

Totally nonnegative matrices and real-rooted polynomials in combinatorics

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based on joint work with
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Motivation: Negative dependence

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- ▶ If the X_i 's model repelling particles, then one would expect different sites i, j to be **negatively correlated**:

$$\mathbb{P}[X_i = 1 \mid X_j = 1] \leq \mathbb{P}[X_i = 1]$$

Quest for a theory of negative dependence

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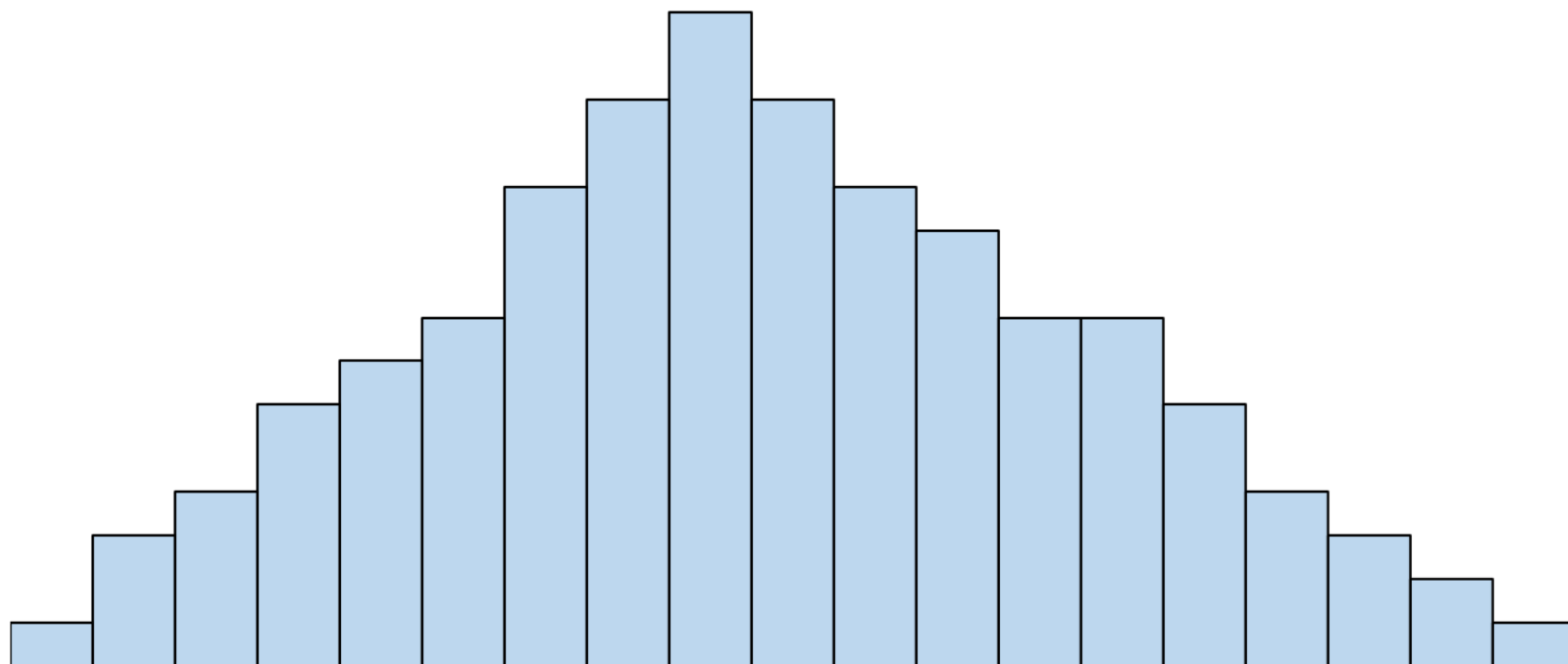
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- ▶ The theory of **Lorentzian polynomials** (B., Huh, 2020) merges the two.
- 3. Using **totally nonnegative matrices** (today).

Univariate negative dependence inequalities

- ▶ Suppose μ is a probability measure on subsets of V , and let X_i be as before. Define the **rank sequence** of μ :
- ▶ $r_k = r_k(\mu) = \mathbb{P}[\sum X_i = k] = \mathbb{P}[\text{exactly } k \text{ sites are occupied}]$.

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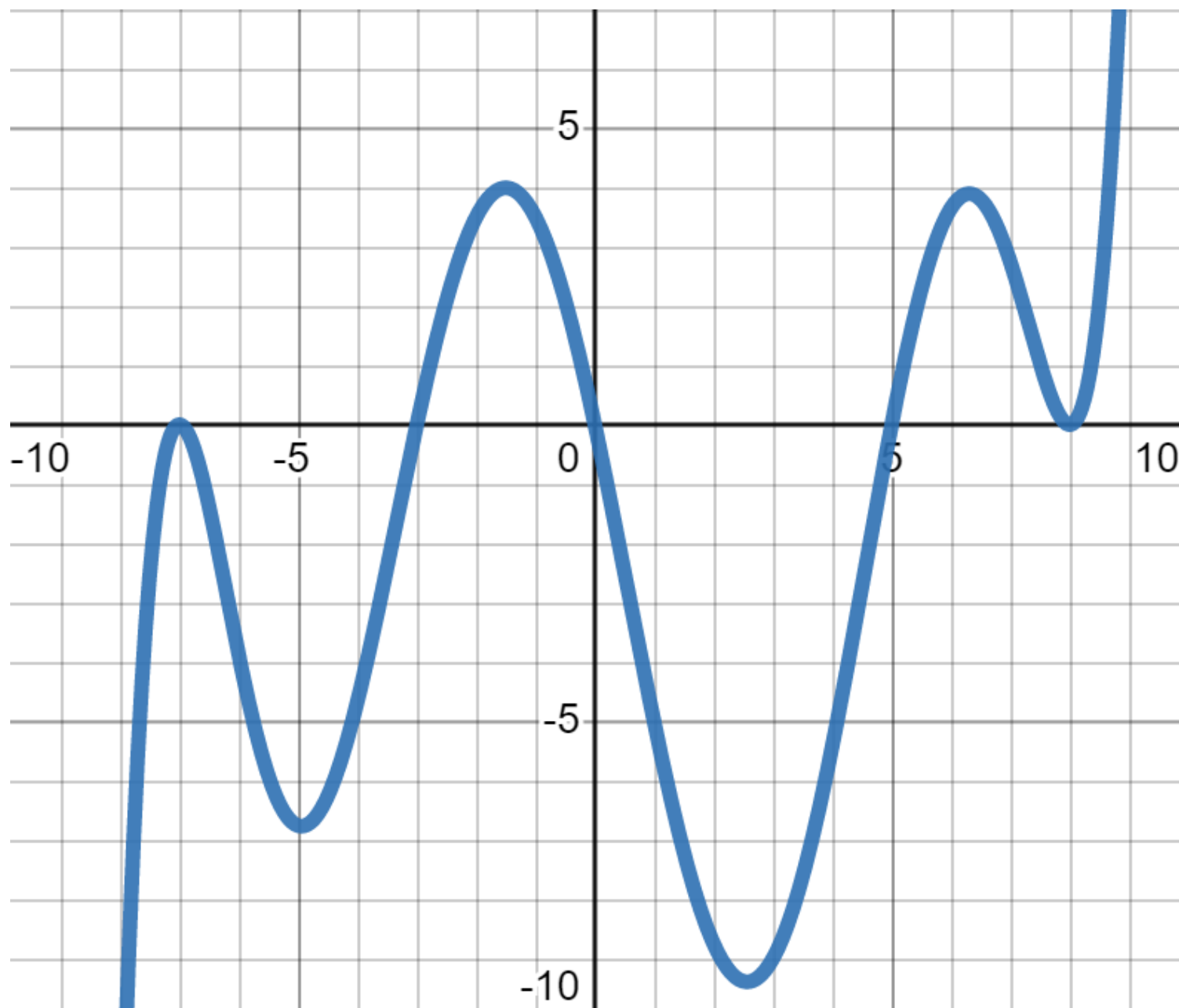
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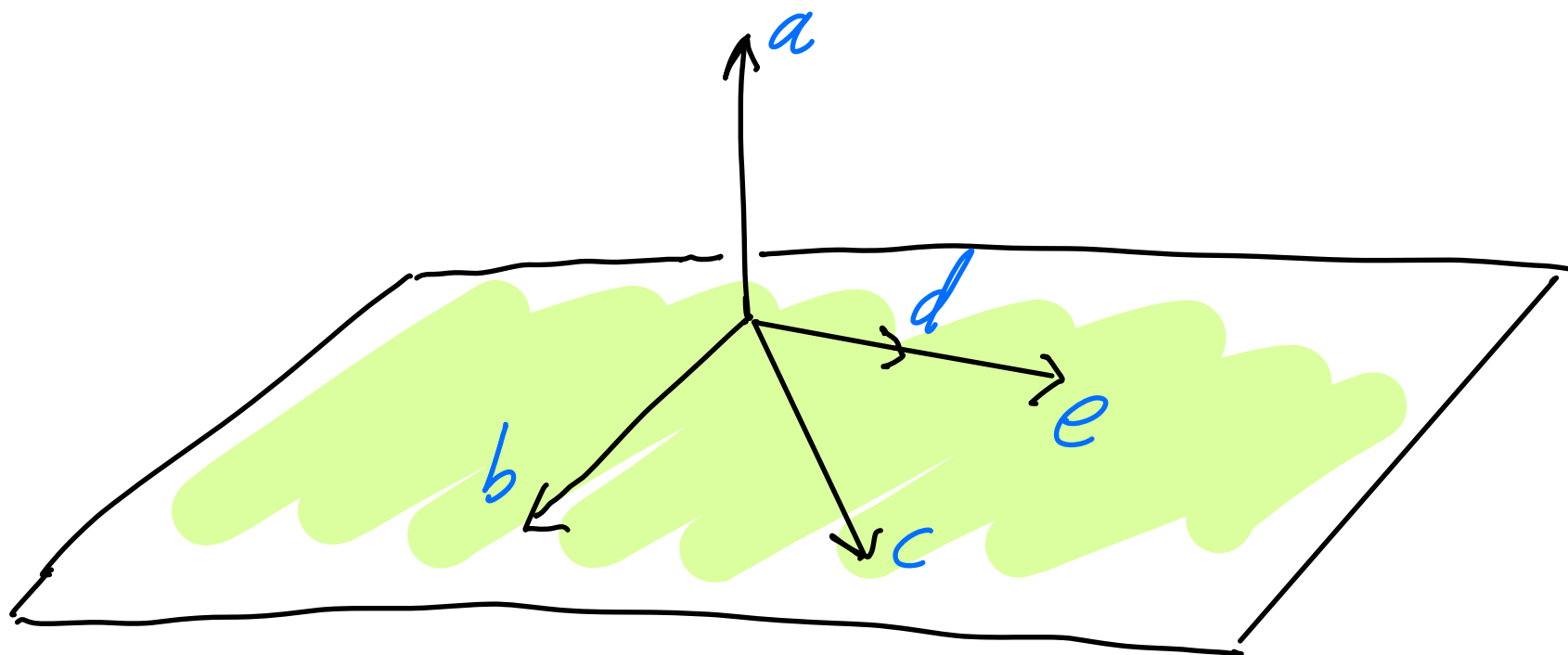
- ▶ The polynomial $r_0 + r_1 t + r_2 t^2 + \cdots + r_n t^n$ is **real-rooted**, i.e., all of its zeros are real.

