Analysis and geometry: crystalline structures

Interesting mathematical results are often obtained when ideas from completely different areas of mathematics come together and interact in a non-trivial way cross-fertilising each other. Current project will focus on crystalline structures – discrete sets generalising conventional crystals and recently introduced quasicrystals. It lies entirely in analysis, more precisely spectral theory, but uses important ideas from algebraic geometry, topology and other areas of mathematics.

Crystalline measures popularised by Yves Meyer [4] are atomic measures supported by certain discrete sets Λ with their Fourier transforms (denoted by $\hat{}$) being again atomic measures:

$$\mu = \sum_{x_j \Lambda} a_j \delta_{x_j} \Rightarrow \hat{\mu}(\ell) = \sum_{\ell_j \in S} b_j \delta_{\ell_j}, \quad a_j, b_j \in \mathbb{C}.$$

The interesting case is when the sets Λ and S are not periodic, otherwise one gets classical Poisson summation formula.

The first non-trivial example of a one-dimensional crystalline measure where both sets Λ and S are uniformly discrete has been constructed in our joint paper with P. Sarnak [3]. Our construction grew up from the trace formula proven two decades earlier for Laplacians on metric graphs [2] and used trigonometric polynomials, stable multivariate polynomials and Dirichlet series. Generalisation for several dimensions can be found in our recent paper [1]. This discovery lead to unexpected developments in several seemingly non-connected areas: spectral theory, crystalline measures, trigonometric polynomials, *etc.*, providing excellent opportunities for PhD projects. Here are few possible projects:

- **Reducibility of trigonometric polynomials.** It is conjectured (H. Shapiro) that any two trigonometric polynomials having infinitely many common zeroes have a trigonometric polynomial as a common factor. One may start by looking at real rooted polynomials and use their connection to multivariate stable polynomials.
- **Spectra of Laplacians on metric graphs** can be described via zero sets of multivariate polynomials [2]. It would be interesting by looking at the structure of these zero sets to obtain geometric interpretations for numerous eigenvalue estimates already derived.
- **Spectral theory for crystalline Schrödinger operators.** It is expected that spectra of Schrödinger operators with potentials given by crystalline measures have fractal structure. One starts by analysing few explicit examples of such operators.

Necessary education: mathematically, the project lies on the border between Fourier analysis, discrete mathematics, spectral theory, and number theory, and therefore knowledge in some of these areas is a requirement, whilst certain acquaintance with other areas is desirable.

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More information and how to apply: Deadline: April 21, 2025

`https://su.varbi.com/en/what:job/jobID:803014/`

References

[1] Lior Alon, Mario Kummer, Pavel Kurasov, and Cynthia Vinzant, *Higher dimensional Fourier quasicrystals from Lee-Yang varieties*, Invent. math. **239** (2024), 321-376.

[2] Pavel Kurasov, *Spectral geometry of graphs*, Operator Theory: Advances and Applications, vol. 293, Birkhäuser/Springer, Berlin, [2024] ©2024. MR4697523

[3] P. Kurasov and P. Sarnak, *Stable polynomials and crystalline measures*, J. Math. Phys. **61** (2020), no. 8, 083501, 13, DOI 10.1063/5.0012286. MR4129870

[4] Yves F. Meyer, *Measures with locally finite support and spectrum*, Proc. Natl. Acad. Sci. USA **113** (2016), no. 12, 3152–3158, DOI 10.1073/pnas.1600685113. MR3482845