

# Analysis on Metric Spaces; Why and How?

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# A Metric Space

- ① A set  $X$  with a function  $d(\cdot, \cdot)$  that satisfies
  - $d(x, y) \geq 0$ ;  $d(x, y) = 0$  iff  $x = y$ ;
  - $d(x, y) = d(y, x)$ ;
  - $d(x, y) \leq d(x, z) + d(z, y)$  (the triangle inequality)
- ②  $B(x, r) = \{y : d(y, x) < r\}$  a basis for topology
- ③ Compactness, connectedness, continuity, uniform continuity, uniform limits of sequences and functions, completeness,...
- ④ But no notion of “directions ”

# Examples of Metric Spaces

- 1  $\mathbb{R}^n$ , manifolds
- 2 ANY subset of a metric space (!)
- 3 Boundaries of open sets in  $\mathbb{R}^n$
- 4 Subsets with intrinsic length distance
- 5 Normed vector spaces, finite/infinite dim, e.g.  $\mathbb{R}^n$  with  $\|\cdot\|_1$
- 6 Cantor sets, fractals
- 7 Finite or infinite graphs
- 8 Finitely generated groups (!)
- 9 Boundary at infinity of hyperbolic spaces/groups
- 10 Limits of manifolds
- 11 Alexandrov spaces
- 12 Finsler and SubRiemannian manifolds
- 13 Gluing, quotienting, compactification, products, etc
- 14  $0 < a < 1$ , then  $(X, d^a)$  a metric space (Snowflake)

# Universal Structures on Metric Spaces

- ① Hausdorff measure(s), Hausdorff dimension
- ② Theory of rectifiable curves
- ③ Many Lipschitz functions on them:  $x \mapsto d(x, x_0)$ , and compose with any Lipschitz function on  $\mathbb{R}^1$ .

$f: X \rightarrow Y$  is Lipschitz if  $\exists C$  s.t.

$$d(f(x), f(y)) \leq Cd(x, y)$$

called bi-Lipschitz if a lower bound holds as well

# Analysis on Metric Spaces

Analysis = *We can take derivatives and derivatives contain some information about the functions.*

Is there hope for a unified theory of “analysis on metric spaces” that

- ① coincides with the classical theories on  $\mathbb{R}^n$ , manifolds, graphs,
- ② leads to nontrivial theories on some genuinely non-Euclidean spaces
- ③ interacts with the geometry nicely, e.g. with quasisymmetric maps
- ④ is stable under limits?

# Sources of Motivation

- ➊ Analysis: variational problems, PDE, weighted Euclidean spaces (degenerate elliptic PDE), nonlinear potential theory
- ➋ Geometry: Quasisymmetric uniformization, quasiconformal mappings, bi-Lipschitz embeddings
- ➌ Dynamics and geometric group theory
- ➍ Philosophically. Deepen our understanding of the old, e.g., (I) differentiability of Lipschitz functions, (II) “Sobolev meets Poincaré”
- ➎ Discover better, universal approaches to the old

# Some Success Stories of Analysis on Metric Spaces

- ① Heinonen-Koskela '97: on spaces with “controlled geometry” QC maps are QS
- ② Cheeger '99: On PI spaces, Lipschitz functions are differentiable a.e.
- ③ Semmes '96: “Good metric spaces without good parameterizations”
- ④ Laakso '98: For every real number  $\alpha \geq 1$ , there exist PI spaces dimension  $\alpha$ .

Items (1) and (2) apply to the examples of Semmes and Laakso.

- (5) Bonk-Kleiner: Certain Gromov boundaries are quasisymmetric to  $\mathbb{S}^2$

# Doubling Measures and Zero-Order Analysis

- There is a need to bring in measures.

Zero-order analysis requires only a mild condition:  $\mu$  is doubling, i.e.

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Consequences:

- Vitali/5r- covering theorem
- Hardy-Littlewood maximal function
- Lebesgue differentiation theorem

# The Hajłasz Space $M^{1,p}$

**Theorem** (Hajłasz '94):  $X$  a bounded open set in  $\mathbb{R}^n$  with “nice” boundary,  $p > 1$ . Then  $u \in W^{1,p}(X)$  if and only if there exists a non-negative  $g \in L^p(X)$  and a null set  $E$  such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$$

whenever  $x$  and  $y$  lie outside  $E$ .

- No derivatives involved!
- The second half is *the definition* of the Sobolev space  $M^{1,p}(X, d, \mu)$  on an arbitrary metric-measure space.

# Some Desired Properties for a Sobolev Class

- ① reduces to classical theory
- ② Lipschitz functions are included
- ③ is a Banach space
- ④ Poincaré inequality
- ⑤ Sobolev embeddings
- ⑥ Locality of derivatives and some calculus (e.g. Leibniz rule)
- ⑦ “Fine” properties of Sobolev functions

- The theory of rectifiable curves works on any metric space.
- But remember that there some spaces with no nontrivial curves.

**Definition** (Heinonen-Koskela '97): Borel  $g: X \rightarrow [0, \infty]$  **an u.g.** of  $f$  if

$$|f(c(0)) - f(c(\ell))| \leq \int_c g \, ds$$

for all rectifiable curves  $c: [0, \ell] \rightarrow X$

- U.g. highly non-stable under limits and choice of a.e. representatives

# The Newtonian-Sobolev Space $N^{1,p}$

- The notion of “a negligible collection” of curves on a mms
- Shanmugalingam '98: relax the requirement in u.g. definition to require only for  $p$ -a.e. curve
- $N^{1,p}$ : a nice theory under doubling and Poincaré inequality

**Poincaré inequality:** for “every”  $f$ , every u.g. of  $f$ , every ball  $B$

$$\int_B |f - f_B| d\mu \leq C(\text{diam } B) \left( \int_{2B} g^p d\mu \right)^{1/p}$$

- PI more about the geometry of  $X$  than about functions! (Semmes)

# Cheeger's Space, the Differentiability Theorem

$H^{1,p}$  = all  $f$  such that there exist  $f_i$  with u.g.  $g_i$  such that

$$f_i \rightarrow f \text{ in } L^p \quad \text{and} \quad \sup_i \|g_i\|_{L^p} < \infty$$

- Equivalent to  $N^{1,p}$  when  $p > 1$ , the space PI – Shanmugalingam
- Indeed,  $H^{1,p} = N^{1,p} = M^{1,p}$ , when  $p > 1$  and the space is PI
- **Theorem** (Cheeger '99): Lipschitz functions on spaces supporting PI are differentiable a.e.

# No Curves? No Problem!

- Well, we have Hajłasz spaces.
- Or... write  $X = \partial Y$ , with a better  $Y$ , a *hyperbolic filling*
- Functions on  $X = \text{traces}$  of functions on  $Y$

# Current Fashion in Analysis on Metric Spaces

- 1 PDE, non-local and fractional variational problems
- 2 Quasisymmetric mappings, conformal dimension
- 3 RCD and optimal transport
- 4 Connections to the probability theory
- 5 Quantitative geometric measure theory (GMT)

Thank you for listening!