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Spectral Theory of PDEs VT 2022

2. problem sheet. Each problem gives max. 4 points.
Submission via e-mail before 18 March, 1pm.

Problem 1
Let $\mathcal{K}, \mathcal{H}$ be Hilbert spaces with $\mathcal{K} \subset \mathcal{H}$ and let $a(\cdot,\cdot) : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ be an additional scalar product on $\mathcal{K}$. Show that the function

$$\mathbb{R} \ni t \mapsto F(t) := \frac{a(u + tv, u + tv)}{\|u + tv\|_{\mathcal{H}}^2} \in \mathbb{R}$$

is differentiable for all $u, v \in \mathcal{K}$ and compute its derivative. In particular, what is $F'(0)$?

Problem 2
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $d \geq 1$.[1]

(i) Prove that there exists an orthonormal basis of eigenfunctions of the Dirichlet Laplacian on Ω which consists of real-valued functions only.[2]

(ii) Let $t\Omega$ be the domain obtained from Ω by scaling with a factor $t > 0$. Prove that the Dirichlet Laplacian eigenvalues satisfy $\lambda_k(t\Omega) = \frac{1}{t^2}\lambda_k(\Omega)$.

Problem 3
Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^d$ be bounded domains with Dirichlet Laplacian eigenvalues $\lambda_k(\Omega), \lambda_k(\widetilde{\Omega})$, respectively. Assume that N disjoint domains that are congruent to Ω fit into $\widetilde{\Omega}$. Show that

$$\lambda_{kN}(\widetilde{\Omega}) \leq \lambda_k(\Omega)$$

holds for all k.

Problem 4
Let $\Omega = (0, \sqrt{\pi}) \times (0, \sqrt{\pi}) \subset \mathbb{R}^2$ and

$$u : \Omega \to \mathbb{R}, \quad u(x, y) = x + y.$$

Compute the Schwarz symmetrization u^* of u. Furthermore, check by hand that the integrals

$$\int_\Omega |u|^2 \mathrm{d}x \qquad \text{and} \qquad \int_{\Omega^*} |u^*|^2 \mathrm{d}x$$

coincide.

Problem 5
Show that...

(i) . . . among all bounded domains $\Omega \subset \mathbb{R}^2$ of fixed area there exists no maximizer of the first Dirichlet Laplacian eigenvalue $\lambda_1(\Omega)$.[3]

(ii) . . . among all bounded domains $\Omega \subset \mathbb{R}^2$ of fixed area there exists no minimizer of the first nontrivial Neumann Laplacian eigenvalue $\mu_2(\Omega)$.

(iii) . . . among all bounded domains $\Omega \subset \mathbb{R}^2$ with fixed perimeter there exists no maximizer of the first nontrivial Neumann Laplacian eigenvalue $\mu_2(\Omega)$.

[1] The statements of this problem are formulated for Dirichlet boundary conditions but are true for the Neumann and Robin cases as well.
[2] Recall that, a priori, the eigenfunctions we get from the discrete spectral theorem are complex-valued.
[3] In other words, $\sup \lambda_1(\Omega) = +\infty$ where Ω ranges over all bounded domains of a given fixed area.