Real zeros are repelling



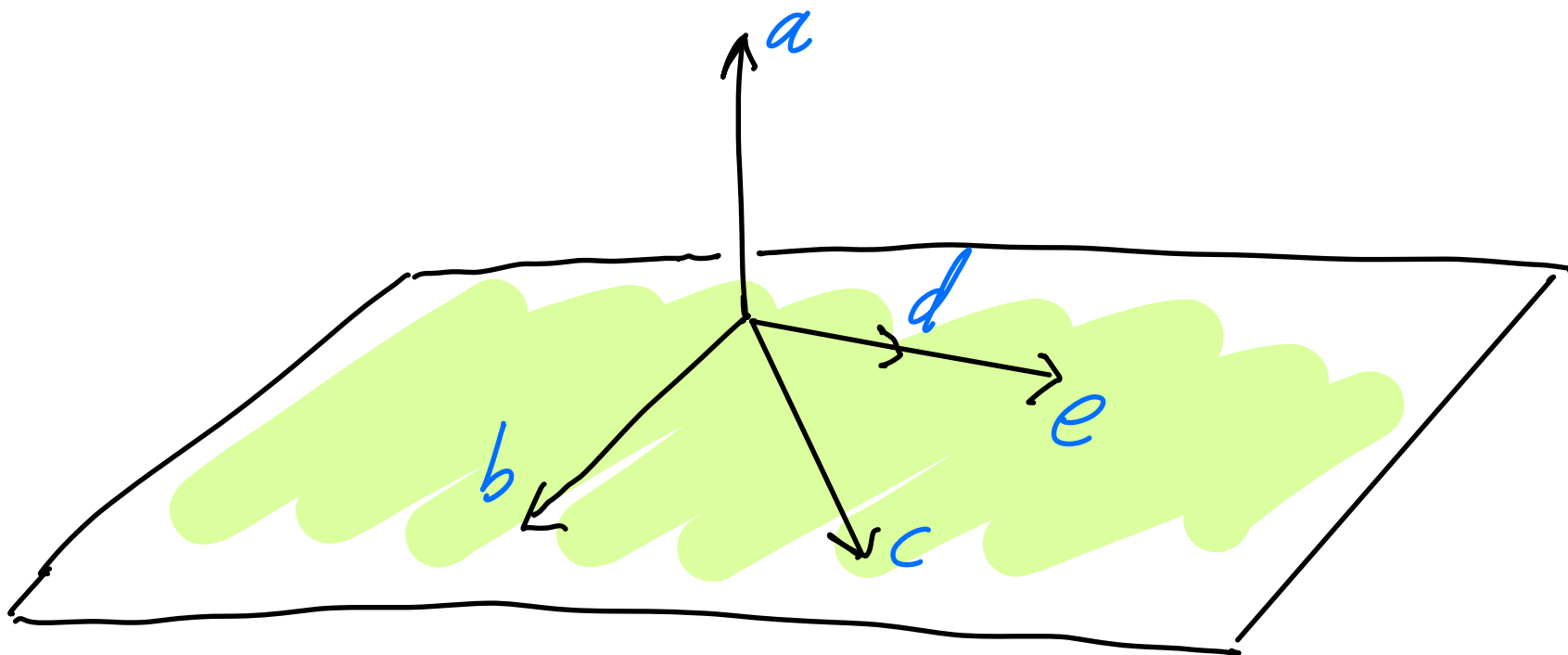
Example from linear algebra

- ▶ Let $V = \{a, b, \dots\}$ be a finite set of vectors in a vector space.
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- ▶ Then $r_k(\mu) = C \cdot f_k$, where f_k is the number of **linearly independent subsets** of V of size k .
- ▶ $(f_0, f_1, f_2, f_3) = (1, 5, 9, 5)$.

Mason's conjecture

► **Mason's strong conjecture** (1972).

The sequence $f_0, f_1, \dots, f_n$, $n = |V|$, satisfies **Newton's inequalities**, i.e.,

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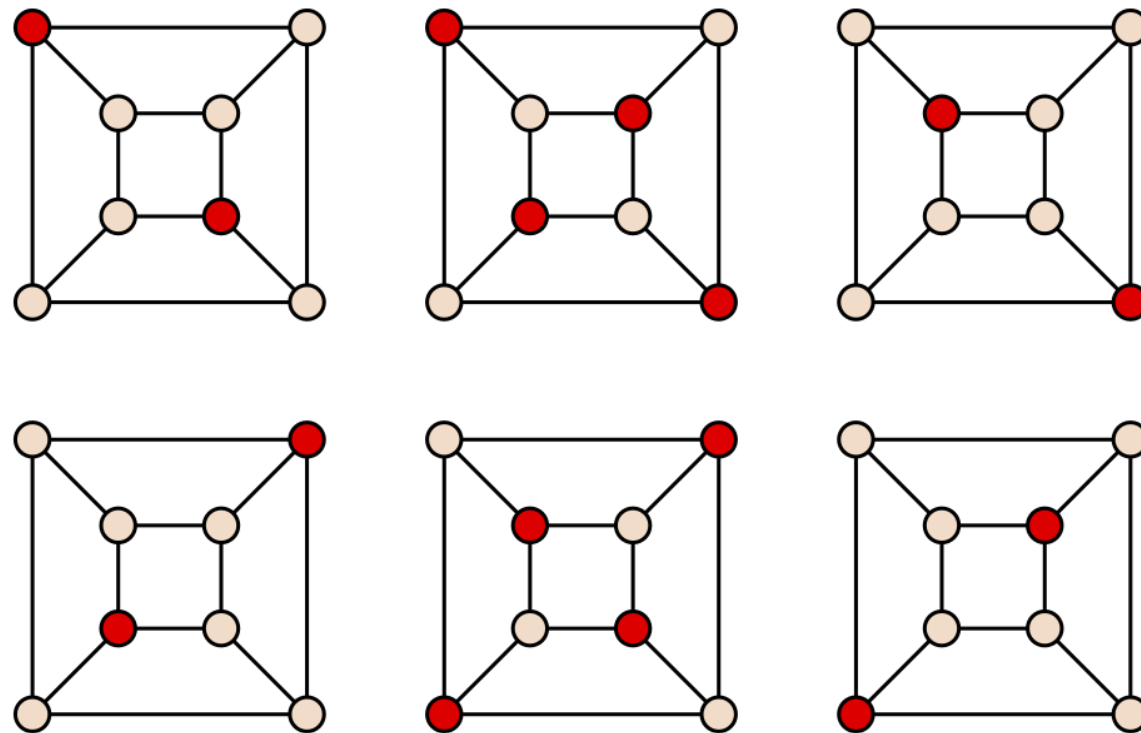
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- ▶ Proved by B., Huh and Anari *et al.* (2020), independently, using **Lorentzian polynomials**.
- ▶ Also

$$\mathbb{P}[X_i = 1 \mid X_j = 1] \leq 2 \cdot \mathbb{P}[X_i = 1].$$

Hard-core repulsive lattice-gas model

- ▶ Let $G = (V, E)$ be a graph.
- ▶ A set S of vertices is called **independent** if S does not contain both vertices of an edge.



