gamma) = c^{[K:\mathbb{Q}]} \prod_{i=1}^l \zeta_K(-m_i),$$

Siegel, Weil, Langlands, Harder ...



Spectral zeta functions

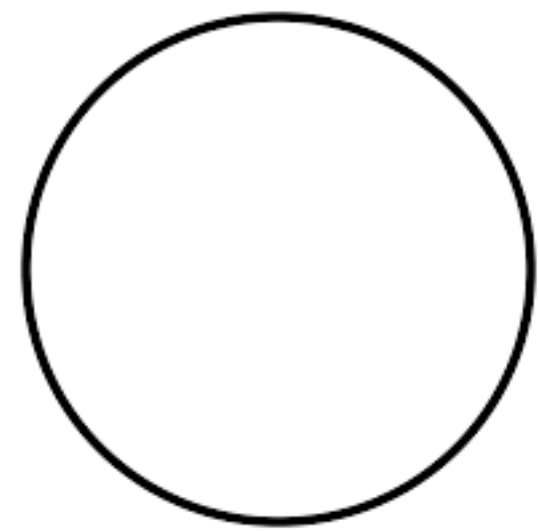
Spectral zeta functions

Carleman 1934; Minakshisundaraman-Pleijel 1949

Given a compact Riemannian manifold X with its Laplacian Δ , let

$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s}$$

Example: $X = \mathbb{R}/\mathbb{Z}$, with $\Delta = -\frac{\partial^2}{\partial x^2}$



$$\zeta_{\mathbb{R}/\mathbb{Z}}(s) = \frac{2}{(2\pi)^{2s}} \zeta(2s).$$

$\sin(2\pi nx)$ and $\cos(2\pi nx)$, $n \geq 0$
are the eigenfunctions of $-\partial^2/\partial x^2$.

$$\lambda = (2\pi n)^2$$

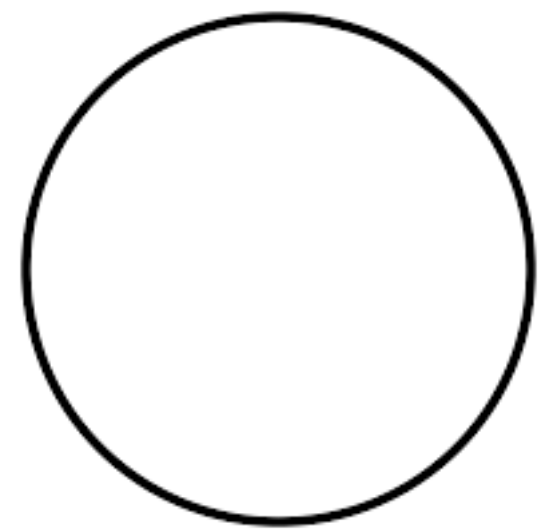
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$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} \qquad \det' \Delta_X := \exp(-\zeta'_X(0)) = "\lambda_1 \lambda_2 \lambda_3 \dots"$$

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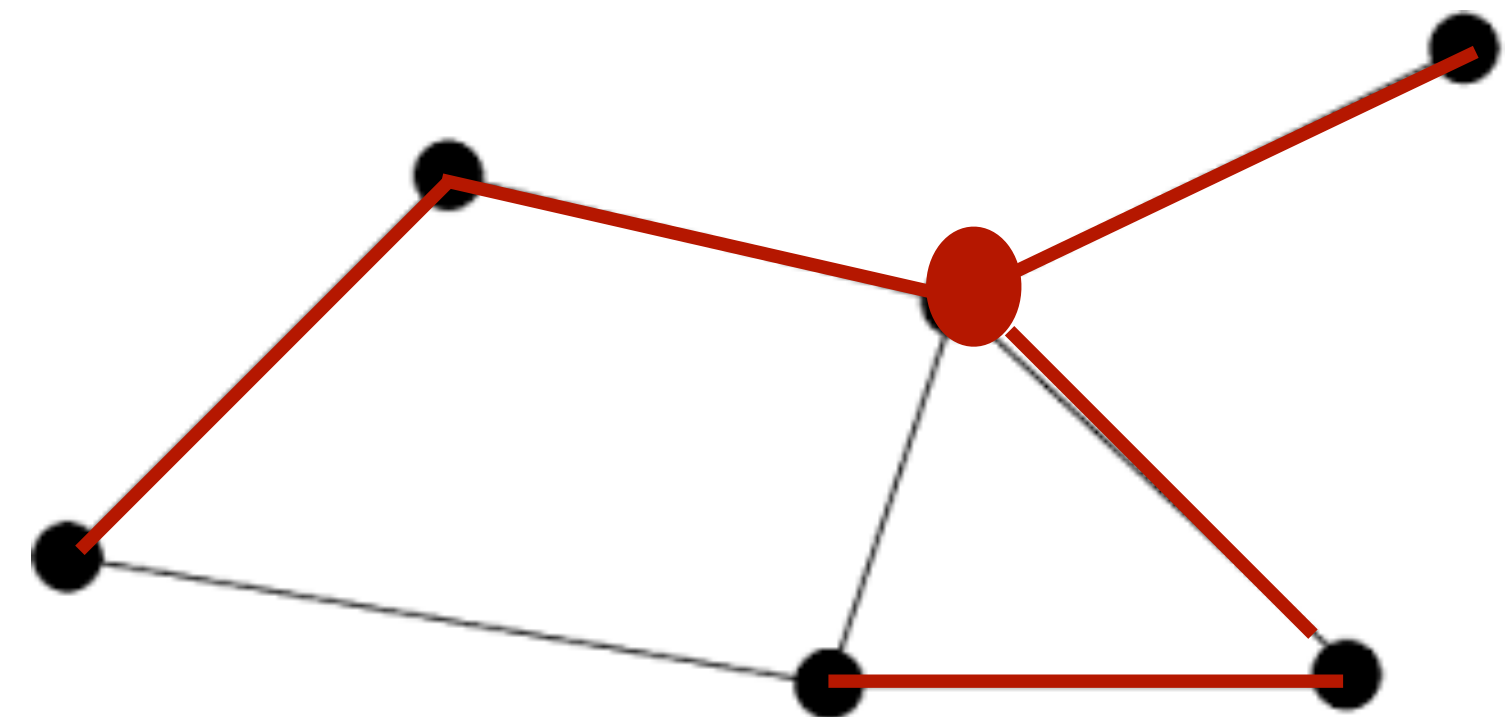
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Graph Laplacian

Let $X = (VX, EX)$ be a countable graph with bounded vertex degree .

