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Spectral Theory of PDEs VT 2022

3. problem sheet. Each problem gives max. 4 points.
Submission via e-mail before 14 April, 1pm.

Problem 1
On the rectangle $\Omega = (0, \pi) \times (0, 1)$, show that each eigenvalue of the Laplacian on Ω with Dirichlet boundary conditions has multiplicity one. Moreover, determine the lowest eigenvalue which is not Courant-sharp.

Problem 2
Is it possible that the Dirichlet Laplacians on two domains of different dimensions are isospectral? If it is, provide an example; otherwise show that it is not possible.

Problem 3
Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \supset \operatorname{dom} T \to \mathcal{H}$ a symmetric operator (i.e. $\operatorname{dom} T$ is dense in $\mathcal{H}$ and $T \subset T^*$. Show that T is self-adjoint (i.e. $T = T^*$) if and only if

$$\operatorname{ran}(T - \lambda) = \mathcal{H} = \operatorname{ran}(T - \overline{\lambda})$$

holds for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$.[1]

Problem 4
Prove that the differential operator in $L^2(\mathbb{R}^2)$ given by

$$Tu = -\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}, \qquad \operatorname{dom} T = H^2(\mathbb{R}^2),$$

is self-adjoint and compute its spectrum.

Problem 5
Prove that the inverse operator of $-\Delta + I$ (where I denotes the identity operator) in $L^2(\mathbb{R}^3)$ is given by

$$\left((-\Delta + I)^{-1} g\right)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} g(y) \mathrm{d}y$$

for $g \in L^2(\mathbb{R}^3)$.

[1] This is actually equivalent to this equation being true for one fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$.