Independent sets in graphs

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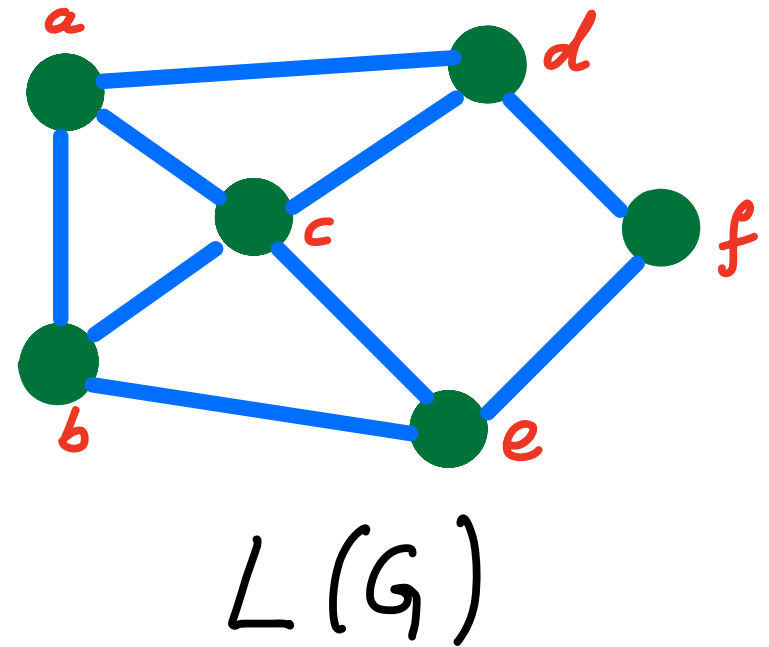
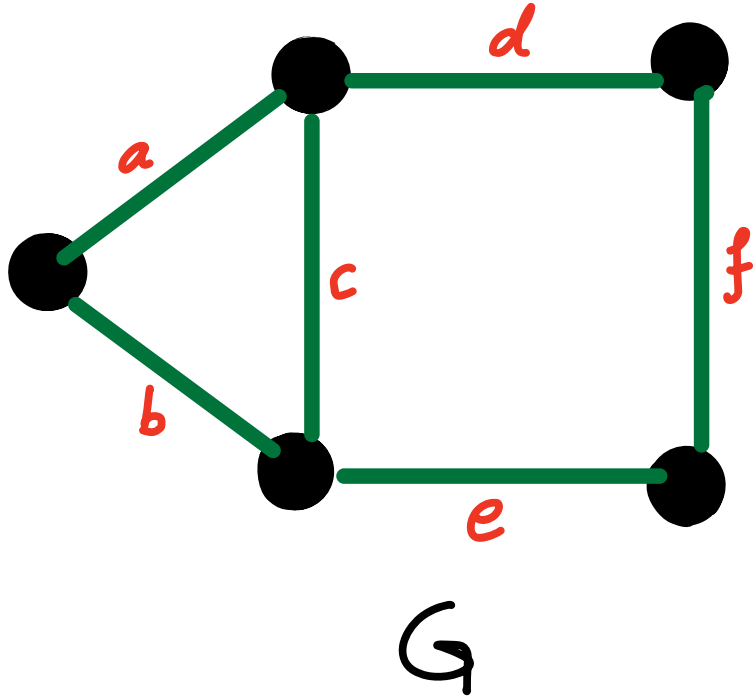
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- ▶ **Alavi, Malde, Schwenk, Erdős (1987):** The sequence $\{i_k(G)\}_{k=0}^{|V|}$ can have arbitrarily many peaks.

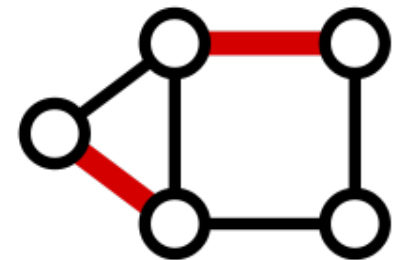
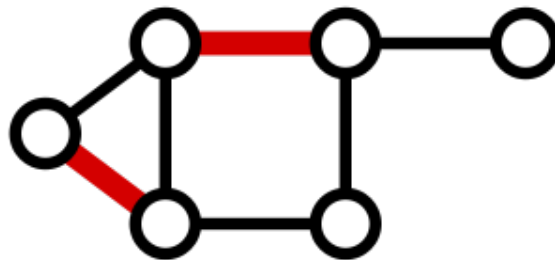
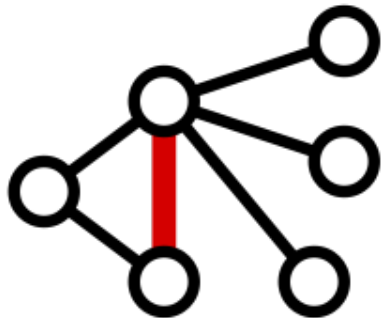
Matchings in graphs

- ▶ The **line graph**, $L(G)$, of the graph $G = (V, E)$ is defined by
 - ▶ The vertices of $L(G)$ are the edges of G
 - ▶ Two vertices in $L(G)$ are adjacent if the corresponding edges in G share a common vertex.



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- ▶ $M \subseteq E$ is a **matching** if no two edges have a common vertex.



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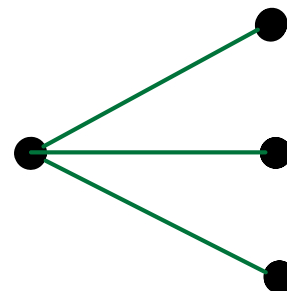
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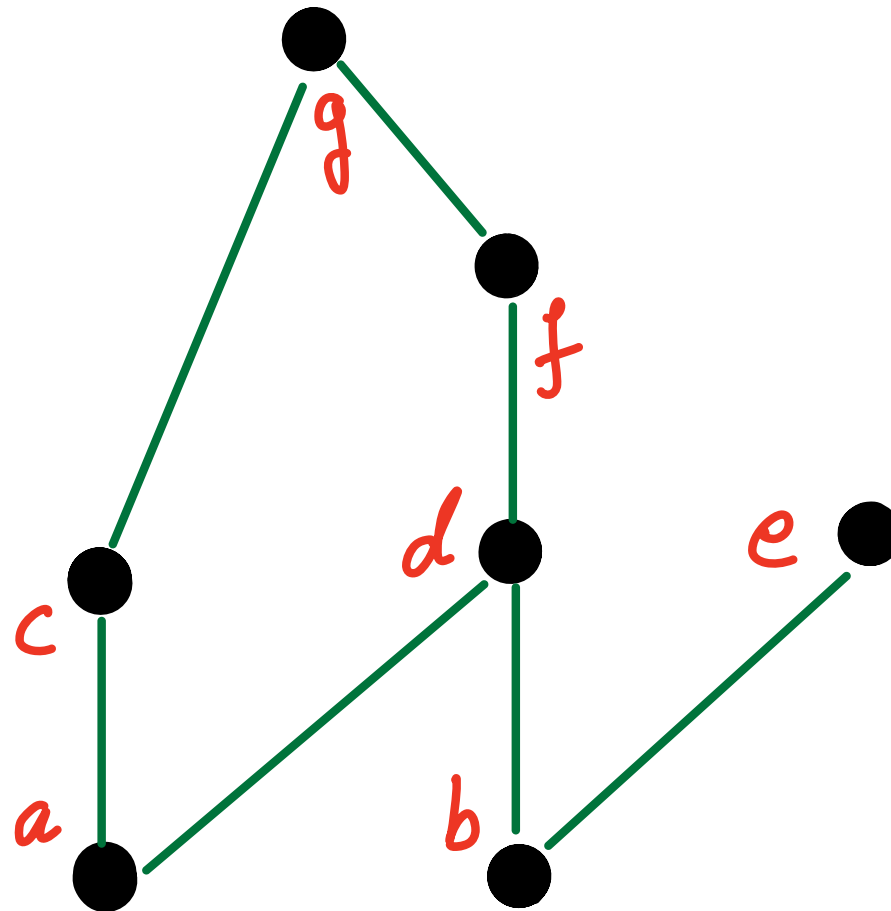
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- ▶ **Theorem** (Chudnovsky–Seymour, 2007). The independence polynomial of a **claw-free graph** is real-rooted.