Its Laplacian on $L^2(VX)$ is

$$\Delta f(x) = d^* df(x) = \sum_{y:[x,y] \in EX} f(x) - f(y) .$$



The Matrix-Tree theorem

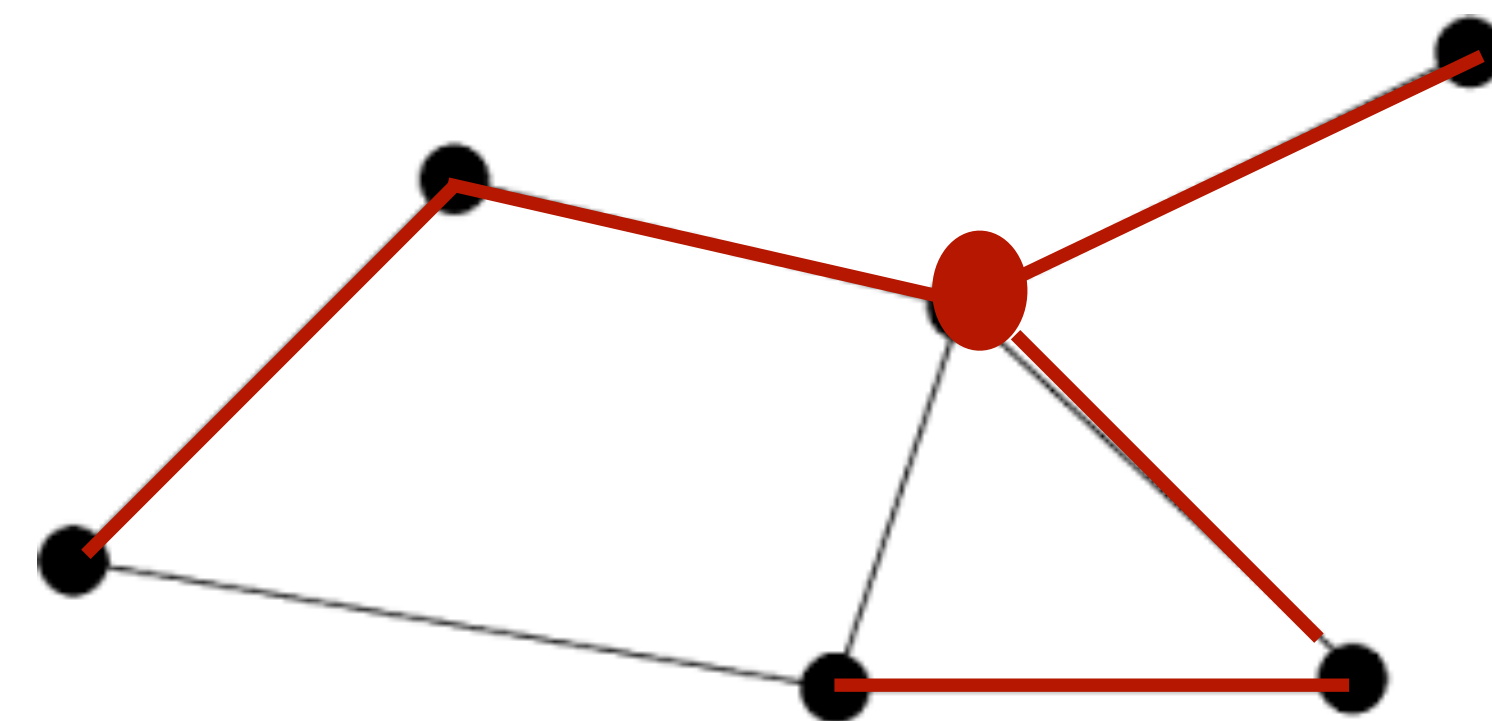
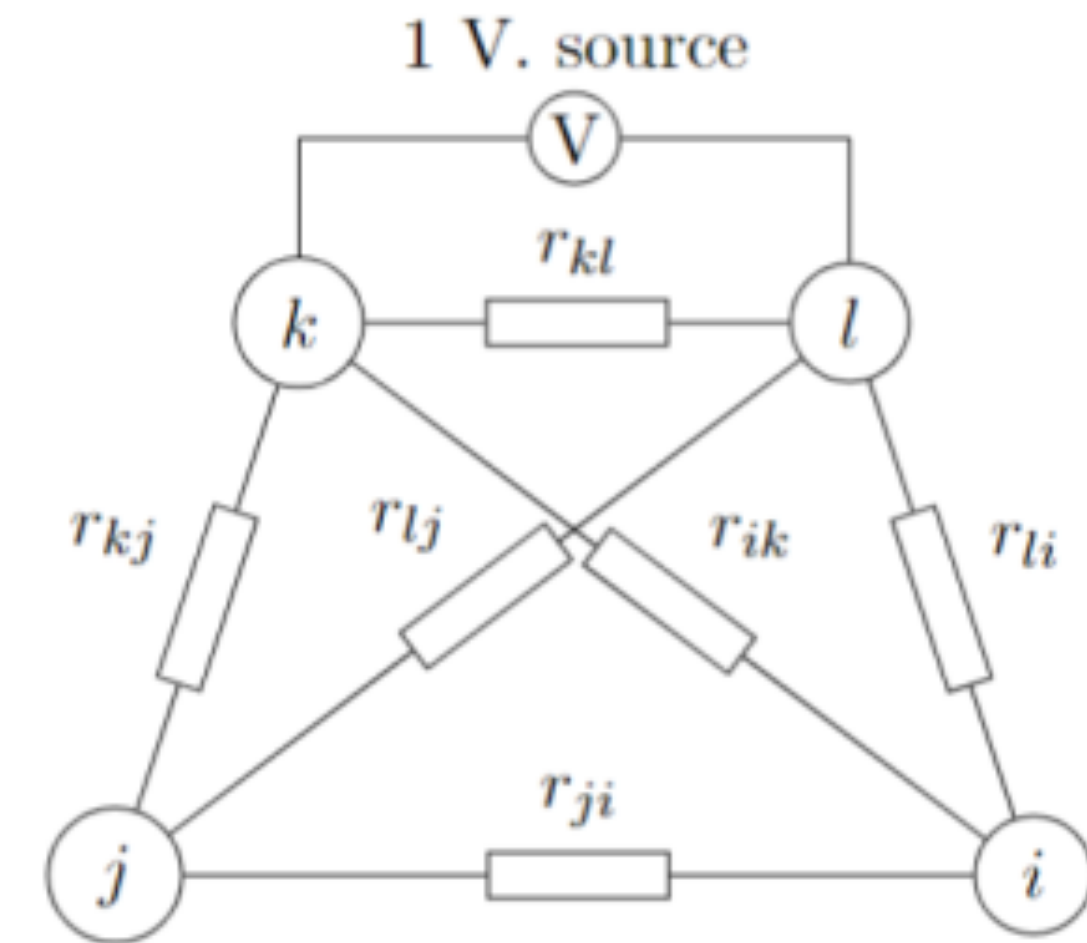
Write $\det(\Delta + x) = p_n x^n + p_{n-1} x^{n-1} + \dots + p_1 x + p_0$,