*No induced
claw*



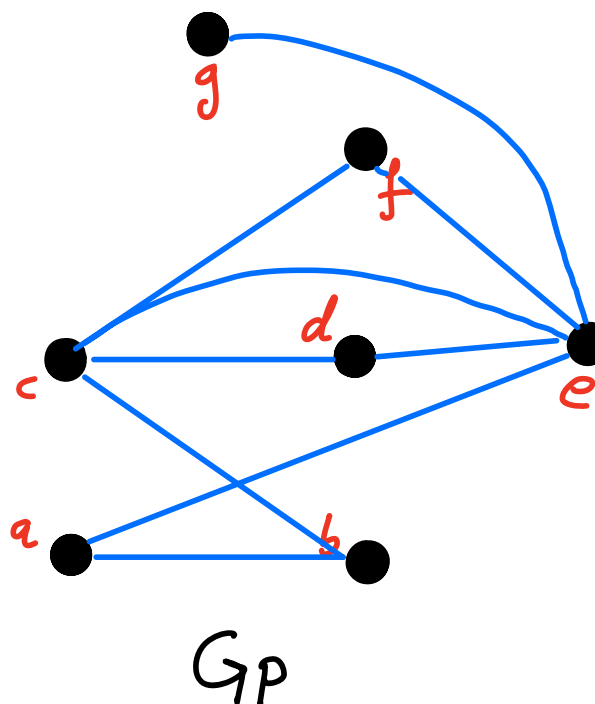
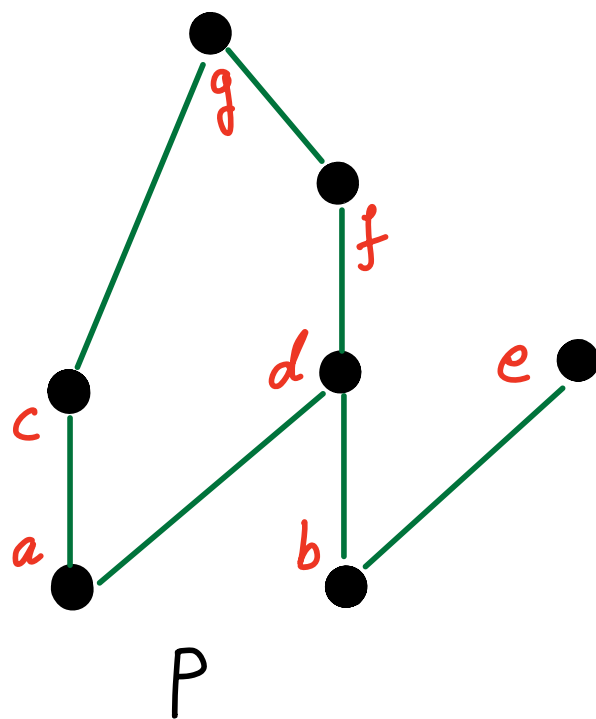
Independent sets from partially ordered sets

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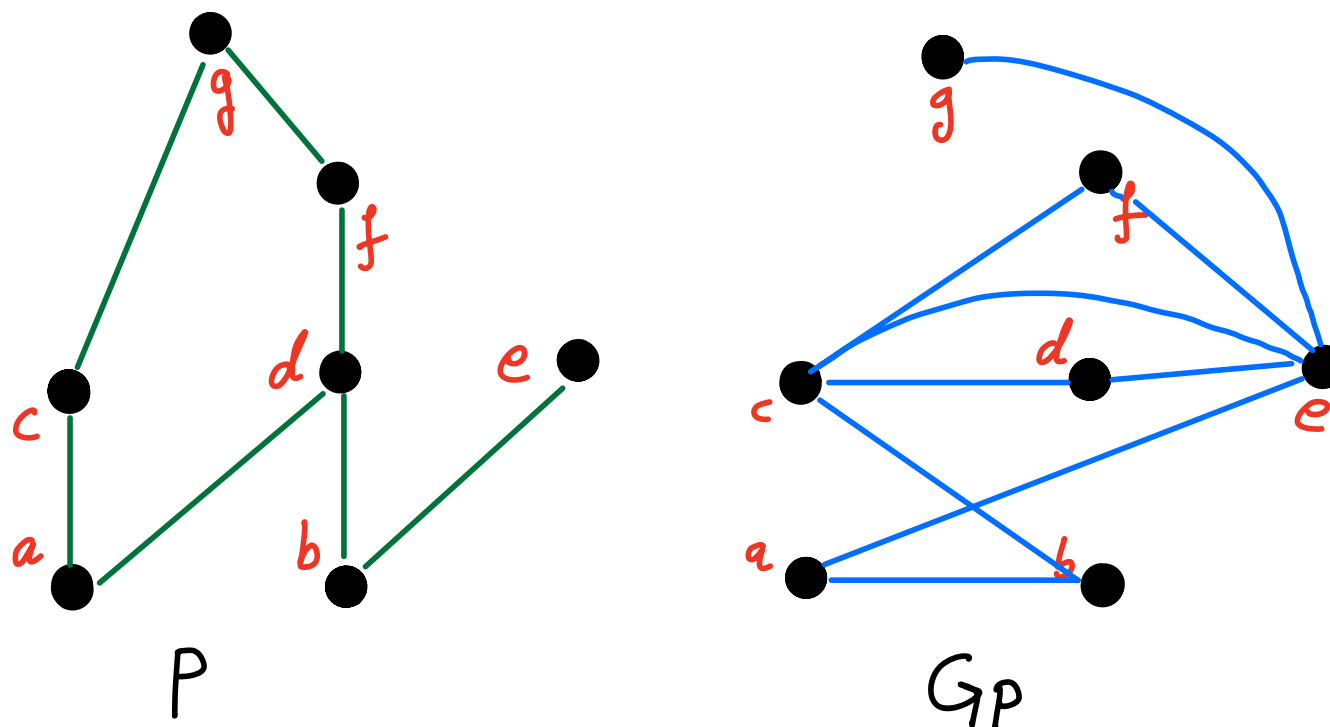
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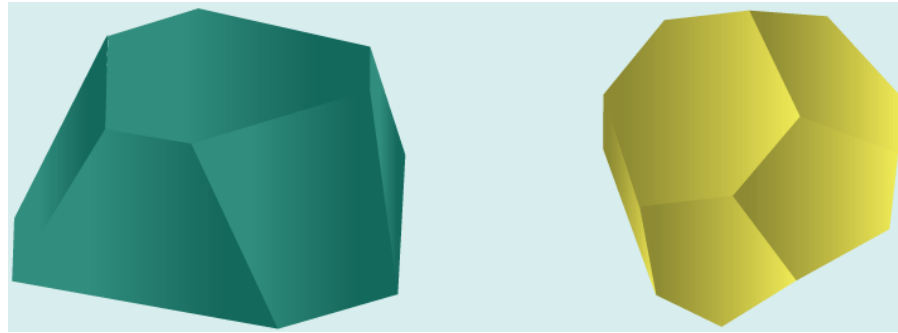
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- ▶ The **chain polynomial** of P is the independence polynomial of G_P :

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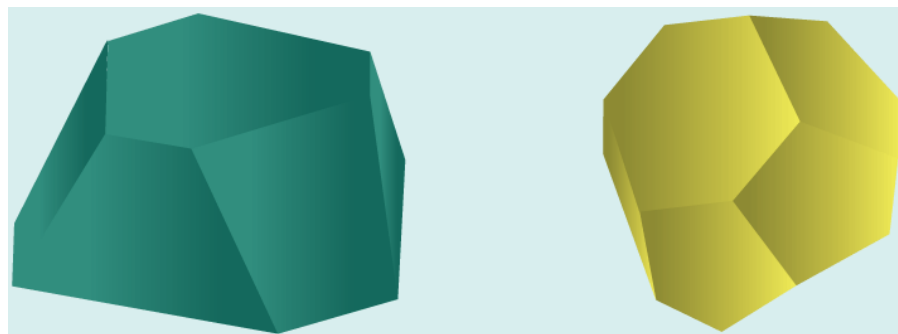
Partially ordered sets from polytopes

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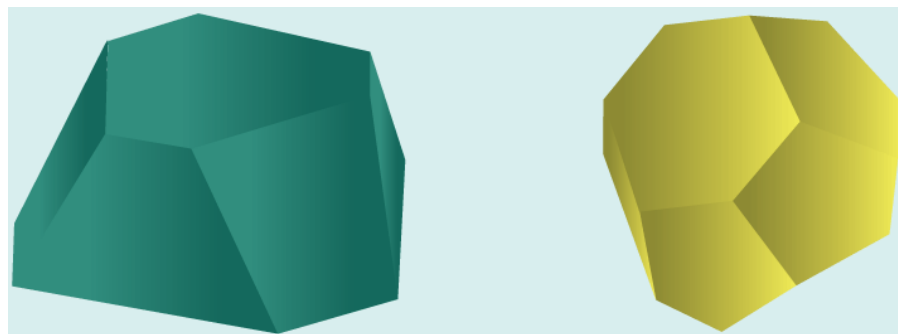
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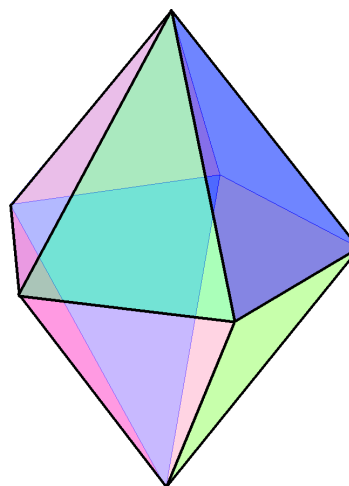
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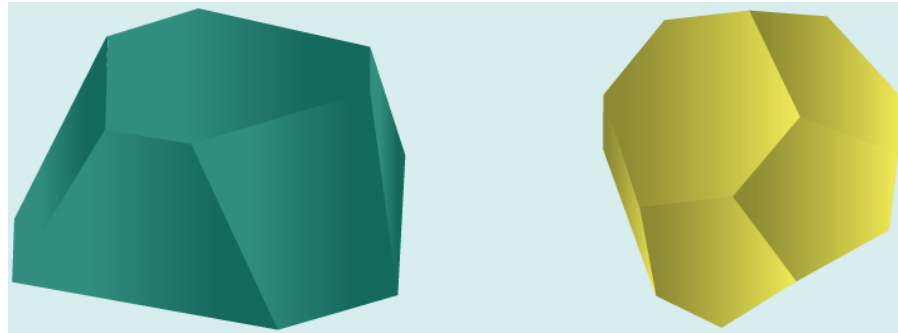


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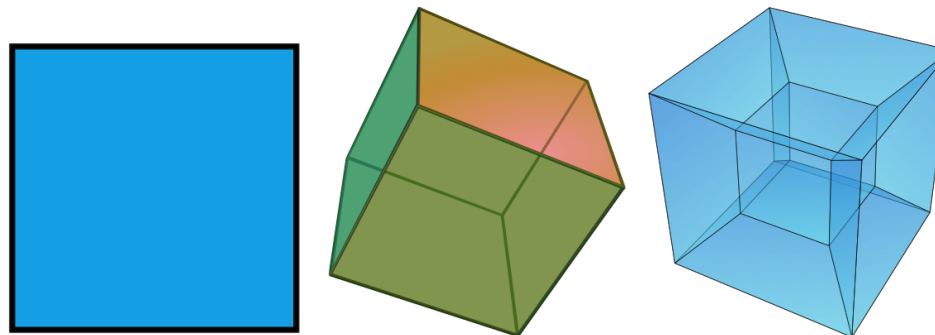


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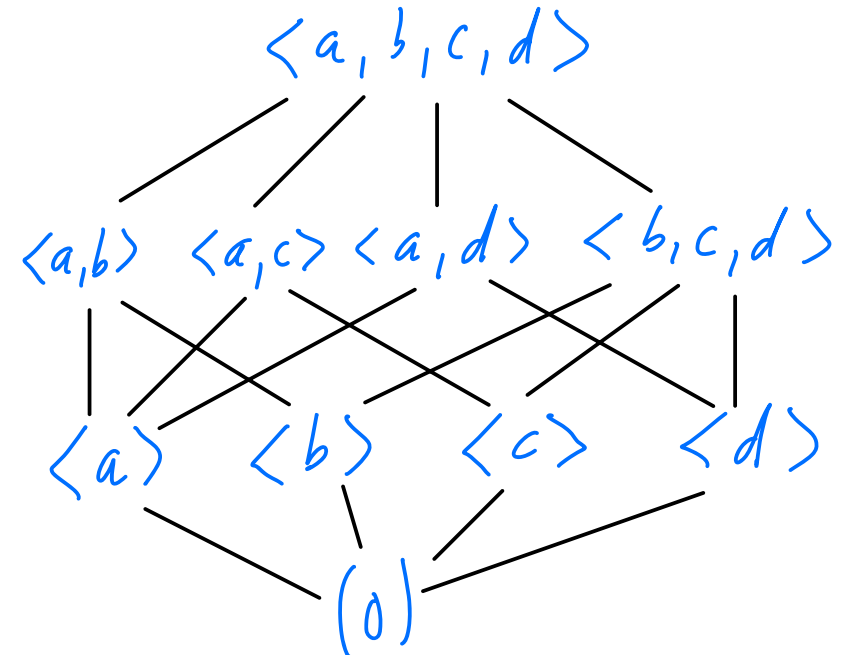
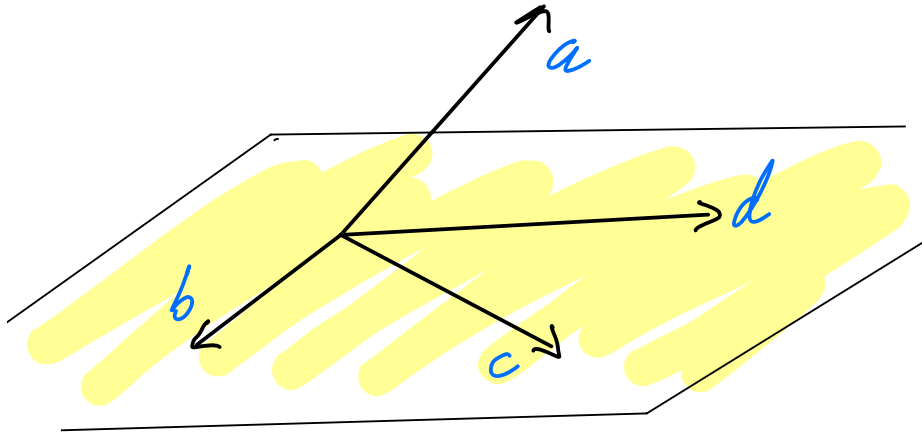


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 - ▶ **Theorem** (Athanasiadis, 2021). True if $\mathcal{P}$ is **cubical**, i.e., all the faces are (hyper-)cubes.