then p_n equals the number of rooted spanning forests of n components.

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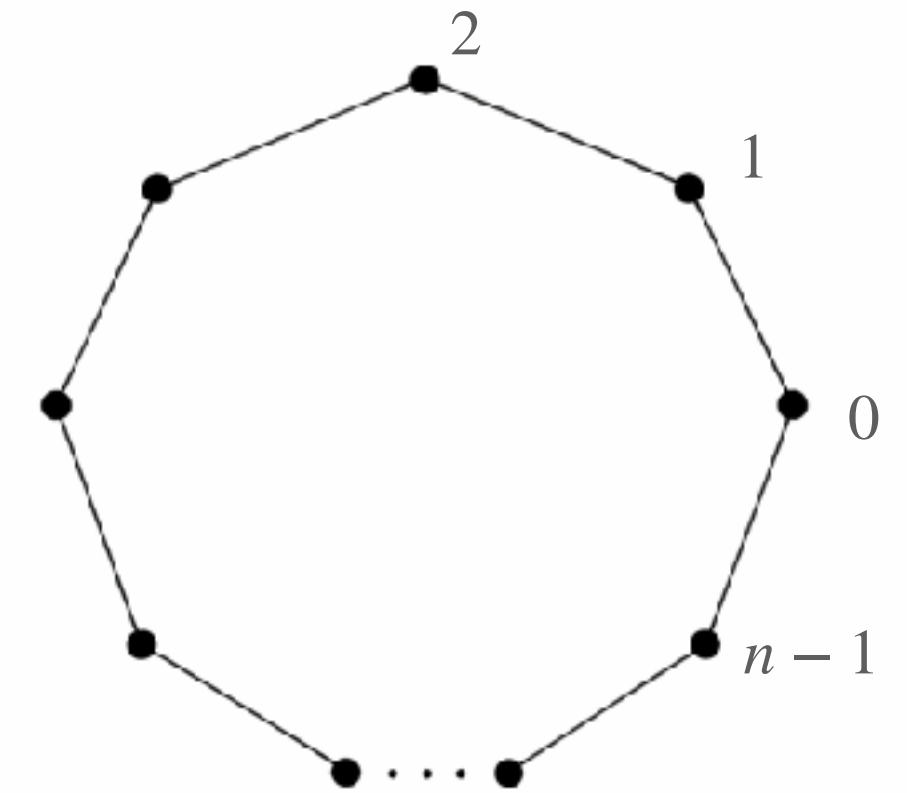
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Spectral zeta function of discrete circles

For $\mathbb{Z}/n\mathbb{Z}$ the Laplacian is: $\Delta f(x) = 2f(x) - f(x-1) - f(x+1)$

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \sum_{k=1}^{n-1} \frac{1}{(4 \sin^2(\pi k/n))^s}$$



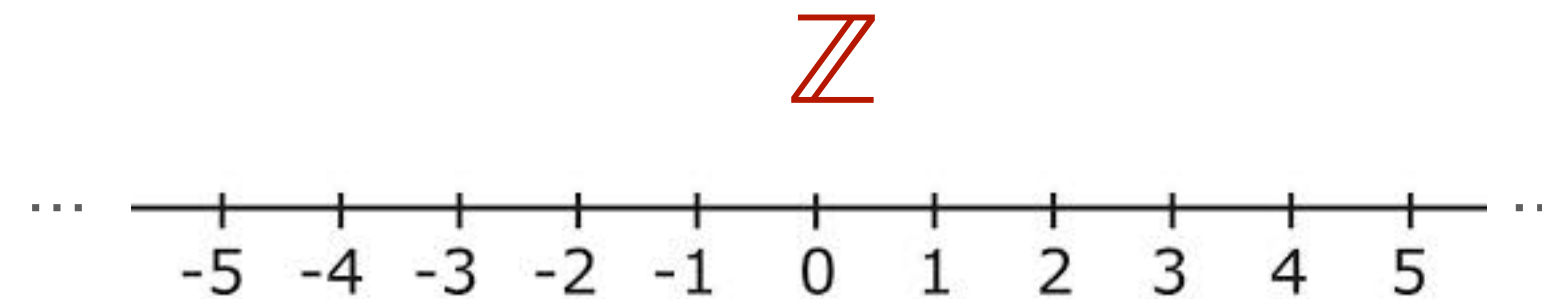
Special values appear in many contexts, for example Verlinde formulas:

$$\dim_{\mathbb{C}} H^0(\mathcal{N}_{g,2,0}, \mathcal{L}^n) = (n+2)^{g-1} 2^{g-1} \zeta_{\mathbb{Z}/(n+2)\mathbb{Z}}(g-1)$$