Geometric lattices

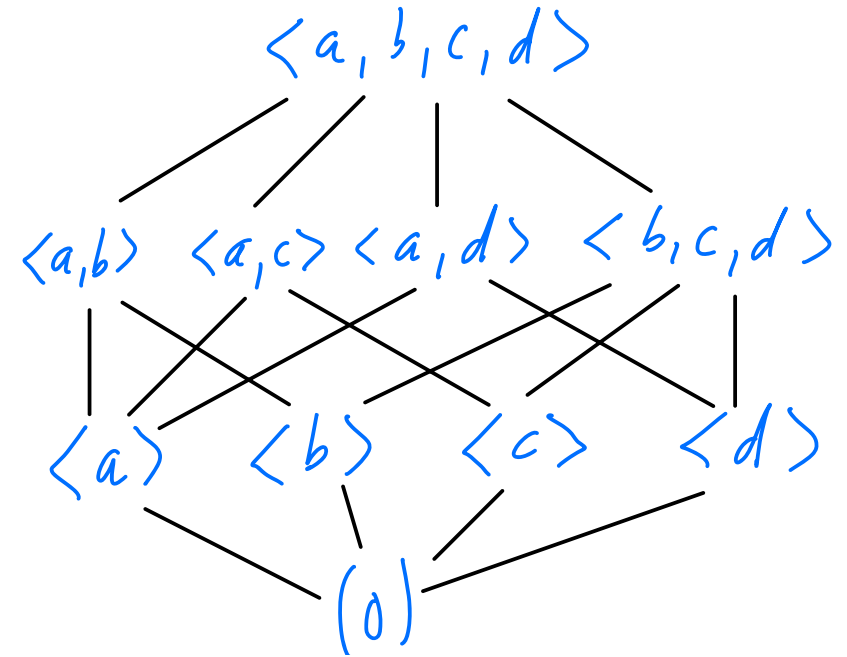
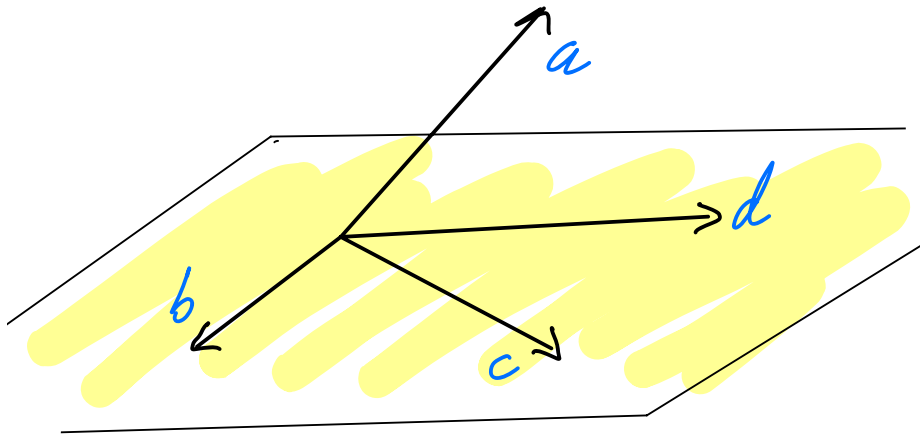
- ▶ Let $V = \{v_1, \dots, v_n\}$ be a set of vectors in a linear space $\mathcal{U}$.
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- ▶ L is a **geometric lattice**.
- ▶ **Conjecture** (Athanasiadis, Kalampogia-Evangelinou, 2023).
The chain polynomial of any geometric lattice is real-rooted.

PLATONIC SOLIDS

TETRAHEDRON

'FOUR SIDED'



△ FIRE

4 FACES
4 POINTS
6 EDGES

OCTAHEDRON

'EIGHT SIDED'



△ AIR

8 FACES
6 POINTS
12 EDGES

HEXAHEDRON

'SIX SIDED'



▽ EARTH

6 FACES
8 POINTS
12 EDGES

ICOSAHEDRON

'TWENTY SIDED'



▽ WATER

20 FACES
12 POINTS
30 EDGES

DODECAHEDRON

'TWELVE SIDED'

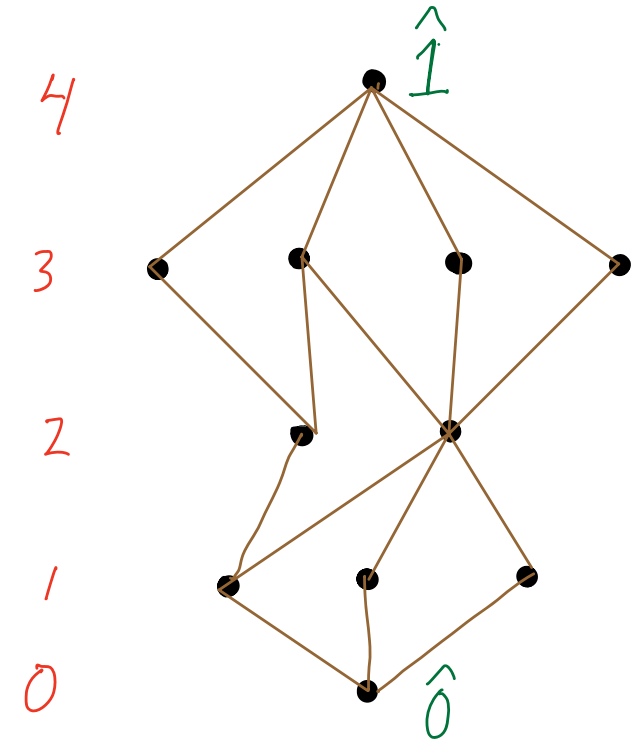


☼ AETHER

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Poset terminology

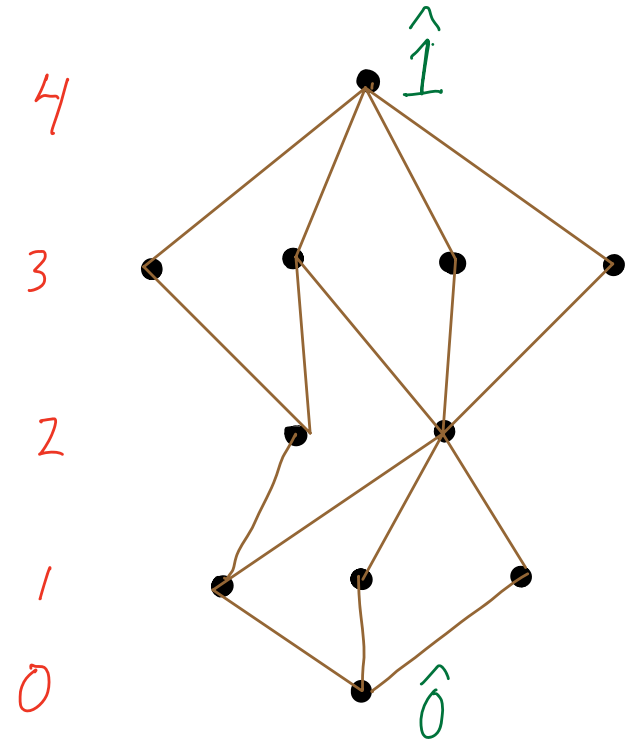
- ▶ Let P be a poset.
- ▶ P has a **least element** $\hat{0}$ if $\hat{0} \leq x$ for all $x \in P$.
- ▶ P has a **greatest element** $\hat{1}$ if $x \leq \hat{1}$ for all $x \in P$.
- ▶ A poset P with $\hat{0}$ is **graded** if there exists a **rank function** $\rho : P \rightarrow \mathbb{N}$ s.t.
 - ▶ $\rho(\hat{0}) = 0$,
 - ▶ $\rho(y) = \rho(x) + 1$ if y covers x .



Poset terminology

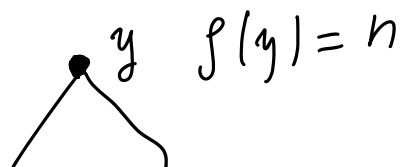
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- ▶ An **interval** in a poset P is a subposet of the form

$$[x, y] = \{z \in P : x \leq z \leq y\}, \quad \text{where } x \leq y \in P.$$



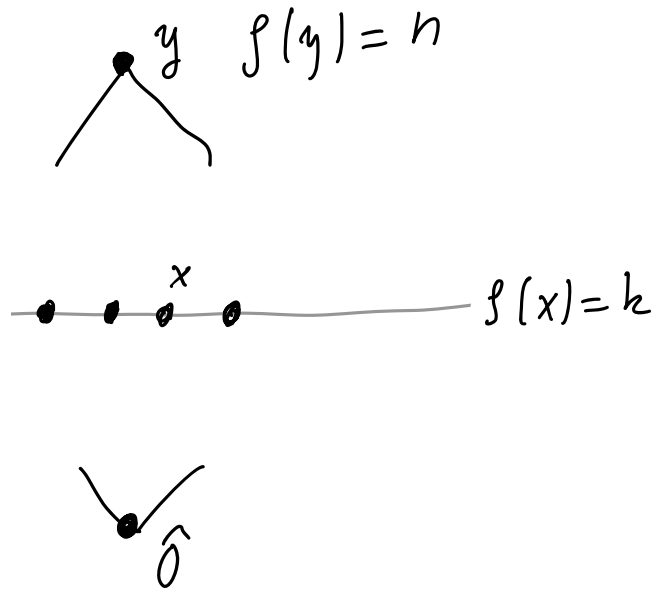
Uniform posets

- ▶ A graded poset P is called **uniform** if there is a matrix $R(P) = (r_{n,k}(P))_{n,k=0}^r$ such that there are exactly $r_{n,k}(P)$ elements of rank k below each element of rank n .



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$$R(P) =$$

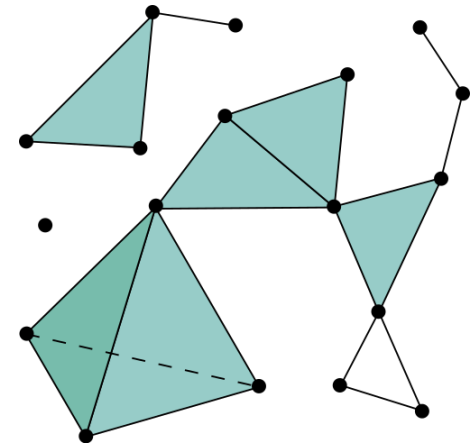
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & * & 1 & 0 & 0 \\ 1 & * & * & 1 & 0 \\ 1 & * & * & * & 1 \end{pmatrix}$$

- ▶ $R(P)$ is **unitriangular**, i.e., lower triangular and all diagonal entries are equal to one.

Example: Simplicial complexes

- Recall that an (abstract) **simplicial complex** on a finite vertex set V is a collection Δ of subsets of V , called **faces**, for which

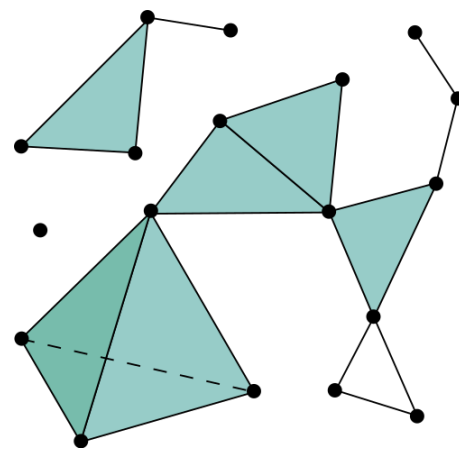
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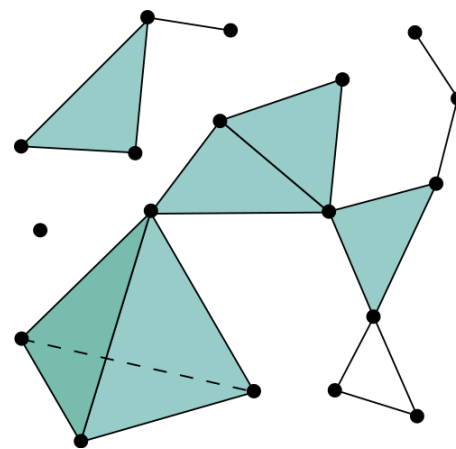


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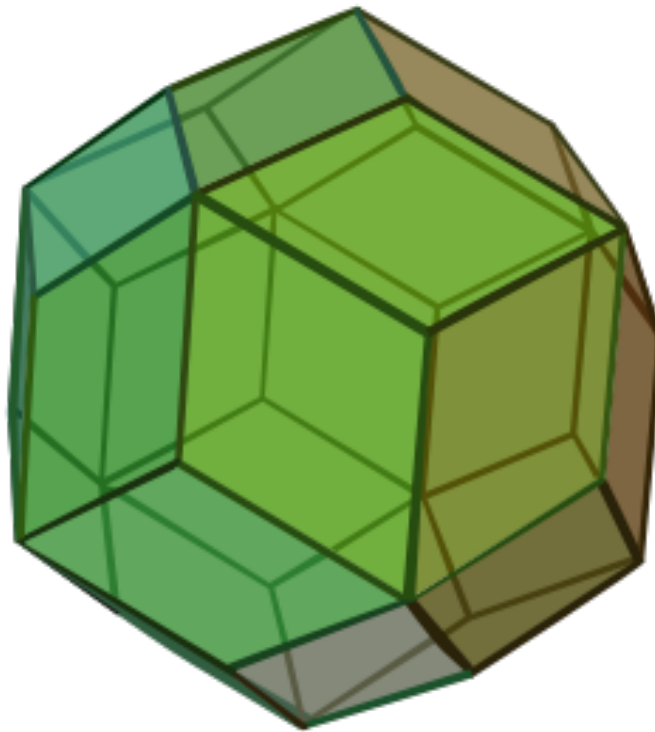
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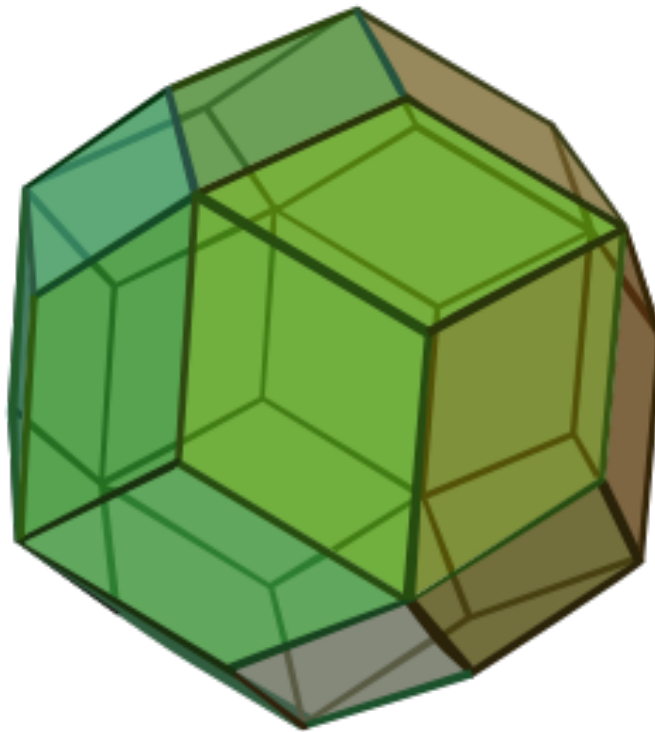
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Example: Projective geometries and q -complexes

- ▶ Let $\mathbb{F}_q$ be a finite field with q elements.
- ▶ Let $\mathbb{B}_n(q)$ be the geometric lattice of all subspaces of $\mathbb{F}_q^n$.

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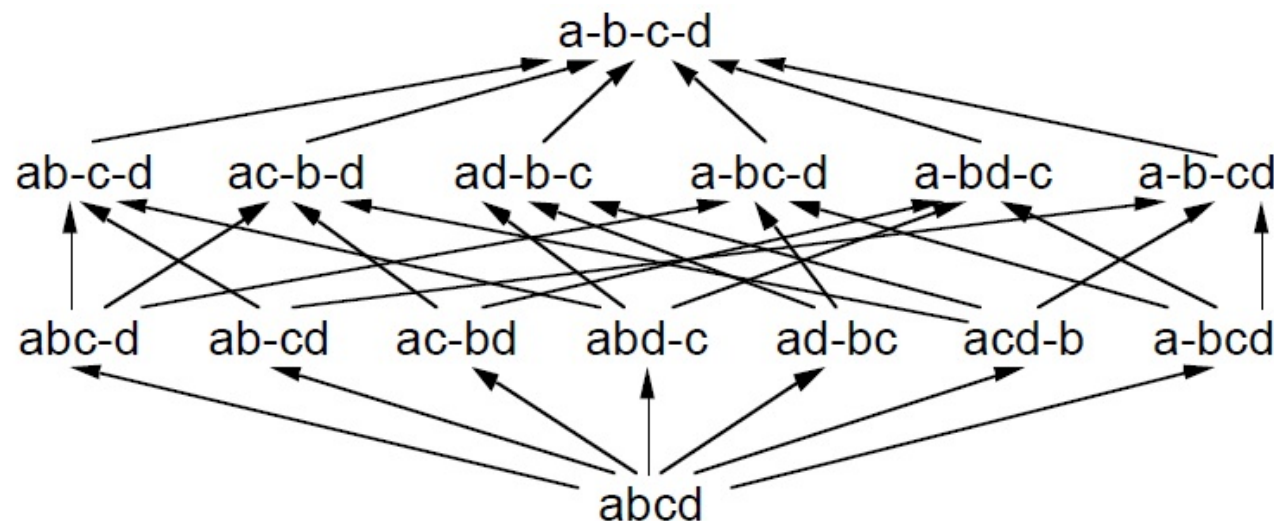
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- ▶ $r_{n,k}(P) = \binom{n}{k}_q$, a **Gaussian binomial coefficient**.