The case of $\zeta_{\mathbb{Z}}(s)$

Graph Laplacian on $\mathbb{Z}$

$$\Delta f(x) = 2f(x) - f(x-1) - f(x+1)$$



$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t} = \frac{1}{2^{2s} \pi^{1/2}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)} = \binom{-2s}{-s} = \prod_{k=1}^\infty \frac{(k-s)^2}{k(k-2s)}.$$

Friedli-K. 2017

Dubout, 2019

K.-Pallich, 2023

$$\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s) \quad \text{for all } s.$$

Appears in Eisenstein series, Selberg zeta function, Gindikin-Karpelevich,...
and special values $s=-n$ in Chebyshev.

The volume of spheres

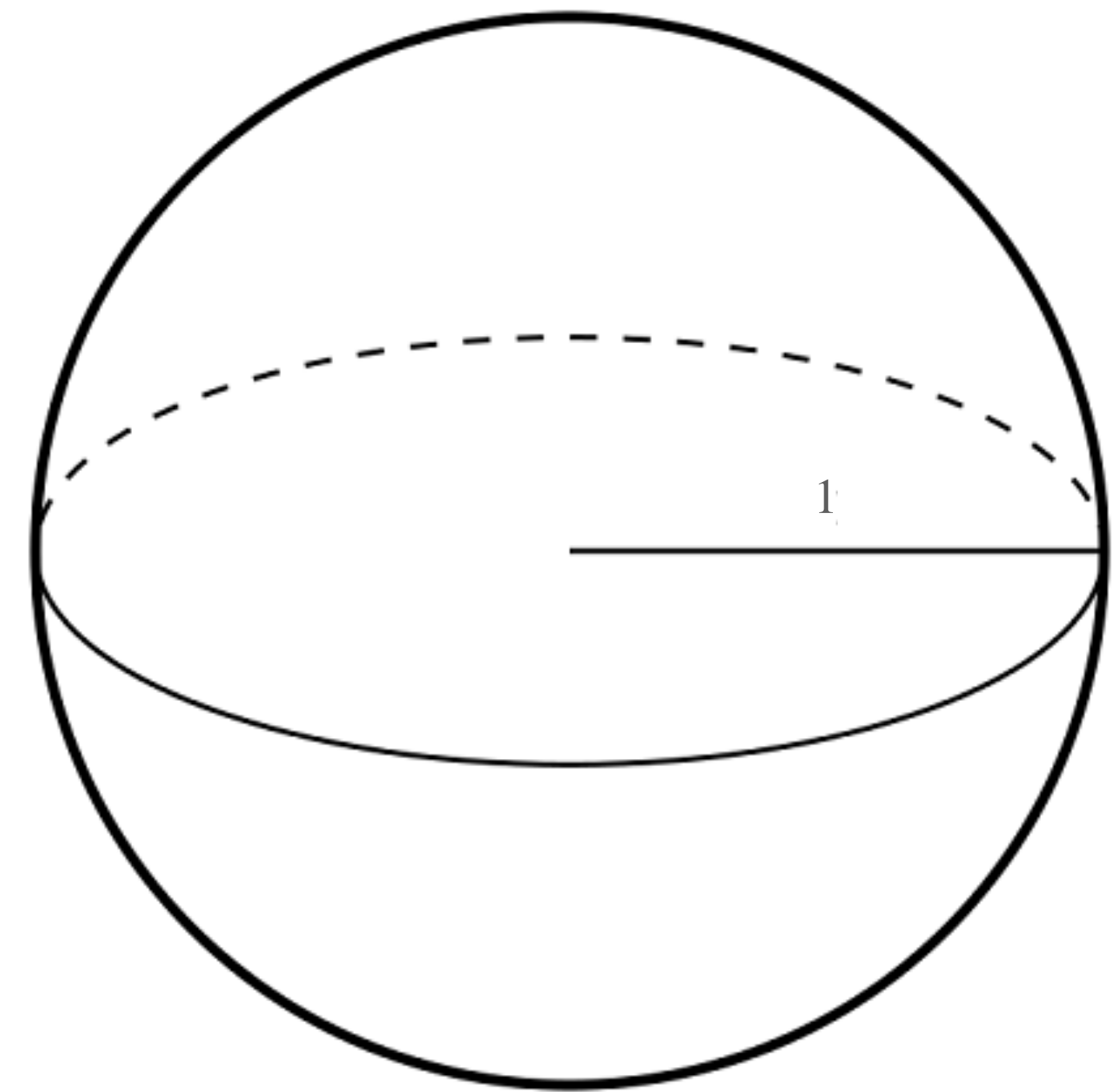
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K.-Pallich, 2023:

$$\text{Vol}(S^n) = 2\tilde{\zeta}_{\mathbb{Z}}(0)\tilde{\zeta}_{\mathbb{Z}}(-1)\dots\tilde{\zeta}_{\mathbb{Z}}(-n+1)$$

$$2, 2\pi, 4\pi, 2\pi^2, \frac{8}{3}\pi^2, \dots$$

$$\tilde{\zeta}_{\mathbb{Z}}(s) := \pi 2^s \zeta_{\mathbb{Z}}(s/2)$$



Asymptotics:
discrete $\rightarrow$ continuous

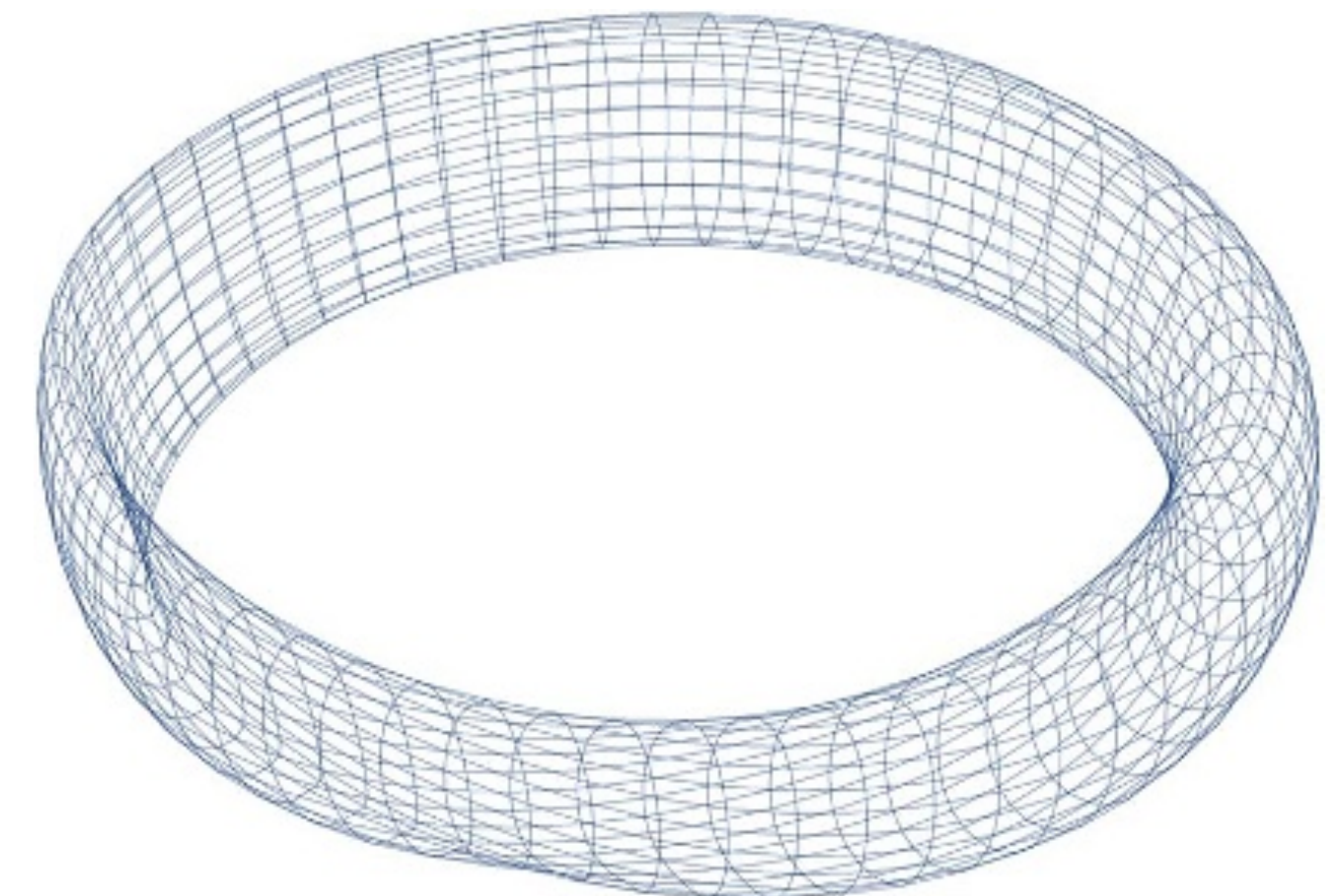
Asymptotics of spectral zeta functions

Chinta-Jorgenson-K. 2010; Friedli-K. 2017

Theorem (Friedli-K. 2017) Let $A = \text{diag}(a_1, \dots, a_d)$ and $A_n := nA$. Then:

$$\zeta_{\mathbb{Z}^d / A_n \mathbb{Z}^d}(s) = \zeta_{\mathbb{Z}^d}(s) \det A n^d + \zeta_{\mathbb{R}^d / A \mathbb{Z}^d}(s) n^{2s} + o(n^{2s}) . \quad n \rightarrow \infty$$

Physics. In the special case of the $\det(\Delta)$. Previously known: $d=2$,
and lead term all d (Kasteleyn, Barber, David-Duplantier, Sokal-Starinets...)