Example: Partition-complexes

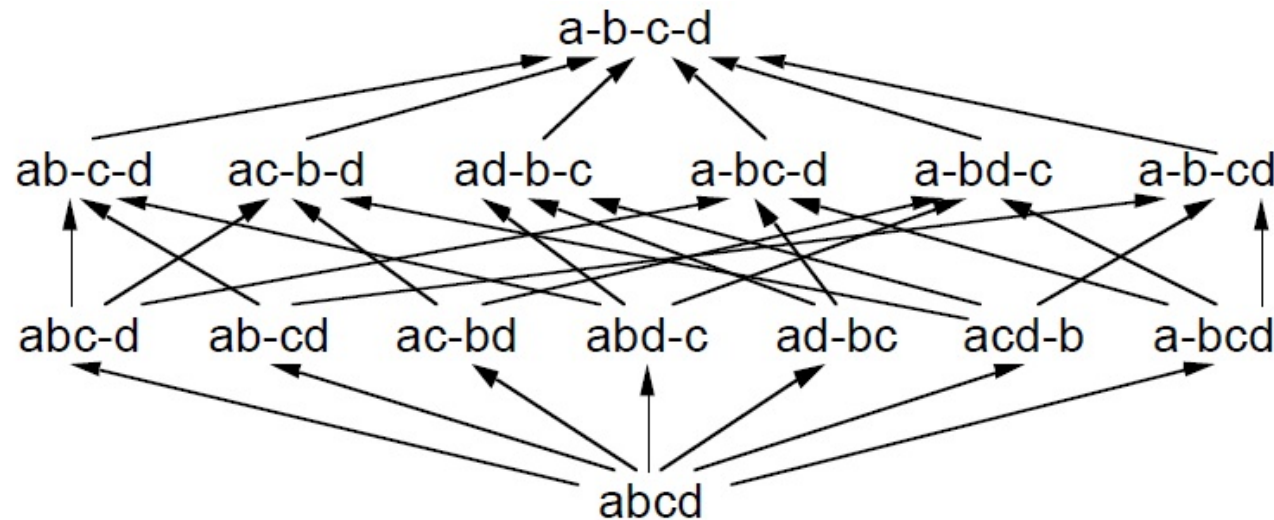
- ▶ Recall that a **partition** of a set S is a collection of disjoint non-empty sets whose union is S .
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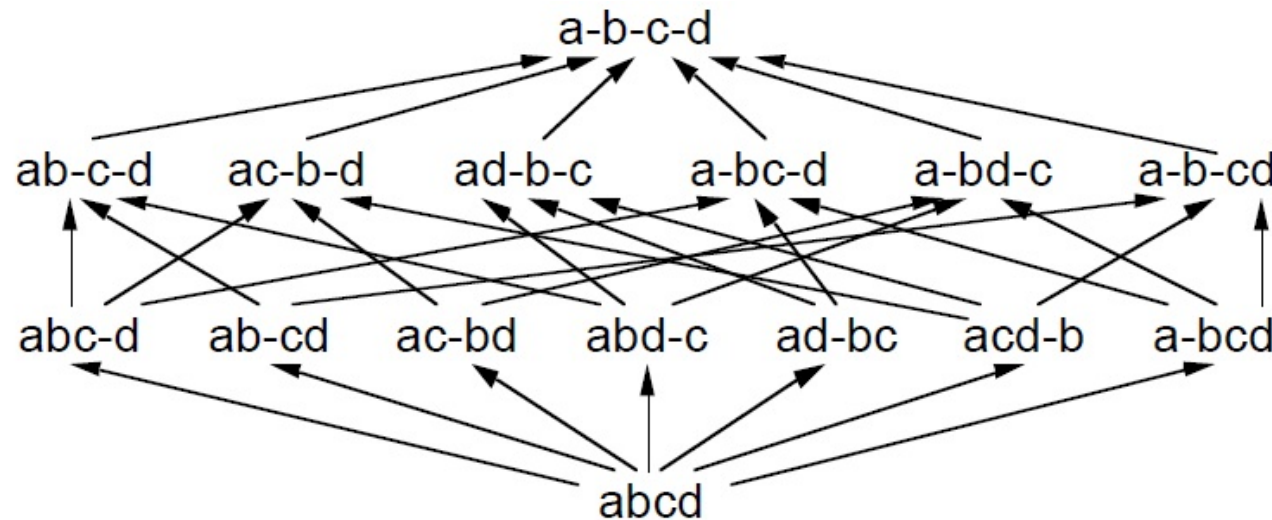
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- ▶ $r_{n,k}(P) = S(n+1, k+1)$, a **Stirling number of the second kind**.

Uniform posets

$$r_{n,k}(\Delta) = \binom{n}{k}, \quad r_{n,k}(\mathbb{B}_n(q)) = \binom{n}{k}_q,$$

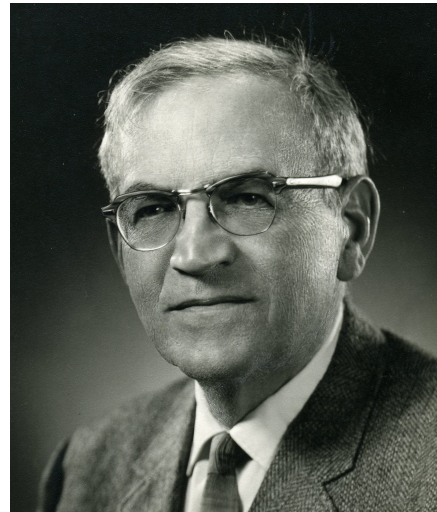
$$r_{n,k}(\Pi_n) = S(n+1, k+1), \quad r_{n,k}(C_n) = \begin{cases} 1 & \text{if } k = 0, \\ 2^{n-k} \binom{n-1}{k-1} & \text{if } k \geq 1. \end{cases}$$

► What do these matrices have in common?



Totally nonnegative matrices

- ▶ A matrix with real entries is **totally nonnegative (TN)** if all its minors are nonnegative.
- ▶ Important in linear algebra, statistics, analysis, geometry, algebraic combinatorics and physics.



Felix Gantmacher, Mark Krein, Isaac Schoenberg, Samuel Karlin

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- ▶ **Theorem** (B., Saud, 2026). The chain polynomial of a TN-poset with $\hat{1}$ is real-rooted.
- ▶ **Theorem** (B., Saud, 2026). The chain polynomial of a uniform geometric lattice is real-rooted.
- ▶ **Theorem** (B., Saud, 2026). The chain polynomial of a paving lattice is real-rooted.

Chain polynomials of matrices

- ▶ Let $R = (r_{n,k})_{n,k=0}^N$ be unitriangular.
- ▶ Define polynomials $p_0(t), p_1(t), \dots, p_N(t)$, by $p_0(t) = 1$ and

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- ▶ We call $p_n(t)$ the n th **chain polynomial** of R .
- ▶ If P is uniform and $R = R(P)$, then

$$p_n(t) = \frac{t}{(1+t)^2} \cdot c_{[\hat{0}, x]}(t), \quad \text{whenever } \rho(x) = n > 1.$$

A general theorem for TN-matrices

- ▶ **Theorem** (B., Saud, 2026). If R is unitriangular and TN, then all zeros of $p_n(t)$ are real and located in the interval $[-1, 0]$.

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- ▶ **Example**. If $r_{n,k} = \binom{n}{k}$, then $p_n(t) = \sum_{k=0}^n k! S(n, k) t^k$.
- ▶ Real-rootedness for this case was first proved by Frobenius.

Pólya frequency sequences

- ▶ $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ is a **Pólya frequency sequence** (PF) if the Toeplitz matrix $(a_{i-j})_{i,j=0}^{\infty}$ is TN, where $a_k = 0$ if $k < 0$.

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ a_4 & a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- ▶ **Theorem** (Aissen, Schoenberg, Whitney, Edrei, 1951).
 $\{a_i\}_{i=0}^{\infty}$ is PF iff

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = C z^N e^{\gamma z} \prod_{i=1}^{\infty} \frac{1 + \alpha_i z}{1 - \beta_i z},$$

where $C, \gamma, \alpha_i, \beta_i$ are nonnegative real numbers, $N \in \mathbb{N}$, and $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty$.

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- ▶ Hence a finite sequence $\{a_i\}_{i=0}^m \subset \mathbb{R}$ is a Pólya frequency sequence iff if all the zeros of $a_0 + a_1 z + \cdots + a_m z^m$ are real and non-positive.

Pólya frequency sequences

- ▶ Landau's Ξ -function is obtained from $\zeta(s)$ by clearing trivial zeroes and poles, and mapping the critical line to the real axis:

$$\Xi(z) = \sum_{n=0}^{\infty} (-1)^n a_n z^{2n}, \quad a_n \geq 0.$$

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- ▶ RH is equivalent to that the Toeplitz matrix $(a_{i-j})_{i,j=0}^{\infty}$ is TN.
- ▶ **Theorem** (Katkova, 2006). All minors of size at most 44 are nonnegative.

Pólya frequency sequences

Theorem (B., Saud, 2026). Suppose

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where $C, \gamma, \alpha_i, \beta_i$ are nonnegative real numbers, $N \in \mathbb{N}$, and $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty$,

and consider

$$\frac{1}{1 - t(f(z) - f(0))} = \sum_{n=0}^{\infty} f_n(t) z^n \in \mathbb{R}[[z, t]].$$

Then $f_n(t)$ is real-rooted, for all n .

A conjecture of Forgács and Tran

- The case when

$$f(z) = \frac{z^N}{\prod_{i=1}^n (1 - \beta_i z)}, \quad \text{where } \beta_i > 0 \text{ for all } i \text{ and } N \in \mathbb{Z}_{>0}.$$

was conjectured by Forgács and Tran (2016).