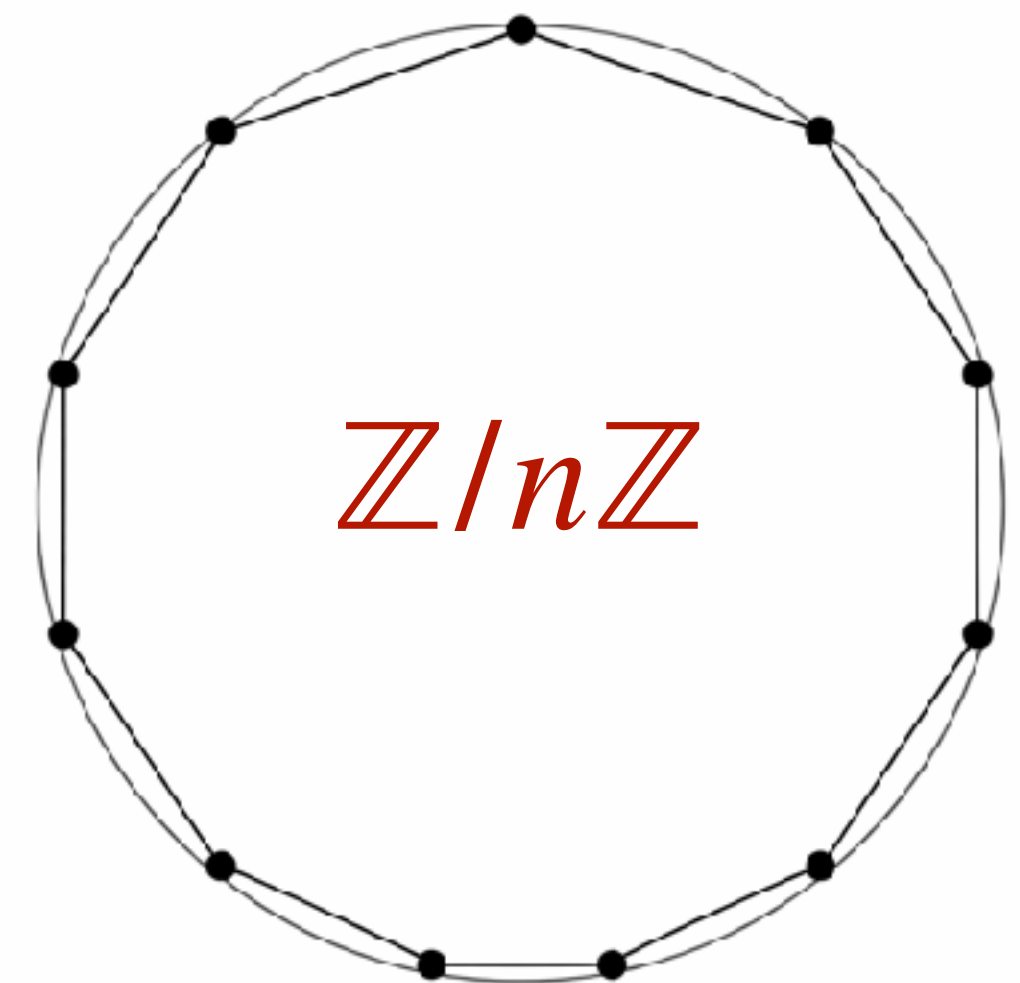
Asymptotics in dimension 1

$$2^{-s} \sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)} = \zeta_{\mathbb{Z}}(s/2)n + 2(2\pi)^{-s}\zeta(s)n^s + \frac{s}{12}(2\pi)^{2-s}\zeta(s-2)n^{s-2} + o(n^{s-2}) \quad n \rightarrow \infty.$$

Also for Dirichlet L-functions (Friedli 2016, K.-Müller 2025)

Theorem (Friedli-K, Friedli , K-Müller)

$$\text{GRH holds iff } \lim_{n \rightarrow \infty} \left| \frac{\xi_n(s, \chi)}{\xi_n(1-s, \bar{\chi})} \right| = 1.$$



Theorem (K-Müller) Let χ be a real, even, non-principal Dirichlet character.

If $L_n(-m, \chi) \geq 0$, then the corresponding Dirichlet L-function has no real zero.

New recursive identities

Identities among special values

K.-Müller, 2025 (Zagier 1996, Xie-Zhao-Zhao 2024)

And analogous
for Dirichlet
L-functions

Corollary 1.3. *For any integers $n, m \geq 1$, the following holds:*

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(m) = \sum_{k=0}^m a_{m-k}(m) \zeta_{\mathbb{R}/\mathbb{Z}}(k) n^{2k}.$$

Explicitly, the case $n = 1$ is the formula (1) above, and in general for $n \geq 1$,

$$\begin{aligned} & \frac{(-1)^{m+1}}{4} \sum_{a=0}^{2m} n^{-a} \binom{2m+1}{a+1} \sum_{j=0}^{a+1} (-1)^j \binom{a+1}{j} \frac{a+1-2j}{a+1} \binom{m+jn+(a-1)/2}{2m+a} \\ &= \sum_{k=0}^m a_{m-k}(m) \frac{\zeta(2k)}{(2\pi)^{2k}} n^{2k}. \end{aligned}$$

The case $n=1$

$$\sum_{k=0}^m a_{m-k}(m) \frac{\zeta(2k)}{(2\pi)^{2k}} = 0. \qquad \left(\frac{z/2}{\sin(z/2)} \right)^{2s} = \sum_{k \geq 0} a_k(s) z^{2k}.$$

Combinatorics to Euler

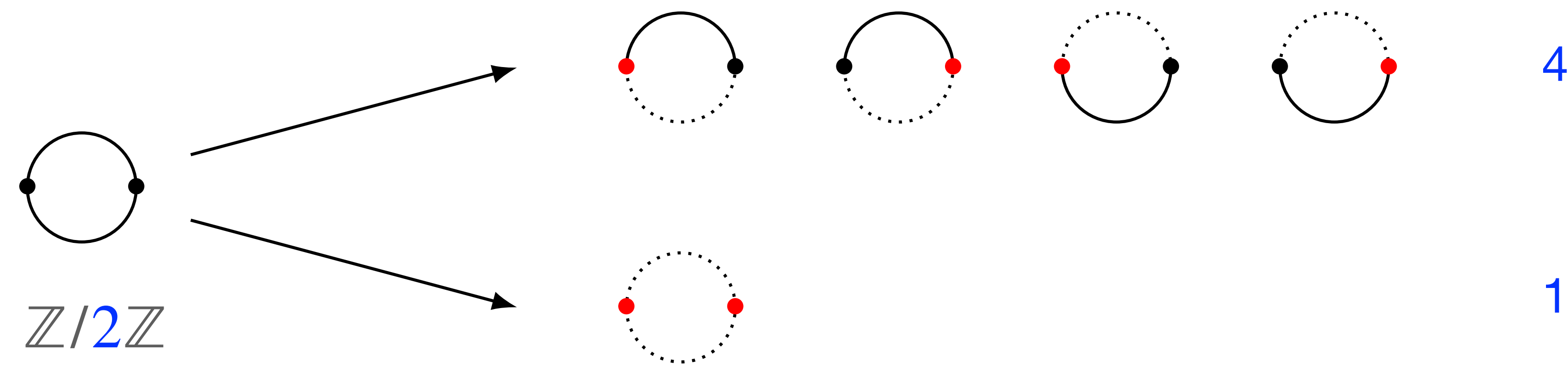
Spanning forests to Euler

Special case $m = 1$:

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(1) = \sum_{k=0}^1 a_{1-k}(1) \zeta_{\mathbb{R}/\mathbb{Z}}(1) n^{2k}$$

$n=2$

Spanning forests to Euler

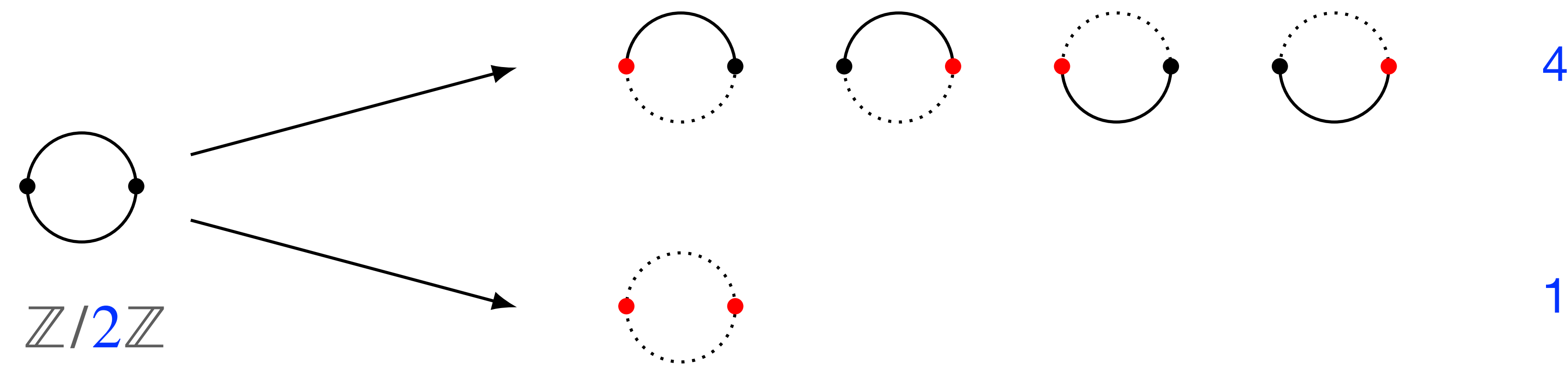


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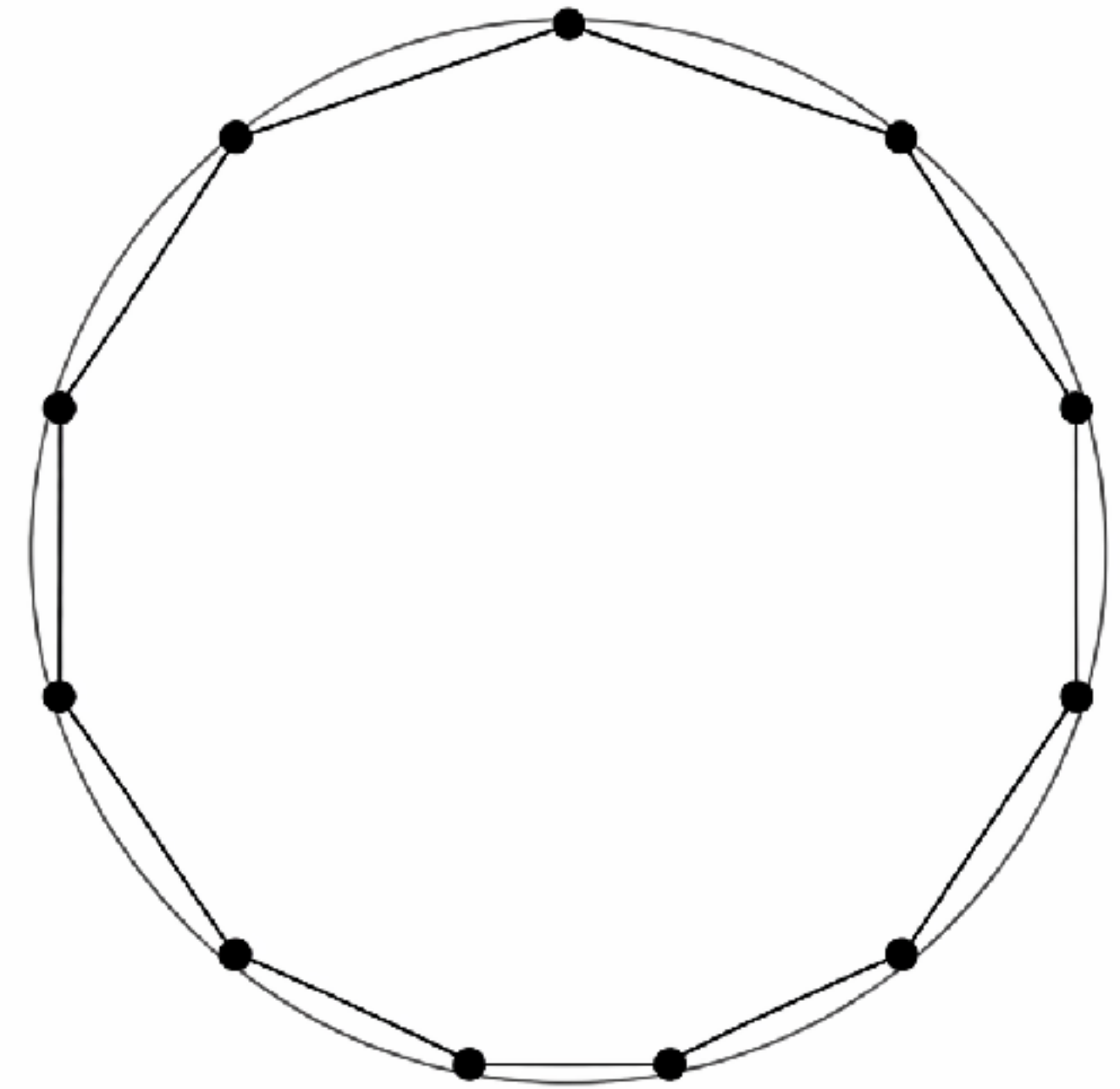
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$n=2$

$$\zeta(2) = \left(\frac{1}{4} - \frac{2}{12} \left(-\frac{1}{2} \right) \right) \frac{(2\pi)^2}{2 \cdot 2^2} = \frac{\pi^2}{6}.$$

Scheme for these special values

1. Pass to the circle and spectral zeta
2. Define analogue for discrete circle
3. Study asymptotics $n \rightarrow \infty$
4. For integral s , $\rightarrow$ symmetric power-sums
5. Relate to elementary symmetric polynomials — Kirchhoff
6. Polynomiality in n
7. Conclude: Exact, finite-terms relations. **Combinatorics $\rightarrow$ Euler**



Summary

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- Spectral zeta functions for graphs; special values in many contexts
- Asymptotics link discrete spectral zetas to classical ones (physics; analytic number theory)
- This gives a (hidden) combinatorial structure to the special values
- Everything also for Dirichlet L-functions
- Infinite families of recursive identities between special values of Dirichlet L-functions

Thank you!

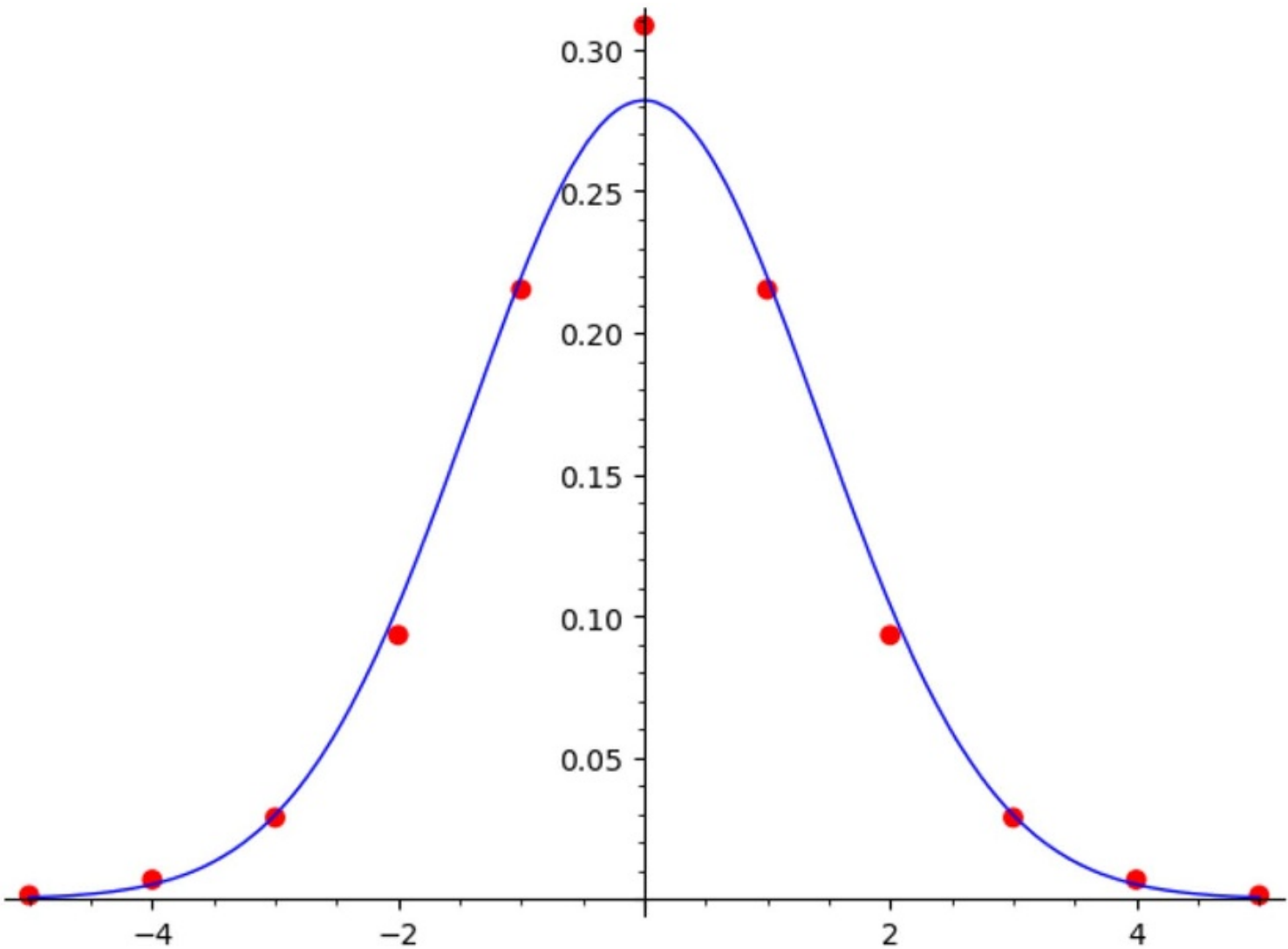
Recall that the Bernoulli numbers B_k are defined via

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{1}{2}x + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}.$$

$$\zeta_X(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} = \int \lambda^{-s} d\mu(\lambda) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}^*(K_X)(t) t^s \frac{dt}{t}$$

where K_X is the heat kernel, i.e. $(\Delta_x + \frac{\partial}{\partial t})K_X(t, x_0, x) = 0$ with $K_X(0, x_0, x) = \delta_{x_0}(x)$.

$$K_{\mathbb{Z}}(t, 0, x) = e^{-2t} I_x(2t)$$



New identities among special values

Theorem (K.-Müller, 2025)

Let χ be an even Dirichlet character mod q . Then for any $n \geq 1, m \geq 0$,

$$L_n(m, \chi) = 2 \sum_{k=0}^m a_{m-k}(m) L(2k, \chi) \left(\frac{qn}{2\pi} \right)^{2k},$$

where $\left(\frac{z/2}{\sin(z/2)} \right)^{2s} = \sum_{k \geq 0} a_k(s) z^{2k}.$

$$L_{\dots}(s, \chi) = \sum_{k \geq 0} \frac{\chi(k)}{\lambda^s}.$$

n=1 due to Xie-Zhao-Zhao, 2024